CHAP5c Oscillator, Filter, and Resonant circuit

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Oscillation Circuit
A second-order linear differential equation has the form

\[ P(t) \frac{d^2y}{dt^2} + Q(t) \frac{dy}{dt} + R(t)y = G(t) \]  

(a)

where \( P(t), \ Q(t), \ R(t) \) and \( G(t) \) are constant functions. If \( G(t) = 0 \) for all \( t \), Equation (a) is homogenous linear equation. Thus, the form of a second-order linear homogeneous differential equation is

\[ P(t) \frac{d^2y}{dt^2} + Q(t) \frac{dy}{dt} + R(t)y = 0 \]  

(b)
Theorem If $y_1$ and $y_2$ are both solutions of the linear homogeneous equation (b) and $c_1$ and $c_2$ are any constants, then the function is also a solution of Equation (b).

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

若 $y_1$ 與 $y_2$ 兩者都是方程式 (b) 的解，則 $y$ 也是方程式 (b) 的解
Solution... $y_1 \checkmark y_2$

Special case，求解...

$P(t) \frac{d^2y}{dt^2} + Q(t) \frac{dy}{dt} + R(t)y = 0$

where $P(t)$, $Q(t)$, and $R(t)$ are constant functions.

$\rightarrow ay'' + by' + cy = 0$ \hspace{1cm} (c)

Where $a$, $b$, and $c$ are constants and $a \neq 0$

General solution？假設equation (c) 具有指數函數解

Let $y = e^{rt}$. We have $y' = re^{rt}$ $y'' = r^2e^{rt}$

……代入方程式（c）
Solution...

将 \( y = e^{rt} \) 代入 equation (c)

\[ ar^2 e^{rt} + bre^{rt} + ce^{rt} = 0 \]

这要为零，才能符合！

或 

\[ (ar^2 + br + c)e^{rt} = 0 \]

\( e^{rt} \) 不为零。因此，\( y = e^{rt} \) 要真的为 equation (c) 的解，\( r \) 必须是 \( ar^2 + br + c = 0 \) 的根。

\[ ar^2 + br + c = 0 \] (d) \( r \) ????

Equation (d) is called the auxiliary equation or characteristic equation of the differential equation \( ay'' + by' + c = 0 \).

(d)为(c)的辅助方程式或称之为特徵方程式
Solution…

The roots $r_1$ and $r_2$ of the **auxiliary equation**

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

**CASE 1.** $b^2 - 4ac > 0$ 第一種可能….  
In this case the roots of the auxiliary equation are real and distinct, so $y_1 = e^{r_1t}$ and $y_2 = e^{r_2t}$ are two linearly independent solutions.

If the roots $r_1$ and $r_2$ of the auxiliary equation $ar^2 + br + c = 0$ are real and unequal, then the general solution of $ay'' + by' + cy = 0$ is $y = c_1 e^{r_1t} + c_2 e^{r_2t}$ 方程式（c）的解可以如此寫……
CASE 2. $b^2 - 4ac = 0$  第二種可能…
In this case, $r_1 = r_2$; that is, the roots of the auxiliary equation are real and equal. Let’s denote by $r$ the common value of $r_1$ and $r_2$. From equation (e), we have

$$r = -\frac{b}{2a}$$

We know that $y_1 = e^{rt}$ is one solution.

Another solution? $y_2 = te^{rt}$

If the auxiliary equation $ar^2 + br + c = 0$ has only one real root $r$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{rt} + c_2te^{rt}$$
CASE 3. \( b^2 - 4ac < 0 \) 第三種可能….
In this case the roots of the auxiliary equation are complex numbers. We can write
\[
r_1 = \alpha + i \beta \quad r_2 = \alpha - i \beta
\]
Where \( \alpha \) and \( \beta \) are real number
\[
\alpha = -\frac{b}{2a} \quad \beta = \frac{\sqrt{4ac - b^2}}{2a}
\]
The solution of the differential equation
\[
y = c_1 e^{r_1 t} + c_2 e^{r_2 t} = \ldots = e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t)
\]
Solution…

If the roots of the auxiliary equation $ar^2 + br + c = 0$ are the complex numbers $r_1 = \alpha + i\beta$, $r_2 = \alpha - i\beta$, then the general solution of $ay'' + by' + cy = 0$ is

$$y = c_1e^{\alpha t} + c_2e^{\alpha t} = \ldots = e^{\alpha t}(c_1\cos \beta tx + c_2\sin \beta t)$$
LC & RLC振盪電路

輸入電壓源？沒有！
合理懷疑

前面已經講過了嗎？
前面，有輸入電壓源。
現在，沒有輸入電壓源。電路是靠著預先儲存在電容的電量來啟動。
現在，我們要了解啟動後，電路電流將如何變動？它的變動與電路上的R、L、C必然有關…
假設開關未接通時電容器上的電量為$Q_0$，開關接通後迴路電流為$i(t)$

