# Chapter 6 Inner Product Spaces

Many geometric notations such as angle, length, and perpendicularity in $R^2$ and $R^3$ may be extended to more general real and complex vector spaces. All of theses ideas are related to the concept of inner product.

線性空間是幾何空間的抽象化，惟幾何空間中具有實質意義的角度、長度與正交等性質，在線性空間中並沒有討論。為了能在線性空間中討論這些問題，需要引入「內積（inner product）」概念。意即，透過 Inner product 將 $R^2$ 與 $R^3$ 內與幾何記號有關的角度、長度與正交等延伸到更廣義的向量空間。

## 6-1 INNER PRODUCTS AND NORMS

### DEFINTION Inner product（內積）

Let V be a vector space over F. An inner product on V is a function that assigns, to every ordered pair of vectors x and y in V, a scalar in F, denoted $\langle x, y\rangle$, such that for all x, y, and z in V and all c in F, the following hold:

在 F 上的向量空間 V 中，其內積為一對應特定常數的函數，註記為$\langle x, y\rangle$，其中，x 與 y 為 V 內一對向量。對 V 內所有向量 x、y、z 與 F 內的純量 c 而言，內積函數必須滿足下列四項公理：

(a) $\langle x+z, y\rangle = \langle x, y\rangle + \langle z, y\rangle$ (additive axiom)

(b) $\langle cx, y\rangle = c\langle x, y\rangle$ (homogeneity axiom)

(c) $\overline{\langle x, y\rangle} = \langle y, x\rangle$ (symmetry axiom)

(d) $\langle x, x\rangle > 0$ if $x \neq 0$ (positive definite axiom)

其中，(c) 在 $F = R$ 時，$\langle x, y\rangle = \langle y, x\rangle$。

若 $a_1, a_2, \ldots, a_n \in F$ 且 $y, v_1, v_2, \ldots, v_n \in V$，則

$$\left\langle \sum_{i=1}^{n} a_i v_i, y \right\rangle = \sum_{i=1}^{n} a_i \langle v_i, y\rangle$$ 是顯而易見的。

## EXAMPLE 1

For $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in $F^n$, define

x 與 y 為 n-tuples，其內積定義為

$< x, y >= \sum_{i=1}^{n} a_i \overline{b_i}$ （Standard inner product）

符合（a）(b)（c）(d) 四項公理？ YES！

是否符合（a）？

令 $z = (c_1, c_2, \ldots, c_n)$，則

$$< x+z, y >= \sum_{i=1}^{n} (a_i + c_i)\overline{b_i} = \sum_{i=1}^{n} a_i \overline{b_i} + \sum_{i=1}^{n} c_i \overline{b_i} =< x, y > + < z, y >$$

Thus, for $x = (1 + i, 4)$ and $y = (2 - 3i, 4 + 5i)$ in $C^2$,

$<x, y> = (1 + i)(2 + 3i) + 4(4 - 5i) = 15 - 15i$.

## EXAMPLE 2

If $<x, y>$ is any inner product on a vector space V and $r > 0$, we may define another inner product by the rule $<x, y>' = r<x, y>$. If $r \leqq 0$, the (d) would not hold.

若 inner product 是以另一種方式定義，就不見得能符合四項公理。

## EXAMPLE 3

Let $V = C([0, 1])$, the vector space of real-valued continuous functions on [0, 1]. For $f, g \in V$, define

V 是[0, 1]上實值連續函數的向量空間，對 V 內任意函數 f 與 g，其內積定義為

$<f, g> = \int_0^1 f(t)g(t)dt$.

符合（a）(b)（c）(d) 四項公理？ YES！

## DEFINITION Conjugate transpose or adjoint

Let $A \in M_{m\times n}(F)$. We define the conjugate transpose or adjoint of A to be the $n\times m$ matrix $A^*$ such that $(A^*)_{ij} = \overline{A_{ji}}$ for all i, j.

定義矩陣的共軛轉置或伴隨矩陣。

## EXAMPLE 4

$$A=\begin{bmatrix} i & 1+2i \\ 2 & 3+4i \end{bmatrix}$$

則 $A^*=\begin{bmatrix} -i & 2 \\ 1-2i & 3-4i \end{bmatrix}$。

若 x 與 y 被視為行向量，則 $\langle x, y\rangle = y^*x$。

## EXAMPLE 5

Let $V = M_{n\times n}(F)$, and define $\langle A, B\rangle = tr(B^*A)$ for $A, B \in V$.

A 與 B 為 V 任何矩陣，內積定義為 $\langle A, B\rangle = tr(B^*A)$

其中，The trace of a matrix A is defined by $tr(A) = \sum_{i=1}^{n} A_{ii}$。

符合（a）（b）（c）（d）四項公理？ YES！

此 inner product definition 稱為「**Frobenius inner product**」。

## DEFINITION Inner product space

A vector space V over F endowed with a specific inner product is called an **INNER PRODUCT SPACE.**

佈於 F 的向量空間 V 賦予特定內積者，稱為內積空間。

If F = C, we call V a COMPLEX INNER PRODUCT SPACE.

