Chap. 4
Force System Resultants
Chapter Outline

- Moment of a Force – Scalar Formation
- Cross Product
- Moment of Force – Vector Formulation
- Principle of Moments
- Moment of a Force about a Specified Axis
- Moment of a Couple
- Simplification of a Force and Couple System
- Further Simplification of a Force and Couple System
- Reduction of a Simple Distributed Loading
Moment of a Force – Scalar Formation

- Moment of a force about a point or axis – a measure of the tendency of the force to cause a body to rotate about the point or axis
- Torque – tendency of rotation caused by $F_x$ or simple moment $(M_o)_z$
Moment of a Force – Scalar Formation

Magnitude

- For magnitude of $M_O$,
  $$M_O = Fd \ (N\cdot m)$$
  where $d =$ perpendicular distance from $O$ to its line of action of force

Direction

- Direction using “right hand rule”
Moment of a Force – Scalar Formation

Resultant Moment

- Resultant moment, \( M_{Ro} = \) moments of all the forces

\[
M_{Ro} = \sum Fd
\]
Cross Product

- Cross Product (vector product)
  \[ \mathbf{C} = \mathbf{A} \times \mathbf{B} \]

- Magnitude
  \[ \mathbf{C} = AB \sin \theta \quad 0^\circ \leq \theta \leq 180^\circ \]

- Direction

\[ \mathbf{C} = (AB \sin \theta) \mathbf{u}_C \]
\[ \mathbf{C} \perp \mathbf{A} \]
\[ \mathbf{C} \perp \mathbf{B} \]
Cross Product

- **Laws of Operation**

\[ A \times B = -B \times A \]
\[ \alpha (A \times B) = (\alpha A) \times B = A \times (\alpha B) = (A \times B)\alpha \]

\[ A \times (B + D) = (A \times B) + (A \times D) \]
\[ A \times (D + B) = (A \times D) + (A \times B) \]
Cross Product

- Cartesian Vector Formulation

\[ \mathbf{A} \times \mathbf{B} = (A_x, A_y, A_z) \times (B_x, B_y, B_z) \]

\[ = (A_y B_z - A_z B_y)i + (A_z B_x - A_x B_z)j + (A_x B_y - A_y B_x)k \]

or

\[ \mathbf{A} \times \mathbf{B} = \begin{vmatrix} i & j & k \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \]

\[ = A_x \begin{vmatrix} j & k \\ B_y & B_z \end{vmatrix} - A_y \begin{vmatrix} i & k \\ B_x & B_z \end{vmatrix} + A_z \begin{vmatrix} i & j \\ B_x & B_y \end{vmatrix} \]

\[ = A_x (B_y - B_z) - A_y (B_x - B_z) + A_z (B_x - B_y) \]

\[ = (A_y B_z - A_z B_y)i + (A_z B_x - A_x B_z)j + (A_x B_y - A_y B_x)k \]
Moment of a Force-Vector Formulation

$$M_O = r \times F = r_B \times F$$

$$M_O = r F \sin \theta = F (r \sin \theta) = Fd$$
Moment of a Force-Vector Formulation

- Transmissibility of a Force  傳遞性

\[ \mathbf{M}_O = \mathbf{r}_A \times \mathbf{F} = \mathbf{r}_B \times \mathbf{F} = \mathbf{r}_C \times \mathbf{F} \]

\(A, B, C\) in the force \(\mathbf{F}\)’s action line
Moment of a Force-Vector Formulation

- Vector Formulation

\[
\mathbf{M}_O = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} i & j & k \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}
\]
Moment of a Force-Vector Formulation

- Resultant Moment of a System of Forces

\[
\sum \mathbf{r} \times \mathbf{F} = \mathbf{M}_{RO}
\]
Example 4.4  Two forces act on the rod. Determine the resultant moment they create about the flange at O. Express the result as a Cartesian vector.

$\mathbf{F}_1 = \{-60\mathbf{i} + 40\mathbf{j} + 20\mathbf{k}\} \text{ kN}$

$\mathbf{F}_2 = \{80\mathbf{i} + 40\mathbf{j} - 30\mathbf{k}\} \text{ kN}$

$\mathbf{A}(0, 5, 0), \mathbf{B}(4, 5, -2)$
Position vectors are directed from point $O$ to each force as shown.