By KVL and KCL

$$v_L + v_C = 0 \quad i_L + i_C = 0$$

$$v_L = L \frac{di_L(t)}{dt} \quad i_C = C \frac{dv_C(t)}{dt}$$

$$\frac{d^2 i(t)}{dt^2} + \frac{1}{LC} i(t) = 0 \quad \text{Let} \quad \omega_0^2 = \frac{1}{LC}$$

$$\Rightarrow \frac{d^2 i(t)}{dt^2} + \omega_0^2 i(t) = 0$$
The complete solution to the differential equation is

\[ i(t) = Ae^{j\omega_0 t} + Be^{-j\omega_0 t} \]

A and B can be solved by considering the initial conditions.

\[ i(t = 0) = ? \]

\[ \frac{di(t = 0)}{dt} = ? \]

LC振盪電路實際上並不存在。
一般的振盪電路多少都會含有電阻，使得系統的振盪造成阻尼衰減。
RLC振盪電路

假設開關未接通時電容器上的電量為 $Q_0$，開關接通後迴路電流為 $i(t)$

$$v_R + v_L + v_C = 0$$

$$R \cdot i(t) + L \cdot \frac{di(t)}{dt} + \frac{1}{C} \int i(t) dt = 0 \quad (1)$$

$$\frac{d^2i(t)}{dt^2} + \frac{R}{L} \frac{di(t)}{dt} + \frac{1}{LC} i(t) = 0 \quad (2)$$

Let

$$\alpha = \frac{R}{2L} \quad \omega_0^2 = \frac{1}{LC}$$

$$\Rightarrow \frac{d^2i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + \omega_0^2 i(t) = 0 \quad (3)$$
α is called the *neper frequency* (納頻率), or *attenuation* (衰減係數), and is a measure of how fast the transient response of the circuit will die away after the stimulus has been removed.

ω₀ is the angular resonance frequency (諧振頻率).

Define

\[ \zeta = \frac{\alpha}{\omega_0} = \frac{R}{2 \sqrt{LC}} \]  Which is the damping factor.
The differential equation (3) has the characteristic equation

\[ r^2 + 2\alpha r + \omega_0^2 = 0 \]

\[ r_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad r_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

The general solution of equation (3)

\[ i(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \]

The coefficients \( A_1 \) and \( A_2 \) are determined by the initial conditions of the specific problem being analyzed. That is, they are set by the values of the currents and voltages in the circuit at the onset of the transient and the presumed value they will settle to after infinite time.
由於R、L、C值的不同，會有三種不同振盪的情形。

- $\zeta > 1$ overdamped response
- $\zeta < 1$ underdamped response
- $\zeta = 1$ critically damped response
Overdamped Response

\[ \zeta > 1 \quad \frac{R}{2L} > \frac{1}{\sqrt{LC}} \Rightarrow i(t) = A_1 e^{-\omega_0 (\zeta + \sqrt{\zeta^2 - 1}) t} + A_2 e^{-\omega_0 (\zeta - \sqrt{\zeta^2 - 1}) t} \]

- The overdamped response is a decay of the transient current without oscillation.

\[ \alpha = \frac{R}{2L} \quad \omega_0^2 = \frac{1}{LC} \]

\[ \zeta = \frac{\alpha}{\omega_0} = \frac{R}{2} \sqrt{\frac{C}{L}} \]
Underdamped Response

\[ \zeta < 1 \quad \frac{R}{2L} < \frac{1}{\sqrt{LC}} \]

\[ \Rightarrow i(t) = B_1 e^{-\alpha t} \cos(\omega_d t) + B_2 e^{-\alpha t} \sin(\omega_d t) \Rightarrow i(t) = B_3 e^{-\alpha t} \sin(\omega_d t + \phi) \]

\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} = \omega_0 \sqrt{1 - \zeta^2} \]

called the damped resonance frequency or the damped natural frequency

The underdamped response is a decaying oscillation at frequency \( \omega_d \). The oscillation decays at a rate determined by the attenuation \( \alpha \). The exponential in \( \alpha \) describes the envelope of the oscillation.
Underdamped Response

\[ x(t) = e^{-\alpha t} \sin(\omega_d t + \varphi) \]

http://upload.wikimedia.org/wikipedia/commons/0/0a/Underdamped_oscillation_xt.png
Critically Damped Response

\[ \zeta = 1 \quad \frac{R}{2L} = \frac{1}{\sqrt{LC}} \Rightarrow i(t) = D_1 e^{-\alpha t} + D_2 te^{-\alpha t} \]

- The critically damped response represents the circuit response that decays in the fastest possible time without going into oscillation.
- This consideration is important in control systems where it is required to reach the desired state as quickly as possible without overshooting.
- \( D_1 \) and \( D_2 \) are arbitrary constants determined by boundary conditions.
Critical damping vs. Over damping

A critically damped system is one in which the system does not oscillate and returns to its equilibrium position without oscillating.