若 F = C 者，該內積空間稱為複內積空間。

If F = R, we call V a REAL INNER PRODUCT SPACE.

若 F = R 者，該內積空間稱為實內積空間。

If V has an inner product $\langle x, y\rangle$ and W is a subspace of V, then W is also an inner product space when the same function $\langle x, y\rangle$ is restricted to the vectors $x, y \in W$.

若 V 擁有一內積函數 $\langle x, y\rangle$ 且 W 為 V 的子空間，則當相同的內積函數 $\langle x, y\rangle$ 被限制是來自 W 時，W 亦為一內積空間。

## SPACE H of continuous complex-valued functions

Space H of continuous complex-valued functions defined on the interval $[0, 2\pi]$ with the inner product

H 為一定義在區間[ 0, 2π]，類似 C([0, 1])的連續複值函數空間，其內積定義為

$$< f, g > = \frac{1}{2\pi}\int_0^{2\pi} f(t)\overline{g(t)}dt$$

## Theorem 6.1

Let V be an inner product space. Then for x, y, z ∈ V and c ∈ F, the following statements are true.

V 為一內積空間，對 V 內的 x、y、z 及 F 內的 c 而言，下列敘述為真：

(a) $<x, y + z> = <x, y> + <x, z>$

(b) $<x, cy> = \bar{c}<x, y>$

(c) $<x, 0> = <0, x> = 0$.

(d) $<x, x> = 0$ if and only if $x = 0$.

(e) If $<x, y> = <x, z>$ for all $x \in V$, then $y = z$.

## DEFINITION Norm（範數） or Length

Let V be an inner product space. For x ∈ V, we define the norm or length of x by $\|x\| = \sqrt{< x, x >}$ .

## EXAMPLE 6 歐式長度定義

Let $V = F^n$. If $x = (a_1, a_2,..., a_n)$ , then

x 為 n-tuples，其長度可定義為：

$$\|x\| = \|(a_1, a_2, ..., a_n)\| = \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2}$$ is the **Euclidean** definition of length.

## Theorem 6.2

Let V be an inner product space over F. Then for all x, y ∈ V and c ∈ F, the following statements are true.

V 為一佈於 F 的內積空間，對 V 內所有 x、y 及 F 內的 c 而言，下列敘述為真：

下列敘述為真：

(a) $\|cx\| = |c| \cdot \|x\|$

(b) $\|x\| = 0$ if and only if x = 0. In any case, $\|x\| \geq 0$.

(c) Cauchy-Schwarz inequality $|<x,y>| \leq \|x\| \cdot \|y\|$.

(d) Triangle inequality $|x+y| \leq \|x\| + \|y\|$ .

## EXAMPLE 7

For $F^n$, we may apply (c) and (d) of Theorem 6.2 to the standard inner product to obtain the following well-known inequalities:

對 $F^n$ 內的 n-tuple，將定理 6.2 的（c）與（d）應用到 standard inner product 可獲得下列知名的不等式：

$$\left|\sum_{i=1}^{n} a_i \overline{b_i}\right| \leq \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2} \left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}$$

$$\left[\sum_{i=1}^{n} |a_i + b_i|^2\right]^{1/2} \leq \left[\sum_{i=1}^{n} |a_i|^2\right]^{1/2} + \left[\sum_{i=1}^{n} |b_i|^2\right]^{1/2}$$

其中，第一個結果，可從 $<x,y> = \|x\| \|y\| \cos\theta$ 來對照。

For $x = (a_1, a_2, \ldots, a_n)$ and $y = (b_1, b_2, \ldots, b_n)$ in $F^n$, define Standard inner product

$$<x,y> = \sum_{i=1}^{n} a_i \overline{b_i}$$

## DEFINITION Orthogonal、Orthonormal

Let V be an inner product space.

Vectors x and y in V are **orthogonal** if <x, y> = 0.

兩向量正交。

A subset of V is **orthogonal** if any two distinct vectors in S are orthogonal.

子集合正交。

A vector x in V is a **unit vector** if $\|x\| = 1$..

單位向量。

A subset S of V is **orthonormal** if S is **orthogonal** and consists entirely of **unit vectors**.

子集合為 orthonormal（單範正交），表示其為正交子集合且由單位向量所組成。

## DEFINITION Normalizaing

The process of multiplying a nonzero vector by the reciprocal of its length is called **normalizing**.

$x \to (1/\|x\|)x$ .