These vectors are

$$r_A = \{5j\} \text{ m}$$

$$r_B = \{4i + 5j - 2k\} \text{ m}$$

The resultant moment about $O$ is

$$\bar{M}_O = \sum (r \times F) = r_A \times F + r_B \times F$$

$$= \begin{vmatrix} i & j & k \\ 0 & 5 & 0 \\ -60 & 40 & 20 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 4 & 5 & -2 \\ 80 & 40 & -30 \end{vmatrix}$$

$$= \begin{vmatrix} 30i - 40j + 60k \end{vmatrix} \text{ kN} \cdot \text{ m}$$
**Principle of Moments – Varignon’s Theorem**

\[ M_0 = r \times F_1 + r \times F_2 + r \times F_3 + r \times F_4 + \ldots \]

\[ = r \times (F_1 + F_2 + \ldots) \]

\[ = r \times F \]

*合力對固定點的力矩
等於各分力對固定點的力矩之總合*
p. 133, 4–4. Two men exert forces of $F = 400$ N and $P = 250$ N on the ropes. Determine the moment of each force about $A$. Which way will the pole rotate, clockwise or counterclockwise?

\[
(M_A)_C = 400 \left( \frac{4}{5} \right) (3.6) = 1152 \text{ N} \cdot \text{m} \quad \text{Ans}
\]

\[
(M_A)_B = 250 (\cos 45^\circ) (5.4) = 954.6 \text{ N} \cdot \text{m} \quad \text{Ans}
\]

Since $(M_A)_C > (M_A)_B$

Clockwise \hspace{1cm} \text{Ans}
4–24. In order to raise the lamp post from the position shown, force \( F \) is applied to the cable. If \( F = 1000 \) N, determine the moment produced by \( F \) about point \( A \).
Geometry: Applying the law of cosines to Fig. a,

\[ BC^2 = 3^2 + 6^2 - 2(3)(6) \cos 105^\circ \]

\[ BC = 7.370 \text{ m} \]

Then, applying the law of sines,

\[ \frac{\sin \theta}{3} = \frac{\sin 105^\circ}{7.37^\circ} \]

\[ \theta = 23.15^\circ \]

**Moment About Point A:** By resolving force \( F \) into components parallel and perpendicular to the lamp pole, Fig. a, and applying the principle of moments,

\[ (\Sigma M)_A = \Sigma Fd; \quad M_A = 1000 \sin 23.15^\circ (6) + 1000 \cos 23.15^\circ (0) \]

\[ = 2358.8 \text{ N} \cdot \text{m} = 2.36 \text{ kN} \cdot \text{m} \text{ (counterclockwise)} \]

**Ans.**

4-1-18
The wheelbarrow and its contents have a center of mass at \( G \). If \( F = 100 \text{ N} \), and the resultant moment produced by force \( F \) and the weight about the axle at \( A \) is zero, determine the mass of the wheelbarrow and its contents.
Resolving force $\mathbf{F}$ into its horizontal and vertical components, Fig. $a$, and applying the principle of moments,

$$\sum (M_R)_A = \Sigma Fd; \quad 0 = 100 \cos 30^\circ (1.15) + 100 \sin 30^\circ (1.5) - M(9.81)(0.3)$$

$$M = 59.3 \text{ kg}$$
The force \( F = \{6i + 8j + 10k\} \) N creates a moment about point \( O \) of \( M_O = \{-14i + 8j + 2k\} \) N\(\cdot\)m. If the force passes through a point having an \( x \) coordinate of 1 m, determine the \( y \) and \( z \) coordinates of the point. Also, realizing that \( M_O = Fd \), determine the perpendicular distance \( d \) from point \( O \) to the line of action of \( F \).
Chap. 4-2

Force System Resultants
Moment of a Force About a Specified Axis

- $M_y$ for $F$ about a specified axis ($y$-axis) is the moment, and is the projection of $M_o$ on the $y$-axis.