In an overdamped system the system does not oscillate and returns to its equilibrium position without oscillating but at a slower rate compared to a critically damped system.

回到平衡位置的速率，critically damped system 比 over damped system 快
\( \omega_d \) VS. \( \omega_0 \)

- **damped resonance frequency** \( \omega_d \)
  - the frequency the circuit will naturally oscillate at if not driven by an external source.

- **resonance frequency or undamped resonance frequency** \( \omega_0 \)
  - the frequency at which the circuit will resonate when driven by an external oscillation.
Effects of damping

- $\zeta = 0.1$
- $\zeta = 0.25$
- $\zeta = 0.5$
- $\zeta = 0.75$
- $\zeta = 1.0$ (Critically damped)
- $\zeta = 1.5$ (Overdamped)

(Under damped)

Steady-state
使用三種不同的電阻，比較response

\[ \zeta = \frac{\alpha}{\omega_0} = \frac{R}{2} \sqrt{\frac{C}{L}} \]

\[ 0.1 \ \mu F \]

\[ 1 \ mH \]

\[ 300 \ \Omega \rightarrow \]

\[ 200 \ \Omega \rightarrow 100 \ \Omega \]
Sinusoidal

Frequency Response
合理懷疑

前面已經講過了嗎？
前面，輸入電壓源的頻率是固定。
現在，我們想藉由一個工具（表示式），通盤了解不同頻率下，電路的輸出與輸入關係。
Frequency Response

- 電路的「sinusoidal frequency response（或frequency response）」用來衡量一個電路在承受到任意頻率（振幅不變，頻率有變）的輸入信號（sinusoidal input）時，會有怎樣的反應（測量電路相對輸出端的響應）？是動詞，也是名詞。
- 只要知道電路的frequency response，只要知道input（包括amplitude, phase, and frequency），就可以推算output。
定義 Frequency Response

$$H_v(j\omega) = \frac{V_L(j\omega)}{E_s(j\omega)}$$

就叫它為電路的 frequency response

以響應的幅度（用分貝）和相位（用弧度）來表示

$$H_v(j\omega) = |H_v(j\omega)| \angle \Phi_H$$
表達 Frequency Response

以響應的幅度（用分貝）和相位（用弧度）來表示

\[ H_v(j\omega) = |H_v(j\omega)| \angle \Phi_H \]

- **Magnitude** \(|H(j\omega)|\) vs. \(\omega/\omega_0\)
- **Phase** \(\Phi_H\) vs. \(\omega/\omega_0\)
EXAMPLE 1/2

\[ Z_{TH} = (Z_S + Z_1) \parallel Z_2 \]

\[ E_{TH} = E_S \frac{Z_2}{Z_S + Z_1 + Z_2} \]
EXAMPLE 2/2

把load放回Thevenin equivalent source circuit可以得到

\[ V_L = \frac{Z_L}{Z_L + Z_T} E_{TH} \]

\[ V_L = \frac{Z_L Z_2}{Z_L (Z_S + Z_1 + Z_2) + Z_2 (Z_S + Z_1)} E_S \]

The frequency response of the circuit:

\[ \frac{V_L(j\omega)}{E_S(j\omega)} = H_V(j\omega) = \frac{Z_L Z_2}{Z_L (Z_S + Z_1 + Z_2) + Z_2 (Z_S + Z_1)} \]
解讀 Frequency Response

\[ H_v(j\omega) = \frac{V_L(j\omega)}{E_s(j\omega)} \]

\[ H_v(j\omega) = |H_v(j\omega)| \angle \Phi_H \]

- \( V_L(j\omega) \) 與 \( E_s(j\omega) \) 間的大小（magnitude）與相位角關係分別為 amplitude-scaled 與 phase-shifted。
- 只要 \( H_v(j\omega) \) 與 \( E_s(j\omega) \) 已知，就可以求 \( V_L(j\omega) \)。

\[ V_L = |H_v| \cdot E_S \quad \Phi_L = \Phi_H + \Phi_S \]
EXAMPLE: Computing the Frequency Response of a Circuit

Compute the frequency response $H_v(j\omega)$ for the circuit.

Known: $R_1 = 1\,k\Omega$, $C = 10\mu F$, $R_L = 10k\Omega$

Find: The frequency response $H_v(j\omega) = \frac{V_L(j\omega)}{E_s(j\omega)}$
Solution

The Thevenin equivalent circuit is

\[ Z_{TH} \]

其中， \( Z_{TH} = \frac{Z_1}{Z_1 + Z_2} \), \( E_{TH} = V_S \frac{Z_2}{Z_1 + Z_2} \)

\[ Z_1 = 10^3 \Omega \quad Z_L = 10^4 \Omega \quad Z_2 = \frac{1}{j\omega C} = \frac{1}{j\omega \times 10^{-5}} \]