正規化，是一種過程，將任一非零向量變成一單位向量。

### EXAMPLE 8 Normalizing process

In $F^3$, {(1, 1, 0)}, (1, -1, 1), (-1, 1, 2)} is an orthogonal set of nonzero vectors, but it is not orthonormal; however, if we normalize the vectors in the set, we obtain the orthonormal set

$$\left\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\right\}.$$

獲得一 Orthonormal set。

### EXAMPLE 9 Orthonormal subset

Recall the inner product space H.

內積空間 H 的 orthonormal subset S，

定義 $S = \{ f_n$: n is an integer$\}$；其中，$f_n(t) = e^{int}$。

For any integer n, let $f_n(t) = e^{int}$, where $0 \leqq t \leqq 2\pi$.

$S = \{ f_n$: n is an integer$\}$ is a subset and orthonormal subset of H.

$$< f_m, f_n > = \frac{1}{2\pi}\int_0^{2\pi} e^{imt}\overline{e^{int}}dt = \dots = 0 \quad \text{and}$$

$$< f_n, f_n > = \frac{1}{2\pi}\int_0^{2\pi} e^{int}\overline{e^{int}}dt = \dots = 1$$

In other words, $<f_m, f_n> = \delta_{mn}$.

## 6-2 THE GRAM-SCHMIDT ORTHOGONALIZATION PROCESS AND ORTHOGONAL COMPLEMENT

The role of standard ordered bases of $C^n$ and $R^n$ ?

前面章節已知 $C^n$ 與 $R^n$ 上標準有序基底的特殊性質：

▶Basis vectors form an orthonormal set.

▶The building blocks of vector spaces.

▶The building blocks of inner product spaces.

## DEFINITION Orthonormal basis

Let V be an inner product space. A subset of V is an orthonormal basis for V if it is an **ordered basis** that is **orthonormal**.

單範正交基底？為一單範正交的一組有序基底。

## EXAMPLE 1

The standard ordered basis for $F^n$ is an **orthonormal basis** for $F^n$.

$F^n$ 的標準有序基底是 $F^n$ 的 **orthonormal basis**。

## EXAMPLE 2

The set $\left\{ \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right), \left(\frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}}\right) \right\}$ is an **orthonormal basis** for $R^2$.

為一單範正交基底？ YES！

## Theorem 6.3

Let V be an inner product space and $S = \{v_1, v_2, \ldots, v_k\}$ be an orthogonal subset of V consisting of nonzero vectors. If $y \in \text{span}(S)$, then

$$y = \sum_{i=1}^{k} \frac{\langle y, v_i \rangle}{\|v_i\|^2} v_i$$

$S = \{v_1, v_2, \ldots, v_k\}$是 V 的 orthogonal subset，y 為 S 生成集的元素。

既然 $y \in \text{span}(S)$，則 y 可寫成 S 的 elements 的 linear combination $y = \sum_{i=1}^{k} a_i v_i$。

其中，$a_i = \frac{\langle y, v_i \rangle}{\|v_i\|^2}$ 即為 Linear combination 的係數。

## Corollary 1 to Theorem 6.3

If, in addition to the hypothses of Theorem 6.3, S is orthonormal and $y \in \text{span}(S)$,, then

在 Theorem 6.3 所列條件外，加上 S is orthonormal subset，則 y 可寫成：

$$y = \sum_{i=1}^{k} \langle y, v_i \rangle v_i$$

**Corollary 2 to Theorem 6.3**

Let V be an inner product space, and let S be an orthogonal subset of V consisting of nonzero vectors. Then S is linear independent.

若 S 是 Theorem 6.3 所稱，非零向量的 orthogonal subset，則 S 是線性獨立。

**EXAMPLE 3**

The orthonormal set $\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\}$ is an **orthonormal basis** for $R^3$. Let x = (2, 1, 3). The coefficients given by Corollary 1 to Theorem 6.3 that express x as a linear combination of the basis vectors are

$\{\frac{1}{\sqrt{2}}(1,1,0), \frac{1}{\sqrt{3}}(1,-1,1), \frac{1}{\sqrt{6}}(-1,1,2)\}$為一 orthonormal basis，利用 Theorem 6.3 的 Corollary 1 表示 x 為基底向量線性組合的係數為

$$a_1 = \frac{1}{\sqrt{2}}(2+1) = \frac{3}{\sqrt{2}},\quad a_2 = \frac{1}{\sqrt{3}}(2-1+3) = \frac{4}{\sqrt{3}},\quad a_3 = \frac{1}{\sqrt{6}}(-2+1+6) = \frac{5}{\sqrt{6}}$$

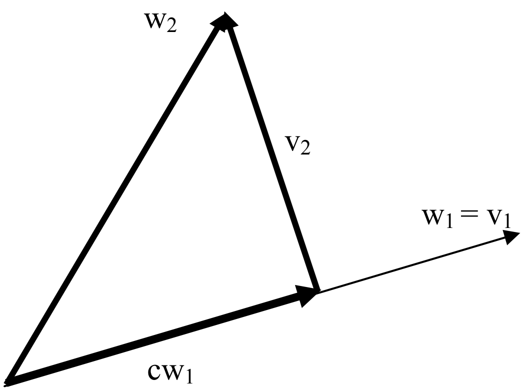
$$(2,1,3) = \frac{3}{2}(1,1,0) + \frac{4}{3}(1,-1,1) + \frac{5}{6}(-1,1,2)$$

**How to cobstruct an orthogonal set from a linearly independent set of vector** s in such a way that both sets generate the same subspace.