1. Let $O$ be any point on the $y$-axis, first find $M_o = \langle r \times F \rangle$

2. Then use the specific axis unit vector ($\mathbf{j}$ in this example) and the dot product with $M_o$ to find $M_y = M_o \cdot \mathbf{j}$

3. $M_y = [\mathbf{j} \cdot (r \times F)] \mathbf{j}$
**Moment of a Force About a Specified Axis**

Let $a-a'$ be an arbitrary axis.

The moment $M_a$ of a force $F$ about an axis $a-a'$ is given by:

$$M_a = M_O \cos \theta = M_O \cdot \underline{u}_a$$

$$= \underline{u}_a \cdot (\underline{r} \times \underline{F})$$

$$= \begin{vmatrix} u_{ax} & u_{ay} & u_{az} \\ r_x & r_y & r_z \\ F_x & F_y & F_z \end{vmatrix}$$

This is known as the **Triple scalar product**.

In Japanese:

純量三倍積
(平行六面體的體積)

$$M_a = M_a \underline{u}_a = \left[ \underline{u}_a \cdot (\underline{r} \times \underline{F}) \right] \underline{u}_a$$
Determine the moment produced by force $\mathbf{F}$ about segment $AB$ of the pipe assembly. Express the result as a Cartesian vector.
Moment About Line AB: Either position vector \( \mathbf{r}_{AC} \) or \( \mathbf{r}_{BC} \) can be conveniently used to determine the moment of \( \mathbf{F} \) about line \( AB \).

\[
\mathbf{r}_{AC} = (3 - 0)i + (4 - 0)j + (4 - 0)k = [3i + 4j + 4k] \text{ m}
\]

\[
\mathbf{r}_{BC} = (3 - 3)i + (4 - 4)j + (4 - 0)k = [4k] \text{ m}
\]

The unit vector \( \mathbf{u}_{AB} \), Fig. 1, that specifies the direction of line \( AB \) is given by

\[
\mathbf{u}_{AB} = \frac{(3 - 0)i + (4 - 0)j + (0 - 0)k}{\sqrt{(3 - 0)^2 + (4 - 0)^2 + (0 - 0)^2}} = \frac{3}{5}i + \frac{4}{5}j
\]

Thus, the magnitude of the moment of \( \mathbf{F} \) about line \( AB \) is given by

\[
M_{AB} = \mathbf{u}_{AB} \cdot \mathbf{r}_{AC} \times \mathbf{F} = \begin{vmatrix}
3 & 4 & 0 \\
\frac{4}{5} & 5 & 0 \\
3 & 4 & 4 \\
-20 & 10 & 15
\end{vmatrix}
\]

\[
= \frac{3}{5}[4(15) - 10(4)] - \frac{4}{5}[3(15) - (-20)(4)] + 0
\]

\[=-88 \text{ N} \cdot \text{m}\]
or

\[ M_{AB} = u_{AB} \cdot \mathbf{r}_{BC} \times \mathbf{F} = \begin{vmatrix} 3 & 4 & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 4 \\ -20 & 10 & 15 \end{vmatrix} \]

\[ = \frac{3}{5}[0(15) - 10(4)] - \frac{4}{5}[0(15) - (-20)(4)] + 0 \]

\[ = -88 \text{ N} \cdot \text{m} \]

Thus, \( M_{AB} \) can be expressed in Cartesian vector form as

\[ M_{AB} = M_{AB} \mathbf{u}_{AB} = -88 \left( \frac{3}{5} \mathbf{i} + \frac{4}{5} \mathbf{j} \right) = [-52.8 \mathbf{i} - 70.4 \mathbf{j}] \text{ N} \cdot \text{m} \quad \text{Ans.} \]
The A-frame is being hoisted into an upright position by the vertical force of $F = 400 \text{ N}$. Determine the moment of this force about the $y$ axis when the frame is in the position shown.
Using $x', y', z$:

$u_y = -\sin 30^\circ i' + \cos 30^\circ j'$

$r_{Ac} = -2 \cos 15^\circ i' + 1j' + 2 \sin 15^\circ k$

$F = 400k$

$$M_y = \begin{vmatrix}
-\sin 30^\circ & \cos 30^\circ & 0 \\
-2 \cos 15^\circ & 1 & 2 \sin 15^\circ \\
0 & 0 & 400
\end{vmatrix} = -200 + 669.2 + 0$$

$M_y = 469.2 \text{ N}\cdot\text{m} \quad \text{Ans}$
Also, using \( x, y, z \).