Using the voltage divider rule and the equivalent circuit, we obtain the following expression:

\[ V_L = E_{TH} \frac{Z_L}{Z_T + Z_L} = H_V E_S \]

\[ H_V(j\omega) = \frac{V_L(j\omega)}{E_S} = \frac{Z_1 Z_2}{Z_L(Z_1 + Z_2) + Z_1 Z_2} = \ldots = \frac{100}{\sqrt{110^2 + \omega^2}} - \arctan\left(\frac{\omega}{100}\right) \]
Filters
A simple RC filter:

\[ H(j\omega) = \frac{V_o}{V_i}(j\omega) \]

其中，\( V_i \) 與 \( V_o \) 分別為輸入與輸出電壓。

\[ V_o(j\omega) = V_i(j\omega) \frac{1}{j\omega C} \frac{1}{R + \frac{1}{j\omega C}} \]
Low Pass Filter $^{2/4}$

The frequency response of the RC filter is

\[
\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega CR}
\]

由此式可以發現：

當 $\omega = 0$ 時，the value of the frequency response function is 1，

即 the filter is passing all of the input.

**WHY?**

當 $\omega = 0$，$\frac{1}{j\omega C} \to \infty$，

the capacitor acts as an open circuit, and $V_o(j\omega = 0) = V_i(j\omega = 0)$

當 $\omega$ 增加，the magnitude of the frequency response 減少。
在濾波器中，會將此頻率點以上（或以下）頻率成份濾掉。

低通濾波器的頻率響應為：

\[ H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{1 + j\omega CR} = \frac{1}{\sqrt{1 + (\omega CR)^2}} e^{-j\arctan(\omega CR)} \]

或 \[ H(j\omega) = |H(j\omega)|e^{j\Phi_H(j\omega)} \]

其中，\[ |H(j\omega)| = \frac{1}{\sqrt{1 + (\omega CR)^2}} = \frac{1}{\sqrt{1 + (\omega / \omega_c)^2}} \]

\[ \Phi_H(j\omega) = -\arctan(\omega CR) = -\arctan\left(\frac{\omega}{\omega_c}\right) \]

其中，\[ \omega_c = \frac{1}{RC} \] is called the cutoff frequency of the filter。\[ \omega_c \] gives an indication of the filtering characteristics of the circuit。
The phasor voltage $V_i = V_i e^{j\phi_i}$ is scaled by the factor $|H|$ and shifted by a phase angle $\Phi_H$ at each frequency.

So that, the resultant output $V_o = V_o e^{j\phi_o}$

$V_o = |H|V_i \quad \Phi_o = \Phi_H + \Phi_i$
The frequency response plots are commonly employed to describe the frequency response of a circuit, since they can provide the effect of a filter on an exciting signal.

This RC filter has the property of "PASSING" signals at low frequency (<<1/RC) and of filtering out signals at high frequency (>>1/RC). Therefore, such a filter is called a LOW-PASS FILTER.
High Pass Filter $1/3$

Consider the circuit:

```
  +--- C ---+  +
     |      |
+--- R ---+  -
  |  |  |  |
V_i  +---  +---  V_o
  |  |  |  |
- -+- -+- -+
```

The frequency response for the high-pass filter:

$$H(j\omega) = \frac{V_o(j\omega)}{V_i(j\omega)} \quad \text{其中，} \quad V_o(j\omega) = V_i(j\omega) \frac{R}{R + \frac{1}{j\omega C}} = V_i(j\omega) \frac{j\omega CR}{1 + j\omega CR}$$
High Pass Filter $^{2/3}$

The frequency response of the RC filter is

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{j\omega CR}{1 + j\omega CR}$$

為了清楚看到上述關係，吾人可將之前所得到的 $H(j\omega)$ 描述成

magnitude and phasor 型態：

$$H(j\omega) = \frac{V_o}{V_i}(j\omega) = \frac{j\omega CR}{1 + j\omega CR} = \frac{\omega CR}{\sqrt{1 + (\omega CR)^2}}e^{j(90 - \arctan(\omega CR))}$$

或 $H(jw) = |H(jw)|e^{j\Phi_H(jw)}$

其中，$$|H(j\omega)| = \frac{\omega CR}{\sqrt{1 + (\omega CR)^2}}$$

$$\Phi_H(j\omega) = 90 - \arctan(\omega CR)$$
High Pass Filter

當 \( \omega \to 0 \), the magnitude of the frequency response will be ZERO！

當 \( \omega \to \infty \), the magnitude of the frequency response will 趨近於 1！

對於 phase shift 而言，當 \( \omega \to 0 \), the phase shift is 90；當 \( \omega \to \infty \), the phase shift tends to ZERO！

其中，\( \omega_c = \frac{1}{RC} \) is called the cutoff frequency of the filter。
**BAND Pass Filter**

- Passing mainly those frequencies within a certain frequency range.

![Diagram of a bandpass filter showing characteristics such as pass band and stop band, with equations and circuit components including resistors and inductors.](image)
A band-pass filter is a device that passes frequencies within a certain range and rejects (attenuates) frequencies outside that range. An example of an analogue electronic band-pass filter is an RLC circuit.