如何從線性獨立子集合建構一組正交集合？

Consider a simple case, $\{w_1, w_2\}$ is a linearly independent subset of an inner product space.

$\{w_1, w_2\}$是內積空間的線性獨立子集合。




We want to construct an orthogonal set from $\{w_1, w_2\}$ that spans the same subspace. The set $\{v_1, v_2\}$？如何找出$\{v_1, v_2\}$，讓$\{v_1, v_2\}$與$\{w_1, w_2\}$生成相同的子空間？

$v_1 = w_1$, $v_2 = w_2 - cw_1$ c?

$0 = \langle v_2, w_1 \rangle = \langle w_2 - cw_1, w_1 \rangle = \langle w_2, w_1 \rangle - c\langle w_1, w_1 \rangle$

$$c = \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2}$$

Thus $v_2 = w_2 - \frac{\langle w_2, w_1 \rangle}{\|w_1\|^2} w_1$

**Theorem 6.4 Extended to finite LI subset**

Let V be an inner product space and $S = \{w_1, w_2, \ldots, w_n\}$ be a linearly independent subset of V. Define $S' = \{v_1, v_2, \ldots, v_n\}$, where $v_1 = w_1$ and

如何由 $S = \{w_1, w_2, \ldots, w_n\}$，建構出 $S' = \{v_1, v_2, \ldots, v_n\}$

$$v_k = w_k - \sum_{j=1}^{k-1} \frac{\langle w_k, v_j \rangle}{\|v_j\|^2} v_j \quad k = 2, \ldots, n$$

Then $S' = \{ v_1, v_2, \ldots, v_n \}$ is an orthogonal set of nonzero vectors such that span(S') =span (S).

S 的生成集與 S'的生成集相同。

# GRAM-SCHMIDT process
## constructing $\{v_1, v_2, \ldots, v_n\}$ by using Theorem 6.4.

利用 Theorem 6.4 建構 $\{v_1, v_2, \ldots, v_n\}$ 的過程稱為 GRAM-SCHMIDT process。

**EXAMPLE 4**

In $R^4$, let $w_1 = (1, 0, 1, 0)$, $w_2 = (1, 1, 1, 1)$, and $w_3 = (0, 1, 2, 1)$. Then $\{w_1, w_2, w_3, w_4\}$ is linearly independent. We use the GRAM-SCHMIDT process to compute the orthogonal vector $v_1$, $v_2$, and $v_3$, and then we normalize these vectors to obtain an orthonormal set.

以 GRAM-SCHMIDT process 由 $w_1 = (1, 0, 1, 0)$ 、$w_2 = (1, 1, 1, 1)$與 $w_3 = (0, 1, 2, 1$ 建構 $v_1$、 $v_2$ 與 $v_3$，進而正規化，以獲得 orthonormal set。

Take $v_1 = w_1 = (1, 0, 1, 0)$

$$v_2 = w_2 - \frac{<w_2, v_1>}{\|v_1\|^2} v_1 = \ldots = (0,1,0,1)$$

$$v_3 = w_3 - \frac{<w_3, v_1>}{\|v_1\|^2} v_1 - \frac{<w_3, v_2>}{\|v_2\|^2} v_2 = \ldots = (-1,0,1,0)$$

To obtain the orthonormal basis

$$u_1 = \frac{1}{\|v_1\|^2} v_1 = \frac{1}{\sqrt{2}}(1,0,1,0)$$

$$u_2 = \frac{1}{\|v_2\|^2} v_2 = \frac{1}{\sqrt{2}}(0,1,0,1)$$

$$u_3 = \frac{1}{\|v_3\|^2} v_3 = \frac{1}{\sqrt{2}}(-1,0,1,0)$$

## EXAMPLE 5

Let $V = P(R)$ with the inner product $<f(x), g(x)> = \int_{-1}^{1} f(t)g(t)dt$, and consider the subspace $P_2(R)$ with the standard ordered basis β. We use the GRAM-SCHMIDT process to replaceβ by an orthogonal basis $\{v_1, v_2, v_3\}$ for $P_2(R)$, and use this orthogonal basis to obtain an orthnormal basis for $P_2(R)$.