Coordinates of point \( C \):

\[
x = 1 \sin 30^\circ - 2 \cos 15^\circ \cos 30^\circ = -1.173 \text{ m}
\]

\[
y = 1 \cos 30^\circ + 2 \cos 15^\circ \sin 30^\circ = 1.832 \text{ m}
\]

\[
z = 2 \sin 15^\circ = 0.518 \text{ m}
\]

\[\mathbf{r}_{AC} = -1.173\mathbf{i} + 1.832\mathbf{j} + 0.518\mathbf{k}\]

\[\mathbf{F} = 400\mathbf{k}\]

\[
M_y = \begin{vmatrix} 0 & 1 & 0 \\ -1.173 & 1.832 & 0.518 \\ 0 & 0 & 400 \end{vmatrix} = 469.2 \text{ N} \cdot \text{m} \quad \text{Ans}
\]
Moment of a Couple 力偶

\[ \mathbf{M}_O = \mathbf{r}_A \times (-\mathbf{F}) + \mathbf{r}_B \times (\mathbf{F}) \]
\[ = (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F} \]
\[ = \mathbf{r}_{AB} \times \mathbf{F} \]
\[ = \mathbf{r} \times \mathbf{F} \]
Moment of a Couple

- **Scalar Formulation**
  \[ M = Fd \]

- **Vector Formulation**
  \[ \mathbf{M} = \mathbf{r} \times \mathbf{F} \]
Moment of a Couple

- Equivalent Couples
  等效

- Resultant Couple Moment
  合力偶  力偶合

\[ M = |50| N \cdot m \]

\[ 200 \text{ N} \quad 0.25 \text{ m} \]

\[ 100 \text{ N} \quad 0.5 \text{ m} \]

\[ M = |50| N \cdot m \]

\[ M_1 \]

\[ M_2 \]

\[ M_3 \]

\[ M_4 \]
Example 4.12

Determine the couple moment acting on the pipe. Segment $AB$ is directed $30^\circ$ below the $x$–$y$ plane.
SOLUTION I (VECTOR ANALYSIS)

Take moment about point O,
\[ \mathbf{M} = \mathbf{r}_A \times (-250\mathbf{k}) + \mathbf{r}_B \times (250\mathbf{k}) \]
\[ = (0.8\mathbf{j}) \times (-250\mathbf{k}) + (0.6\cos30^\circ\mathbf{i} + 0.8\mathbf{j} - 0.6\sin30^\circ\mathbf{k}) \times (250\mathbf{k}) \]
\[ = \{-130\mathbf{j}\}\text{N.cm} \]

Take moment about point A
\[ \mathbf{M} = \mathbf{r}_{AB} \times (250\mathbf{k}) \]
\[ = (0.6\cos30^\circ\mathbf{i} - 0.6\sin30^\circ\mathbf{k}) \times (250\mathbf{k}) \]
\[ = \{-130\mathbf{j}\}\text{N.cm} \]
Take moment about point A or B, 
\[ M = Fd = 250N(0.5196m) \]
\[ = 129.9N.cm \]
Apply right hand rule, \( M \) acts in the \(-\mathbf{j}\) direction \( \mathbf{M} = \{-130\mathbf{j}\}N.cm \)
p.155  Problem 76
Determine the required magnitude of force $\mathbf{F}$ if the resultant couple moment on the frame is 200 N·M, clockwise.
\( (M_c)_1 = F \left( \frac{4}{5} \right) (0.2) + F \left( \frac{3}{5} \right) (0.2) = 0.28F \)

\( (M_c)_2 = -1500 \cos 30^\circ \ (0.4) - 1500 \sin 30^\circ \ (0.4) = -819.62 \text{ N} \cdot \text{m} = 819.62 \text{ N} \cdot \text{m} \)