BAND Pass Filter

Frequency Response

Gain = \frac{V_{out}}{V_{in}}

Stop Band

Pass Band

Stop Band

Filter Response

Slope = +20dB/Decade

Slope = -20dB/Decade

-3dB (45°)

http://www.electronics-tutorials.ws/filter/filter_4.html
The related frequency response function for the filter is:

\[ H(j\omega) = \frac{V_o}{V_i} (j\omega) \]

其中，

\[ V_o(j\omega) = V_i(j\omega) \frac{R}{R + \frac{1}{j\omega C} + j\omega L} \]

\[ = V_i(j\omega) \frac{j\omega CR}{1 + j\omega CR + (j\omega)^2 LC} \]

*RLC band-pass filter. The circuit preserves frequencies within a band.*
The frequency response of the RLC filter is:

\[
\frac{V_o(j\omega)}{V_i} = \frac{j\omega CR}{1 + j\omega CR + (j\omega)^2 LC} = \frac{R}{L} \frac{j\omega}{-\omega^2 + R \frac{j\omega}{L} + \frac{1}{LC}}
\]

\[|H(j\omega)| = \frac{R}{L} \frac{\omega}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L} \omega\right)^2}}\]

\[\angle H(j\omega) = \Phi_H = 90^\circ - \arctan \frac{R}{L} \frac{\omega}{\frac{1}{LC} - \omega^2}\]
At resonance frequency, the frequency response function will be real.

\[ j\omega L + \frac{1}{j\omega C} = 0 \Rightarrow \omega_0 = \frac{1}{\sqrt{LC}} \Rightarrow f_0 = \frac{1}{2\pi\sqrt{LC}} \]

\[ |H(j\omega)| \text{ will be Max. at } |H(j\omega_0)| \]

\[ |H(j\omega_0)| = \sqrt{\left(1 - \frac{\omega_0^2}{\omega_0^2} \right)^2 + \left(\frac{R}{L} \omega_0 \right)^2} = 1 \]
Set \( \left| H(j\omega) \right|_{\text{max}} \) to find cutoff frequencies.

\[
\frac{1}{\sqrt{2}} = \sqrt{\left( \frac{1}{LC} - \omega_c^2 \right)^2 + \left( \frac{R}{L} \omega_c \right)^2}
\]

\[
\omega_{c_1} = -\frac{R}{2L} + \sqrt{\left( \frac{R}{2L} \right)^2 + \left( \frac{1}{LC} \right)^2}
\]

\[
\omega_{c_2} = \frac{R}{2L} + \sqrt{\left( \frac{R}{2L} \right)^2 + \left( \frac{1}{LC} \right)^2}
\]

\[
\omega_0 = \sqrt{\omega_{c_1} \cdot \omega_{c_2}} = \frac{1}{\sqrt{LC}}
\]
Bandwidth

\[ BW = \omega_{c2} - \omega_{c1} = \frac{R}{L} \]

The Quality factor \( Q \)

\[ Q = \frac{\omega_0}{BW} = \frac{1}{R} \sqrt{\frac{L}{C}} \]

Where \( \omega_{c1} \) and \( \omega_{c2} \) are the two frequencies that determine the pass-band (band-width) of the filter (即可以通過input signal的頻率範圍)。
BAND Pass Filter $^{9/10}$

Band-pass filter amplitude response

Band-pass filter phase response

Radian frequency (logarithmic scale)
BAND Pass Filter

- 当 $\omega \rightarrow 0$，capacitor 形同 open-circuit，the response of the filter is ZERO！
- 当 $\omega \rightarrow \infty$，inductor 形同 open-circuit，the response of the filter is ZERO！
- 因此，低频部分与高频部分的 input signal 是无法通过 band-pass filter。
Q值越大，the sharpness of the resonance越大，band-pass filter變得更具選擇性，能通過band-pass filter的input signal的頻率將僅侷限於resonant frequency附近。

Bandwidth，或width of the pass-band，一個用來衡量選擇範圍的指標：
$$BW = \frac{\omega_0}{Q} = \frac{R}{L}$$

Q越大，BW越小；Q越小，BW越大。
**Example 5.14**

**EXAMPLE 5.14** Sketch the output frequency characteristic for the network in Fig. 5.69.

![Circuit Diagram]

\[ E = 40 \text{ mV} \]
**Example 5.14**

**Solution:** The resonant frequency is

\[
 f_s = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{(6.28)\sqrt{(1 \times 10^{-3})}(0.4 \times 10^{-6})} \\
 = \frac{1}{6.28\sqrt{4 \times 10^{-10}}} = \frac{1}{6.28(2 \times 10^{-5})} = \frac{10^4}{1.256} \\
= 7.96 \text{ kHz}
\]

\[
 BW = \frac{R}{2\pi L} = \frac{2 \Omega + 20 \Omega}{(6.28)(1 \times 10^{-3})} = \frac{22}{6.28} \times 10^3 \text{ Hz} = 3.5 \text{ kHz}
\]

\[
 f_1 = f_s - \frac{BW}{2} = 7.96 \text{ kHz} - \frac{3.5 \text{ kHz}}{2} = 6.21 \text{ kHz}
\]

\[
 f_2 = f_s + \frac{BW}{2} = 7.96 \text{ kHz} + \frac{3.5 \text{ kHz}}{2} = 9.71 \text{ kHz}
\]

\[
 V_{\text{max}} = \frac{(20 \Omega)(40 \text{ mV})}{20 \Omega + 2 \Omega} = \frac{800 \text{ mV}}{22} = 36.36 \text{ mV}
\]
Example 5.14

The pass-band output appears in Fig. 5.70.