以 GRAM-SCHMIDT process 由 standard ordered basis $\beta = \{1, x, x^2\}$建構 $v_1$、 $v_2$ 與 $v_3$，進而正規化，獲得 orthonormal set。

Take $v_1 = 1$. Then $\|v_1\|^2 = \int_{-1}^{1} 1^2 dt = 2$, and $<x, v_1> = \int_{-1}^{1} t \cdot 1dt = 0$.

Thus $v_2 = x_2 - \frac{<v_1, x>}{\|v_1\|^2} v_1 = x - \frac{0}{2} = x.$

Furthermore $<x^2, v_1> = \int_{-1}^{1} t^2 \cdot 1dt = \frac{2}{3}$ and $<x^2, v_2> = \int_{-1}^{1} t^2 \cdot tdt = 0.$

Therefore

$$v_3 = x^2 - \frac{<x^2, v_1>}{\|v_1\|^2} v_1 - \frac{<x^2, v_2>}{\|v_2\|^2} v_2 = \ldots = x^2 - \frac{1}{3}.$$

We conclude that $\{1, x, x^2 - \frac{1}{3}\}$ is an orthogonal basis for $P_2(R)$.

To obtain the orthonormal basis

$$u_1 = \frac{1}{\|v_1\|^2} v_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1^2 dt}} = \frac{1}{\sqrt{2}}$$

$$u_2 = \frac{1}{\|v_2\|^2} v_2 = \frac{x}{\sqrt{\int_{-1}^{1} t^2 dt}} = \sqrt{\frac{3}{2}} x$$

$$u_3 = \frac{1}{\|v_3\|^2} v_3 = \sqrt{\frac{5}{8}}(3x^2 - 1)$$

**Legendre polynomials**

Applying the GRAM-SCHMIDT orthogonalization process to the basis $\{1, x, x^2, \ldots\}$ for P(R), we obtain an orthogonal basis whose elements are called the Legendre polynomials.

當 $\beta = \{1, x, x^2, \ldots.\}$，則建構所得的正交基底其元素稱為 Legendre polynomials.

## Theorem 6.5

Let V be a nonzero finite-dimensional inner product space. Then V has an orthonormal basis β. Furthermore, if $\beta = \{v_1, v_2, \ldots, v_n\}$ and $x \in V$, then

V 為非零有限維度的內積空間，β 為 orthonormal basis，則 V 內的 x 可寫成

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i$$ 一種 linear combination。

## EXAMPLE 6

We use Theorem 6.5 to represent the polynomial $f(x) = 1 + 2x + 3x^2$ as a linear combination of the vectors in the orthonormal basis $\{u_1, u_2, u_3\}$ for $P_2(R)$ obtained in EXAMPLE 5. Observe that

利用 Theorem 6.5 將 f(x)表達成 Example 5 所得到的 orthonormal basis $\{u_1, u_2, u_3\}$ 的線性組合。

$$\langle f(x), u_1 \rangle = \int_{-1}^{1} \frac{1}{\sqrt{2}}(1 + 2t + 3t^2) dt = 2\sqrt{2}$$

$$\langle f(x), u_2 \rangle = \int_{-1}^{1} \sqrt{\frac{3}{2}} t(1 + 2t + 3t^2) dt = \frac{2\sqrt{6}}{3}$$

$$\langle f(x), u_3 \rangle = \int_{-1}^{1} \sqrt{\frac{5}{8}}(3t^2 - 1)(1 + 2t + 3t^2) dt = \frac{2\sqrt{10}}{5}$$

Therefore $f(x) = 2\sqrt{2}u_1 + \frac{2\sqrt{6}}{3}u_2 + \frac{2\sqrt{10}}{5}u_3$

## Corollary to Theorem 6.5

Let V be a finite-dimensional inner product space with an orthonormal basis β = {$v_1$, $v_2$,…, $v_n$} . Let T be a linear operator on V, and let $A = [T]_\beta$.

Then for any i and j, $A_{ij} = <T(v_j), v_i>$

V 為一有限維度內積空間，β 為 orthonormal basis，T 為 V 上線性運算子，A 為其矩陣表示式，則矩陣內的元素 Aij 為 $A_{ij} = <T(v_j), v_i>$ 。

## DEFINITION Fourier coefficients

Let β be an orthonormal subset of an inner product space V, and let x ∈ V. We define the **Fourier coefficients** of x relative to β to be the scalars <x, y>, where y ∈ β.

β 為 V 內一組 orthonormal subset，V 內的 x 相對於 β 的 **Fourier coefficients 為** <x, y>。

Let $f_n(t) = e^{int}$, where $0 \leqq t \leqq 2\pi$. S = { $f_n$: n is an integer}. The nth Fourier coefficient for a continuous function f ∈ V relative to S is

$c_n = <f, f_n> = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$.