The resultant couple moment acting on the beam is required to be 200 N·m, clockwise. Thus,

\( (M_c)_R = (M_c)_1 + (M_c)_2 \)

\(-200 = 0.28F - 819.62 \)

\( F = 2213 \text{ N} \) \hspace{1cm} \text{Ans.} \)
Determine the required magnitude of couple moments so that the resultant couple moment is $\mathbf{MR} = \{-300\mathbf{i} + 450\mathbf{j} - 600\mathbf{k}\}$ N\cdot m.
\[ M_1 = M_1 j \]
\[ M_2 = M_2 (-\cos 30^\circ i - \sin 30^\circ k) = -0.8660 M_2 i - 0.5 M_2 k \]
\[ M_3 = -M_3 k \]

The resultant couple moment is given by

\[ (M_c)_R = \Sigma M; \]
\[ (M_c)_R = M_1 + M_2 + M_3 \]
\[ (-300i + 450j - 600k) = M_1 j + (-0.8660 M_2 i - 0.5 M_2 k) + (-M_3 k) \]
\[ -300i + 450j - 600k = -0.8660 M_2 i + M_1 j - (0.5 M_2 + M_3) k \]

Equating the i, j, and k components yields

\[ -300 = -0.8660 M_2 \quad M_2 = 346.41 \text{ N} \cdot \text{m} = 346 \text{ N} \cdot \text{m} \quad \text{Ans.} \]

\[ M_1 = 450 \text{ N} \cdot \text{m} \quad \text{Ans.} \]

\[ 600 = -0.5(346.41) + M_3 \quad M_3 = 427 \text{ N} \cdot \text{m} \quad \text{Ans.} \]

4-2-19
If $F_1 = 500 \text{ N}$ and $F_2 = 1000 \text{ N}$, determine the magnitude and coordinate direction angles of the resultant couple moment.
\[ \mathbf{r}_1 = [-0.6 \mathbf{k}] \text{ m} \quad \mathbf{r}_2 = [0.6 \mathbf{k}] \text{ m} \quad \mathbf{r}_3 = [0.6 \mathbf{k}] \text{ m} \]

The force vectors \( \mathbf{F}_1, \mathbf{F}_2, \) and \( \mathbf{F}_3 \) are given by

\[ \mathbf{F}_1 = [500 \mathbf{j}] \text{ N} \quad \mathbf{F}_2 = [1000 \mathbf{i}] \text{ N} \]

\[ \mathbf{F}_3 = \mathbf{F}_3 \mathbf{u} = 1250 \left[ \frac{(0 - 0.9) \mathbf{i} + (1.2 - 0) \mathbf{j} + (0.6 - 0.6) \mathbf{k}}{\sqrt{(0 - 0.9)^2 + (1.2 - 0)^2 + (0.6 - 0.6)^2}} \right] = [-750 \mathbf{i} + 1000 \mathbf{j}] \text{ N} \]

Thus,

\[ \mathbf{M}_1 = \mathbf{r}_1 \times \mathbf{F}_1 = (-0.6 \mathbf{k}) \times (500 \mathbf{j}) = [300 \mathbf{i}] \text{ N} \cdot \text{m} \]

\[ \mathbf{M}_2 = \mathbf{r}_2 \times \mathbf{F}_2 = (0.6 \mathbf{k}) \times (1000 \mathbf{i}) = [600 \mathbf{j}] \text{ N} \cdot \text{m} \]

\[ \mathbf{M}_3 = \mathbf{r}_3 \times \mathbf{F}_3 = (0.6 \mathbf{k}) \times (-750 \mathbf{i} + 1000 \mathbf{j}) = [-600 \mathbf{i} - 450 \mathbf{j}] \text{ N} \cdot \text{m} \]