![Diagram](image)

- $V_{C_{\text{max}}} = 36.36 \text{ mV}$
- $BW = 3.5 \text{ kHz}$
- $(\text{Pass band})$
- $f_1 = 6.21 \text{ kHz}$
- $f_2 = 9.71 \text{ kHz}$
- $f_p = 7.96 \text{ kHz}$

**FIG. 5.70**
Bode Plot

- Bode plot is another way to express frequency response. Its horizontal axis is frequency on logarithm scale, and the vertical axis is amplitude of the frequency response in units of decibels (dB).

Low-pass and high-pass filter's dB magnitude plots
What’s Decibel (dB)

- The ratio $\frac{V_{\text{out}}}{V_{\text{in}}}$ is given in units of dB

Where

$$\left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|_{\text{dB}} = 20 \log_{10} \left| \frac{V_{\text{out}}}{V_{\text{in}}} \right|$$
**Bode Plot**

Low frequency part (low-pass filter) 及 high frequency part (high-pass filter) 為一水平線，在該範圍內其 amplitude response 大小為 1。此外，在 cutoff frequency，$w_0$，附近，Response 開始下降或上升的斜率為 ±20dB/decade（-為 low-pass filter，+為 high-pass filter）。
What’s Decade

A decade 是指在 \( f_1 \sim f_2 \) 的頻率範圍內，\( \frac{f_2}{f_1} = 10 \)。

因此，\( \pm 20\text{dB/decade} \) 表示：

\[
\left| H(jw) \right|_{\text{dB}} = -20\text{dB}
\]

\[
-20 = 20 \log_{10} |H(jw)| \quad \Rightarrow \quad |H(jw)| = 0.1
\]

Frequency 每增加 10，Gain 減少 10！
EXAMPLE by MATHLAB

RLC band-pass filter. The circuit preserves frequencies within a band.

\[ R = 143.25 \ \Omega, \quad L = 2.533 \text{ mH}, \quad C = 1 \mu \text{F} \]

Plot \( f = 50 \text{ to } 100 \text{ KHz} \).

- **Magnitude** \( |H(j\omega)| \) vs. \( \omega / \omega_b \)
- **Phase** \( \Phi_H \) vs. \( \omega / \omega_b \)

R=143.25 \ \Omega , L=2.533 \text{ mH}, C=1 \mu \text{ F},
Plot f=50~100 \text{ KHz}. 改變 R ？
R = 143.25 Ω, L = 2.533 mH, C = 1 μF, Q = 0.3513, $f_0 = 1.987 \times 10^4$ Hz
$R = 50 \ \Omega$, $L = 2.533 \ \text{mH}$, $C = 1 \ \mu \text{F}$, $Q = 1.006578$, $f_o = 1.987 \times 10^4 \ \text{Hz}$
$R = 250 \, \Omega, \quad L = 2.533 \, \text{mH}, \quad C = 1 \, \mu\text{F}, \quad Q = 0.201315, \quad f_0 = 1.987 \times 10^4 \, \text{Hz}$
Resonant Circuit
Series RLC Circuit

$I = \frac{V}{Z}$

$V_R = IR$

$V_C = IX_C$

$V_L = IX_L$

$Z = \sqrt{R^2 + (X_L - X_C)^2}$

Phase $\phi = \tan^{-1}\left[\frac{X_L - X_C}{R}\right]$

$X_C = \frac{1}{\omega C}$

$X_L = \omega L$

$Z = R + j(X_L - X_C)$
前面已經講過了嗎？
前面，輸入電壓源的頻率是固定。
現在，輸入電壓源的頻率非固定，導致 total impedance $Z$ 與電流均非固定，且與頻率有關。
現在，諧振電路，就是要找到一個特定的頻率，讓電路進入諧振狀態。在該狀態下， total impedance $Z$ 最小（串聯）或最大（並聯）。
\[ Z = R + j(X_L - X_C) \]
\[ Z = R + j(\omega L - \frac{1}{\omega C}) \]
\[ |Z| = \sqrt{R^2 + (X_L - X_C)^2} \]

某個\( \omega \)，\( Z \)最小。
**Current**

\[
E = V \angle 0^\circ
\]

\[
Z = R + j(\omega L - \frac{1}{\omega C}) = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \angle \phi
\]

\[
I(j\omega) = \frac{E}{Z} \quad i(t) = \sqrt{2}I_m \sin(\omega t - \phi)
\]

\[
I_m = \frac{V}{\sqrt{R^2 + \left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)^2}} \quad \phi = \tan^{-1}\left(\frac{\omega L - \frac{1}{\omega C}}{R}\right)
\]
Current vs. Frequency

- $I_{\text{max}}$
- 0.707$I_{\text{max}}$
- $f_1$ to $f_2$
- BW
The resonance of a series RLC circuit occurs when the inductive and capacitive reactances are equal in magnitude but cancel each other because they are 180 degrees apart in phase.