定義 V 內函數 f 相對於 S = { $f_n$: n is an integer}的 Fourier coefficients 為 $c_n = <f, f_n> = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-int}dt$ 。

## EXAMPLE 7

Let S = { $f_n$: n is an integer}. In EXAMPLE 9 of Section 6.1. S was shown to be an orthonormal set in H. We compute the Fourier coefficients of f(t) = t relative S. Using integration by parts, we have,

f(t) = t 相對於 S 的 Fourier coefficients。

for n ≠ 0,

$< f, f_n > = \frac{1}{2\pi}\int_0^{2\pi} t\overline{e^{int}}dt = \ldots = \frac{-1}{in}$ and,

for n = 0,

$$< f,1 >= \frac{1}{2\pi}\int_0^{2\pi} t(1)dt = \ldots = \pi .$$

## DEFINITION Orthogonal complement 正交補集

Let S be a nonempty subset of an inner product space V. We define $S^{\perp}$ ( read "S perp" to be of all vectors in V that are orthogonal to every vector in S; that is, $S^{\perp} = \{ x \in V: \langle x,y\rangle = 0 \text{ for all } y \in S\}$. The set $S^{\perp}$ is called the **orthogonal complement** of S.

$S^{\perp}$是所有在 V 內且與 S 內每一向量正交的向量組成的集合。

## EXAMPLE 8

$\{0\}^{\perp} = V$ and $V^{\perp} = \{0\}$ for any inner product space V.

## EXAMPLE 9

If $V = R^3$ and $S = \{e_3\}$, the $S^{\perp}$ equals to the xy-plane.

NEXT..

Consider the problem in $R^3$ of finding the distance from a point P to a plane W. The desired distance is clearly given by $\|y-u\|$. The vector $z = y - u$ is orthogonal to every vector in W, and so $z \in W^{\perp}$.

在 $R^3$ 中找出一點 P 到一平面 W 的距離？若令 y 為由 0 到 P 的向量，則問題可以說成：在平面 W 上找最靠近 y 的向量 u，且距離為$\|y-u\|$。由下圖可知，向量 $z = y - u$ 與 W 上的每一向量均正交，所以 $z \in W^{\perp}$。

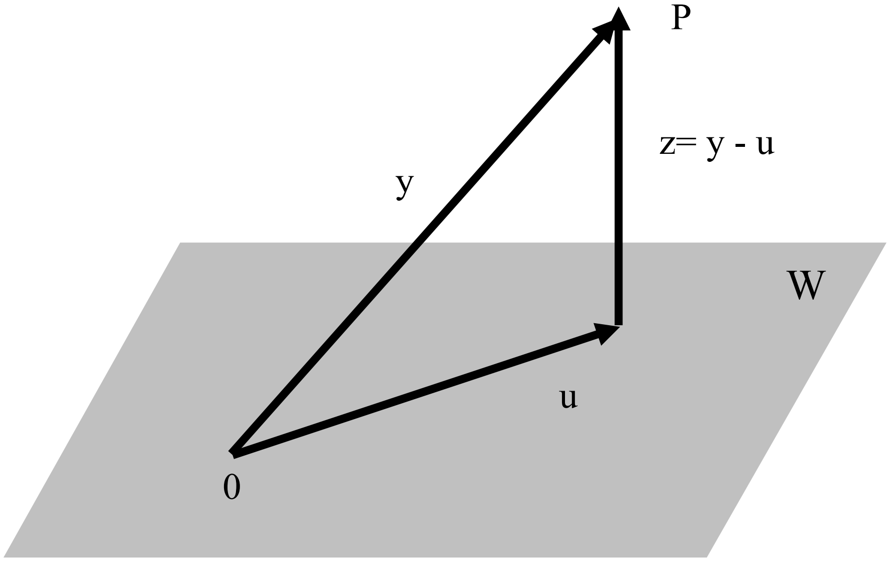
**HOW to find u ?**

## Theorem 6.6 FINDING u

Let W be a finite-dimensional subspace of an integer product space V, and let $\in V$. Then there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$. Furthermore, if

$\{v_1, v_2, \ldots, v_k\}$ is an orthonormal basis for W, then

$$u = \sum_{i=1}^{n} < y, v_i > v_i$$




## Corollary to Theorem 6.6

In the notation of Theorem 6.6, the vector u is the unique vector in W that is "closest" to y; that is, for any x $\in$ W, $\|y-x\| \geq \|y-u\|$, and this inequality is an equality if and only of $x = u$.

與 y 最接近者為 u。即對於任意 x 而言，$\|y-x\| \geq \|y-u\|$。

**The vector u is called the orthogonal projection of y on W.**

**向量 u 為 y 在 W 上的正交投影。**

## EXAMPLE 10

Let $V = P_3(R)$ with the inner product

$< f(x), g(x) >= \int_{-1}^{1} f(t)g(t)dt$ for all f(x), g(x) $\in$ V.