**Resultant Moment:** The resultant couple moment is given by

\[ (\mathbf{M}_C)_R = \Sigma \mathbf{M}_C; \quad (\mathbf{M}_C)_R = \mathbf{M}_1 + \mathbf{M}_2 + \mathbf{M}_3 = (300 \mathbf{i}) + (600 \mathbf{j}) + (-600 \mathbf{i} - 450 \mathbf{j}) = [-300 \mathbf{i} + 150 \mathbf{j}] \text{ N} \cdot \text{m} \]
The magnitude of the couple moment is

\[(M_C)_R = \sqrt{[(M_C)_R]^2_x + [(M_C)_R]^2_y + [(M_C)_R]^2_z} \]
\[= \sqrt{(-300)^2 + (150)^2 + (0)^2} \]
\[= 335.41 \text{ N} \cdot \text{m} = 335 \text{ N} \cdot \text{m} \]

The coordinate angles of \((M_C)_R\) are

\[\alpha = \cos^{-1}\left(\frac{[(M_C)_R]^2_x}{(M_C)_R}\right) = \cos^{-1}\left(\frac{-300}{335.41}\right) = 153.4^\circ \]

\[\beta = \cos^{-1}\left(\frac{[(M_C)_R]^2_y}{(M_C)_R}\right) = \cos^{-1}\left(\frac{150}{335.41}\right) = 63.4^\circ \]

\[\gamma = \cos^{-1}\left(\frac{[(M_C)_R]^2_z}{(M_C)_R}\right) = \cos^{-1}\left(\frac{0}{335.41}\right) = 90^\circ \]
Chap. 4-3
Force System Resultants
Simplification of a Force and Couple System

- An equivalent system is when the *external effects* are the same as those caused by the original force and couple moment system.
- External effects of a system is the *translating and rotating motion* of the body.
- Or refers to the *reactive forces* at the supports if the body is held fixed.

![Diagram](image-url)
Equivalent System

- Point O is On the Line of Action
  - [Images showing the equivalence of forces when O is on the line of action]

- Point O Is Not On the Line of Action
  - [Images showing the moment arm and the resultant force when O is not on the line of action]
Equivalent System

- Resultant of a Force and Couple System

Force Summation: \( F_R = \sum F \)

Moment Summation: \( M_{RO} = \sum M_C + \sum M_o \)
Ex. 4-15, replace the system by an equivalent resultant force and couple moment acting on point O.

\[
\begin{align*}
\mathbf{F}_1 &= -800 \mathbf{kN} \\
\mathbf{u}_{CB} &= \frac{(-0.15, 0.1, 0)}{\sqrt{(-0.15)^2 + 0.1^2}} \\
\mathbf{F}_2 &= 300 \cdot \mathbf{u}_{CB} = (-249.6, 166.4, 0) \mathbf{N} \\
\mathbf{M} &= (0, -400, 300) \mathbf{N} \cdot \mathbf{m} \\
\sum \mathbf{F} &= (-250, 166.4, -800) \mathbf{N} \\
\mathbf{M}_{RO} &= \mathbf{M} + \mathbf{r}_C \times \mathbf{F}_1 + \mathbf{r}_B \times \mathbf{F}_2 \\
&= (0, -400, 300) + \begin{vmatrix}
i & j & k \\
-0.15 & 0.1 & 1 \\
-249.6 & 166.4 & 0 \\
\end{vmatrix} \\
&= (-166.4, -650, 300) \mathbf{N} \cdot \mathbf{m}
\end{align*}
\]
Replace the force system acting on the pipe assembly by a resultant force and couple moment at point $O$. Express the results in Cartesian vector form.

\[ F_1 = \{-20i - 10j + 25k\} \text{ N} \]
\[ F_2 = \{-10i + 25j + 20k\} \text{ N} \]
Equivalent Resultant Force: The resultant force $F_R$ can be determined from

$$F_R = \sum F; \quad F_R = F_1 + F_2$$

$$= (-20i - 10j + 25k) + (-10i + 25j + 20k)$$

$$= [-30i + 15j + 45k] \text{ N} \quad \text{Ans.}$$

Equivalent Resultant Couple Moment: The position vectors $r_{OA}$ and $r_{OB}$, are

$$r_{OA} = (0.15 - 0)i + (0.2 - 0)j + (0 - 0)k = [0.15i + 0.2j] \text{ m}$$

$$r_{OB} = (0.15 - 0)i + (0.4 - 0)j + (0.2 - 0)k = [0.15i + 0.4j + 0.2k] \text{ m}$$