串聯RLC電路發生共振的條件？電容與電感的reactance相等！
Resonance Series RLC Circuit

\[ I = \frac{V}{Z} \]

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} \]

\[ X_c = \frac{1}{\omega C} \quad X_L = \omega L \]

Phase \[ \phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right) \]

At series resonance:

\[ Z = R \quad \omega_0 = \frac{1}{\sqrt{LC}} \]

\[ X_C = X_L \quad \text{Phase} = \phi = 0 \]
Resonant Frequency

The condition of resonance

\[ X_L = X_C \]

\[ 2\pi f_s L = \frac{1}{2\pi f_s C} \]

\[ f_s^2 = \frac{1}{4\pi^2 LC} \]

\[ f_s = \frac{1}{2\pi \sqrt{LC}} \] (hertz)
At Resonance

The impedance of the network

\[ Z_T = R + j(X_L - X_C) = R + j0 \]

\[ Z_T = R \quad \text{ (at resonance)} \]

The power delivered

\[ P_{\text{max}} = I_{\text{max}}^2 R \]

\[ I_{\text{max}} = \frac{E}{Z_T} = \frac{E}{R} \]

\[ P = (0.707I_{\text{max}})^2 R = 0.5I_{\text{max}}^2 R = 0.5P_{\text{max}} \]
電流降低到0.707 \times peak\ value水準時的頻率f_1與f_2稱為half-power或cutoff或band frequencies。f_1、f_2與共振頻率f_0等距離。兩者的間距稱為BW。電流降低到0.707 \times peak\ value水準時的功率；為共振頻率下的功率的一半。
在 $I_m$ 的峰值兩側各有一個半功率點 $f_1$ 及 $f_2$

$$I_m = \frac{V_m}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fL}\right)^2}}$$

$f_1$ 及 $f_2$ 滿足

$$\frac{V_m}{\sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fL}\right)^2}} = \frac{V_m}{\sqrt{2R}}$$

解出 $f$，可得：

$$f_{1,2} = \frac{\pm R + \sqrt{R^2 + \frac{4L}{C}}}{4\pi L}$$

頻寬定義為

$$BW = f_2 - f_1 = \frac{R}{2\pi L}$$

0.707的位置
The quality factor $Q$ is defined by

$$Q = \frac{\omega_0}{\Delta \omega} \quad \omega_2 - \omega_1 = \frac{R}{L} = \Delta \omega$$

where $\Delta \omega$ is the width of the resonant power curve at half maximum. Since that width turns out to be $\Delta \omega = R/L$, the value of $Q$ can also be expressed as

$$Q = \frac{\omega_0 L}{R}$$

R越大，Q越小
A "quality factor" Q is a measure of that selectivity, and we speak of a circuit having a "high Q" if it is more narrowly selective.

Q值越大，the sharpness of the resonance变得更具選擇性，也就是說能通過band-pass filter的input signal的頻率將僅侷限於resonant frequency附近。
Selectivity vs. R

- The selectivity of a circuit is dependent upon the amount of resistance in the circuit.

\[ R \text{ 越大，} Q \text{ 越小} \]
Example 5.11

Example 5.11 For the resonance curve in Fig. 5.59, determine:

a. The resonant frequency.

b. The bandwidth.

c. The $Q$ of the network.

d. The inductance of the network if the resistance of the network is 4 $\Omega$.

Solution:

a. $f_s = 11.4 \text{ kHz} + (12.6 \text{ kHz} - 11.4 \text{ kHz})/2 = 11.4 \text{ kHz} + 0.6 \text{ kHz} = 12 \text{ kHz}$

b. $BW = 12.6 \text{ kHz} - 11.4 \text{ kHz} = 1.2 \text{ kHz}$

c. $BW = \frac{f_s}{Q_s}$ and $Q_s = \frac{f_s}{BW} = \frac{12,000 \text{ Hz}}{1200 \text{ Hz}} = 10$

d. $Q_s = \frac{X_L}{R}$ and $X_L = QR = (10)(4 \text{ $\Omega$}) = 40 \text{ $\Omega$}$

$X_L = 2\pi f_s L$ and $L = \frac{X_L}{2\pi f_s} = \frac{40 \text{ $\Omega$}}{(6.28)(12 \times 10^3 \text{ Hz})} = \frac{40 \times 10^{-3}}{75.36} \text{ H}$

$= 0.531 \text{ mH}$

FIG. 5.59
Example 5.12

**Example 5.12** For the series RLC circuit in Fig. 5.60, determine:

a. $X_L$ for resonance.

b. The $Q$ of the network at resonance.

c. The bandwidth.

d. The power delivered at resonance.

e. The power delivered at the half-power frequencies (HPFs).

f. The general shape of the resonance curve.