We compute the orthogonal projection $f_1(x)$ of $f(x) = x^3$ on $P_2(R)$.

計算 f(x)在 $P_2(R)$上的正交投影？

By EXAMPLE 5,

$\{u_1, u_2, u_3\} = \{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{5}{8}}(3x^2-1)\}$ is an orthonormal basis for $P_2(R)$.

For these vectors, we have

$$< f(x), u_1 > = \int_{-1}^{1} t^3 \frac{1}{\sqrt{2}} dt = 0$$

$$< f(x), u_2 > = \int_{-1}^{1} t^3 \sqrt{\frac{3}{2}} t dt = \frac{\sqrt{6}}{5}$$

$$< f(x), u_3 > = \int_{-1}^{1} t^3 \sqrt{\frac{5}{8}} (3t^2-1) dt = 0$$

Hence $f_1(x) = < f(x), u_1 > u_1 + < f(x), u_2 > u_2 + < f(x), u_3 > u_3 = \frac{3}{5}x$

**Theorem 6.7**

Suppose that $S = \{v_1, v_2, \ldots, v_k\}$ is an orthonormal set in an n-dimensional inner product space V. Then

(a) S can be extened to an orthonormal basis $\{v_1, v_2, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_n\}$ for V.

(b) If W = span(S), then $S_1 = \{v_{k+1}, v_{k+2}, \ldots, v_n\}$ is an orthonormal basis for $W^{\perp}$.

(c) If W is any subspace of V, then $\dim(V) = \dim(W) + \dim(W^{\perp})$.

# 6-3 THE ADJOINT OF A LINEAR OPERATOR

The conjugate transpose of a matrx A： 矩陣 A 的共軛轉置矩陣為 A*。

Now, we define a related linear operator T on V called the adjoint of T, whose matrix representation with respect to any orthonormal basis β for V is $[T]_{\beta}^{*}$

定義 linear operator T 的伴隨運算子 $[T]_{\beta}^{*}$ 。

**Theorem 6.8**

Let V be a finite-dimensional inner product space over F, and let g: V→F be a linear transformation. Then there exits a unique vector y ∈ V such that g(x) = <x, y> for all x ∈ V.

V 內存在唯一向量 y，使得 g(x) = <x, y>。其中，x 為 V 中任一向量。

**EXAMPLE 1**

Define g: $R^2 \rightarrow R$ by $g(a_1, a_2) = 2a_1 + a_2$; clearly g is linera transformation.

g 是由 $R^2 \rightarrow R$ 的一個 linear transformation，定義為 $g(a_1, a_2) = 2a_1 + a_2$。

Let $\beta = \{e_1, e_2\}$, and let $y = g(e_1)e_1 + g(e_2)e_2 = 2e_1 + e_2 = (2, 1)$. Then $g(a_1, a_2) = \langle (a_1, a_2), (2, 1) \rangle = 2a_1 + a_2$.

## Theorem 6.9

Let V be a finite-dimensional inner product space, and let T be a linear operator on V. Then there exists a unique function $T^*$: $V \rightarrow V$ such that $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all x, y $\in$ V. Furthermore, $T^*$ is linear.

對 V 內所有 x 與 y 而言存在唯一函數 $T^*$，使得 $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$。

Theorem 6.9 所稱 Linear operator $T^*$，為 T 的伴隨（Adjoint of the operator T）。

T* is the unique operator on V satisfying $\langle T(x), y \rangle = \langle x, T^*(y) \rangle$ for all x, y $\in$ V. Note that we also have

$$\langle x, T(y) \rangle = \overline{\langle T(y), x \rangle} = \overline{\langle y, T^*(x) \rangle} = \langle T^*(x), y \rangle .$$

## Theorem 6.10

Let V be a finite-dimensional inner product space, and let β be an orthonormal basis for V. If T is a linear operator on V, then

$$[T^*]_\beta = [T]_\beta^*.$$

## Corollary to Theorem 6.10

Let A be an nn matriox. Then $L_{A^*} = (L_A)^*$.

## EXAMPLE 2

Let T be the linear operator on $C^2$ defined by $T(a_1, a_2) = (2ia_1 + 3a_2, a_1 - a_2)$. If β is the standard ordered basis for $C^2$, then

$$[T]_\beta = \begin{bmatrix} 2i & 3 \\ 1 & -1 \end{bmatrix}.$$

So

$$[T^*]_\beta = [T]_\beta^* = \begin{bmatrix} -2i & 1 \\ 3 & -1 \end{bmatrix}.$$

Hence $T^*(a_1, a_2) = (-2ia_1 + a_2, 3a_1 - a_2)$

**Theorem 6.11**

Let V be a finite-dimensional inner product space, and let T and U be linear operator on V. Then

(a) $(T + U)^* = T^* + U^*$;

(b) $(cT)^* = \bar{c}T^*$ for any $c \in F$;

(c) $(TU)^* = U^*T^*$;

(d) $T^{**} = T$;

(e) $I^* = I$.