Thus, the resultant couple moment about point $O$ is

$$M_w = \sum F_O; \quad (M_R)_O = r_{OA} \times F_1 + r_{OB} \times F_2$$

$$= \begin{vmatrix} i & j & k \\ 0.15 & 0.4 & 0.2 \\ -10 & 25 & 20 \end{vmatrix} + \begin{vmatrix} i & j & k \\ 0.15 & 0.4 & 0.2 \\ -10 & 25 & 20 \end{vmatrix}$$

$$= [8i - 8.75j + 10.25k] \text{ N} \cdot \text{m} \quad \text{Ans.}$$
Further Reduction of a Force and Couple System

\[ M_{RO} = F_R d \]
Further Simplification of a Force and Couple System

Concurrent Force System

- A concurrent force system is where lines of action of all the forces intersect at a common point $O$

\[
F_R = \sum F
\]
Further Simplification of a Force and Couple System

Coplanar Force System

- Lines of action of all the forces lie in the same plane
- Resultant force of this system also lies in this plane
Further Simplification of a Force and Couple System

Parallel Force System

- Consists of forces that are all parallel to the $z$ axis
- Resultant force at point $O$ must also be
Further Simplification of a Force and Couple System

Reduction to a Wrench (扳手)

\[ \overrightarrow{M}_R = \overrightarrow{M}_\parallel + \overrightarrow{M}_\perp \]

\[ d = \frac{M_\perp}{F_R} \]
p. 180, 4-124. Replace the force and couple moment system acting on the overhang beam by a resultant force, and specify its location along $AB$ measured from point $A$. 
\( (F_r)_x = 5\, \text{KN} \)

\( (F_r)_y = 49.98\, \text{KN} \)
**Equivalent Resultant Force:** Forces $\mathbf{F}_1$ and $\mathbf{F}_2$ are resolved into their $x$ and $y$ components, Fig. a. Summing these force components algebraically along the $x$ and $y$ axes,

$$
\sum (F_R)_x = \Sigma F_x; \quad (F_R)_x = 26 \left( \frac{5}{13} \right) - 30 \sin 30^\circ = -5 \text{ kN} = 5 \text{ kN} \quad \leftarrow \\
+ \uparrow (F_R)_y = \Sigma F_y; \quad (F_R)_y = -26 \left( \frac{12}{13} \right) - 30 \cos 30^\circ = -49.98 \text{ kN} = 49.98 \text{ kN} \quad \downarrow 
$$

The magnitude of the resultant force $\mathbf{F}_R$ is given by

$$
F_R = \sqrt{(F_R)_x^2 + (F_R)_y^2} = \sqrt{5^2 + 49.98^2} = 50.23 \text{ kN} = 50.2 \text{ kN} \quad \text{Ans.}
$$

The angle $\theta$ of $\mathbf{F}_R$ is

$$
\theta = \tan^{-1} \left[ \frac{(F_R)_y}{(F_R)_x} \right] = \tan^{-1} \left[ \frac{49.98}{5} \right] = 84.29^\circ = 84.3^\circ \quad \text{Ans.}
$$

**Location of Resultant Force:** Applying the principle of moments, Figs. a and b, and summing the moments of the force components algebraically about point A,

$$
\sum (M_R)_A = \Sigma M_A; -49.98(d) = 30 \sin 30^\circ(0.3) - 30 \cos 30^\circ(2) - 26 \left( \frac{5}{13} \right)(0.3) - 26 \left( \frac{12}{13} \right)(6) - 45
$$

$$
d = 4.79 \text{ m} \quad \text{Ans.}
$$
p.182, 4-140. Replace the three forces acting on the plate by a wrench. Specify the magnitude of the force and couple moment for the wrench and the point $P(y, z)$ where its line of action intersects the plate.
Resultant Force Vector:

\[ \mathbf{F}_R = \{ -40i - 60j - 80k \} \text{ kN} \]

\[ F_R = \sqrt{(-40)^2 + (-60)^2 + (-80)^2} = 107.70 \text{ kN} = 108 \text{ kN} \quad \text{Ans} \]

\[ \mathbf{u}_{F_R} = \frac{-40i - 60j - 80k}{107.70} \]
\[ = -0.3714i - 0.5571j - 0.7428k \]