**Solution:**

a. $X_L = X_C = 2 \, \text{k} \Omega$

b. $Q = \frac{X_L}{R} = \frac{2000 \, \Omega}{50 \, \Omega} = 40$

c. $BW = \frac{f_s}{Q} = \frac{30,000 \, \text{Hz}}{40} = 750 \, \text{Hz}$

d. $P_{\text{max}} = I_{\text{max}}^2 R = \left(\frac{E}{R}\right)^2 R = \frac{(60 \, \text{V})^2}{50 \, \Omega} = \frac{3600}{50} \, \text{W} = 72 \, \text{W}$

e. $P_{\text{HPF}} = \frac{P_{\text{max}}}{2} = \frac{72 \, \text{W}}{2} = 36 \, \text{W}$

f. See Fig. 5.61.
Parallel RLC Circuit

\[
Z = \frac{1}{Y} = \frac{1}{G + j(\omega C - \frac{1}{\omega L})} = \frac{1}{G + j(B_C - B_L)}
\]

\[
I = \frac{V}{Z} \quad G = \frac{1}{R}
\]
Focus on the condition

\[ Q_P = \frac{X_L}{R} > 10 \]

\[ Z = \frac{1}{\frac{1}{R + j\omega L} + j\omega C} = \frac{R + j\omega L}{-\omega^2 LC + j\omega RC} \ldots \]
Impedance

\[ Z_{T_{\text{max}}} = Q_p^2 R \]

\[ Q_p = \frac{X_L}{R} \]

\[ Z = \frac{1}{\frac{1}{R + j\omega L} + j\omega C} = \frac{R + j\omega L}{-\omega^2 LC + j\omega RC} \]
Voltage

\[ V_{C_{\text{max}}} = I_s Z_{\text{max}} \]

\[ 0.707 \, V_{C_{\text{max}}} \]

\[ f_1 \quad f_p \quad f_2 \]

\[ \text{BW} \]
The resonance of a parallel RLC circuit occurs when the inductive and capacitive reactances are equal in magnitude but cancel each other because they are 180 degrees apart in phase.
BANDWIDTH

\[ V_C = IZ_T \]

\[ f_p = \frac{1}{2\pi\sqrt{LC}} \quad (Q_p \geq 10) \]

\[ BW = \frac{f_p}{Q_p} \]
EXAMPLE 5.13 For the tank circuit in Fig. 5.65, determine:

a. $X_C$ for resonance.
b. The impedance at resonance.
c. The resonant frequency if $L = 10 \mu H$.
d. The cutoff frequencies.
e. The shape of the resonant curve.
Example 5.13

Solution:

a. \[ Q_p = \frac{X_L}{R} = \frac{40 \ \Omega}{1 \ \Omega} = 40 > 10; \text{ therefore, } X_C = X_L = 40 \ \Omega \]

b. \[ Z_{T_{\text{max}}} = Q_p^2 R = (40)^2 \times 1 \ \Omega = 1600(1 \ \Omega) = 1600 \ \Omega \]

c. \[ X_L = 2\pi f_p L \Rightarrow f_p = \frac{X_L}{2\pi L} = \frac{40 \ \Omega}{(6.28)(10 \times 10^{-6} \ \text{H})} = 636.94 \ \text{kHz} \]

d. \[ \text{BW} = \frac{f_p}{Q_p} = \frac{636,940 \ \text{Hz}}{40} = 15,923.50 \ \text{Hz}, \text{ and } \]

\[ f_1 = f_p - \frac{\text{BW}}{2} = 636,940 \ \text{Hz} - \frac{15,923.50 \ \text{Hz}}{2} = 628,978 \ \text{Hz} \]

\[ f_2 = f_p + \frac{\text{BW}}{2} = 637,000 \ \text{Hz} + \frac{15923.50 \ \text{Hz}}{2} = 644,902 \ \text{Hz} \]
Example 5.13

e. The resonant curve appears in Fig. 5.66.

\[ V_{C_{\text{max}}} = I Z_{T_{\text{max}}} = (20 \text{mA})(1.6 \text{ k}\Omega) = 32 \text{ V} \]

\[ V_C = 0.707 \, V_{C_{\text{max}}} = (0.707)(32 \text{ V}) = 22.26 \text{ V} \]

\[ \text{BW} = 15,923.50 \text{ Hz} \]

\[ f_1 = 628.978 \quad f_2 = 644.902 \]

\[ f_p = 636.94 \]

FIG. 5.66