**Corollary to Theorem 6.11**

Let A and B be n×n matrice. Then

(a) $(A + B)^* = A^* + B^*$;

(b) $(cA)^* = \bar{c}A^*$ for any $c \in F$;

(c) $(AB)^* = B^*A^*$;

(d) $A^{**} = A$;

(e) $I^* = I$.

# Approximation of function

Many problems in the physical sciences and engineering involve approximating given functions by polynomials or by trigonometric functions.

利用多項式或三角函數來近似一已知函數。

For example, it may be necessary to approximate $f(x) = e^x$ by a linear function of the form $\mathbf{g(x) = a + bx}$ over the interval [0,1], or by a trigonometric function of the form $\mathbf{h(x) = a + b \times \sin x + c \times \cos x}$ over the interval $[-\pi, +\pi]$.

Introducing the techniques for approximating functions.

## Least-Squares Approximation

Let C[a, b] be the vector space of continuous functions over the interval [a, b], f be an element of C[a, b], and W be a subspace of C[a, b]. The function g in W such that $\int_0^1 [f(x)-g(x)]^2 dx$ is a minimum is called the **least-squares approximation** to f.

C[a, b]是佈於[a, b]由連續函數所組成的向量空間，f 為向量空間 C[a, b]的元素，W 為向量空間 C[a, b]的子空間，若 W 內一函數 g 能使得 $\int_0^1 [f(x)-g(x)]^2 dx$ 最小者，稱為 f 的 **least-squares approximation**。

This approximation is called the least-squares approximation since the distance formula is based on squaring.

If U is a subspace of $R^n$ and if x defines a point in $R^n$, then the element of U that is closed to x is $proj_U x$.

U 內最接近 x 者，寫成 $proj_U x$。U 是 $R^n$ 的子空間，x 定義 $R^n$ 的一點。

Hence, the least squares approximation to f in the subspace W is $g = proj_W f$.

W 內最接近 f 者為 g，寫成 $g = proj_W f$。

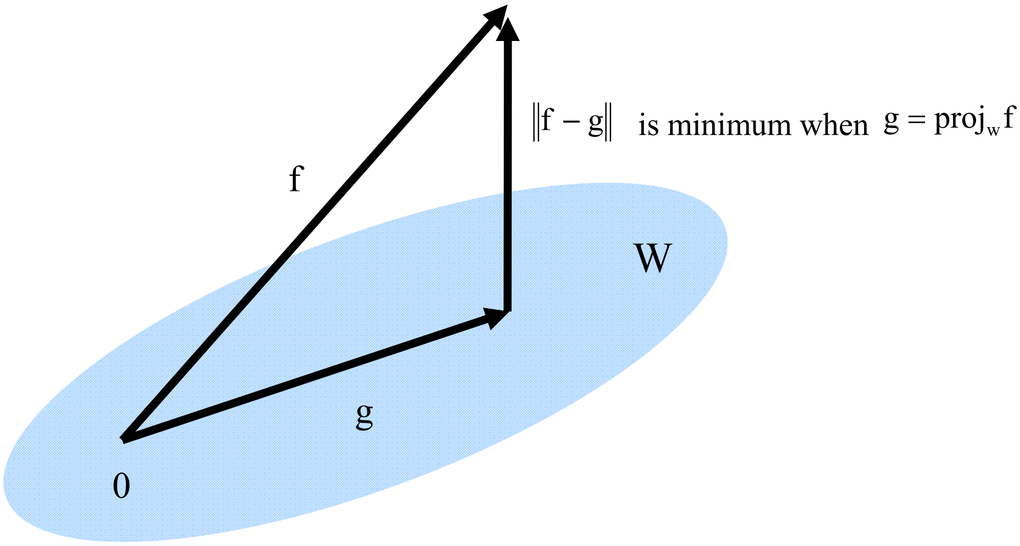
We build on the definition of $proj_U x$ to get an expression for $proj_W f$.

▶If $\{u_1,\ldots,u_m\}$ is an orthonormal basis for U, then we know that

$$proj_U x = (x\cdot u_1)u_1 + (x\cdot u_2)u_2 + \ldots + (x\cdot u_m)u_m$$

▶Let $\{g_1,\ldots,g_m\}$ is an orthonormal basis for W. Replacing the dot product of $R^n$ by the inner product of the function space we get

$$proj_W f = \langle f, g_1\rangle g_1 + \langle f, g_2\rangle g_2 + \ldots + \langle f, g_n\rangle g_n$$



$\|f - g\|$ is minimum when $g = \text{proj}_w f$
f
W
g
0