Resultant Moment: The line of action of \( \mathbf{M}_R \) of the wrench is parallel to the line of action of \( \mathbf{F}_R \). Assume that both \( \mathbf{M}_R \) and \( \mathbf{F}_R \) have the same sense. Therefore, \( \mathbf{u}_{M_R} = -0.3714i - 0.5571j - 0.7428k \).

\[ (M_R)_x = \Sigma M_x; \quad -0.3714M_R = 60(3 - z) + 80y \quad [1] \]
\[ (M_R)_y = \Sigma M_y; \quad -0.5571M_R = 40z \quad [2] \]
\[ (M_R)_z = \Sigma M_z; \quad -0.7428M_R = 40(3 - y) \quad [3] \]

Solving Eqs. [1], [2], and [3] yields:

\[ M_R = -156.0 \text{ N\cdot m} \quad z = 2.172 \text{ m} \quad y = 0.103 \text{ m} \quad \text{Ans} \]

The negative sign indicates that the line of action for \( \mathbf{M}_R \) is directed in the opposite sense to that of \( \mathbf{F}_R \).
Reduction of a Simple Distributed Loading
Reduction of a Simple Distributed Loading

- **Magnitude**
  \[ F_R = \int_0^L w(x)dx = \int_A dA = A \]

- **Location**
  \[ \sum M_O = M_{RO} : \]
  \[ \int xw(x)dx = \bar{x}F_R \]
  \[ \Rightarrow \bar{x} = \frac{\int_A xdA}{\int_A dA} = \frac{\int x dA}{A} \]
p.189, 4-144. Replace the distributed loading by an equivalent resultant force and specify its location, measured from point A.
\[ + \downarrow F_R = \Sigma F; \quad F_R = 1600 + 900 + 600 = 3100 \text{ N} \]

\[ F_R = 3.10 \text{ kN} \downarrow \quad \text{Ans} \]

\[ + \uparrow M_{PA} = \Sigma M_A; \quad x(3100) = 1600(1) + 900(3) + 600(3.5) \]

\[ x = 2.06 \text{ m} \quad \text{Ans} \]
Wind has blown sand over a platform such that the intensity of the load can be approximated by the function \( w = (0.5x^3) \text{ N/m} \). Simplify this distributed loading to an equivalent resultant force and specify the magnitude and location of the force, measured from A.

\[
\begin{align*}
\text{dA} &= w \, dx \\
F_R &= \int dA = \int_0^{10} \frac{1}{2} x^3 \, dx \\
&= \left[ \frac{1}{8} x^4 \right]_0^{10} \\
&= \frac{1250}{8} \text{N} \\
F_R &= 1.25 \text{kN} \quad \text{Ans}
\end{align*}
\]

\[
\begin{align*}
\int x \, dA &= \int_0^{10} \frac{1}{2} x^4 \, dx \\
&= \left[ \frac{1}{10} x^5 \right]_0^{10} \\
&= \frac{10000}{10} \text{N} \cdot \text{m} \\
x &= \frac{10000}{1250} = 8.00 \text{ m} \quad \text{Ans}
\end{align*}
\]
p.192, 4-160. The distributed load acts on the beam as shown. Determine the magnitude of the equivalent resultant force and specify its location, measured from point A.
\[ F_R = \int w(x) \, dx = \int_0^{10} \left( -\frac{2}{15} x^2 + \frac{17}{15} x + 4 \right) \, dx = 52.22 = 52.2 \, \text{kN} \quad \text{Ans} \]

\[ \bar{x} = \frac{\int x \, w(x) \, dx}{\int w(x) \, dx} = \frac{\int_0^{10} x \left( -\frac{2}{15} x^2 + \frac{17}{15} x + 4 \right) \, dx}{52.22} = \frac{244.44}{52.22} \quad \text{Ans} \]

\[ \bar{x} = 4.68 \, \text{m} \quad \text{Ans} \]