Chapter 5 Diagonalization

This chapter is concerned with the so-called diagonalization problem. For a given linear operator T on a finite-dimensional vector space V, we seek answers to the following questions.

1. Does there exist an ordered basis β for V such that $[T]_\beta$ is a diagnoal matrix?
2. If such a basis exists, how can it be found?

A solution to the diagonalization problem leads naturally to the concepts of eigenvalue and eigenvector.

給定有限維度向量空間 V 的線性運算子，是否存在有序基底 β 可使得$[T]_\beta$為一對角矩陣？該基底如何找出來？要解決對角問題，自然得引進 Eigenvalue 與 Eigenvector 的觀念。

5-1 Eigenvalues and Eigenvectors

DEFINTION 5.1 Diagonalizable（可對角化）

A linear operator T on a finite-dimensional vector space V is called diagonalizable if there is an ordered basis β for V such that $[T]_\beta$ is a diagnoal matrix. A square matrix A is called diagonalizable is L_A is diagonalizable.

在有限維度向量空間 V 中，一有序基底 β 可以使得$[T]_\beta$成為一對角矩陣，則該線性運算子 T 被稱為「可對角化」。若 L_A 可對角化，則 A 稱為可對角化。

DEFINITION 2.12

Suppose that V and W are finite-dimensional vector spaces with ordered bases β = {v_1, v_2,… , v_n} and γ = {w_1, w_2,… , w_n}, respectively. Let T: V→W be linear. Then for each j, 1 ≦j≦n, there exist unique scalars a_{ij}∈F, 1≦i≦m, such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } 1 \leqq j \leqq n$$

We call the m×n matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$.

If V = W and β = γ, then we write $A = [T]_\beta$.

設 V 與 W 分別為有限維度的向量空間，$\beta = \{v_1, v_2, \ldots, v_n\}$ 與 $\gamma = \{w_1, w_2, \ldots, w_n\}$ 分別為 V 與 W 的有序基底，且 T: V→W 為 V 映至 W 的線性轉換。

對每一個 j（$1 \leqq j \leqq n$）而言，存在唯一的純量 $a_{ij} \in F$（$1 \leqq i \leqq m$），使得 $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$。

將 m×n 的矩陣 A 定義為 $A_{ij} = a_{ij}$，並稱呼 A 為線性轉換 T 的矩陣表達方式。在 V 與 W 分別以 β 與 γ 作為有序基底下，A 可註記為 $A = [T]_\beta^\gamma$。

若 V = W 且 β = γ，則 $A = [T]_\beta$。

提示：$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ 為「將 v_j 的像 $T(v_j)$ 表達成有序基底 γ 的線性組合」。

We want to determine when a linear operator T on a finite-dimensional vector space V is diagonalizable and, if so, how to obtain an ordered basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for V such that $[T]_\beta$ is a dignoal matrix. If $D = [T]_\beta$ is a dignoal matrix, then for each vector $v_j \in \beta$, we have

$$T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j \quad \text{where } \lambda_j = D_{jj}.$$

Conversely, if $\beta = \{v_1, v_2, \ldots, v_n\}$ is an ordered basis for V such that $T(v_j) = \lambda_j v_j$ for some scalars $\lambda_1, \lambda_2, \ldots \lambda_n$, then clearly

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

當有限維度向量空間 V 內的線性運算子 T 可以對角化時，如何找到有序基底 $\beta = \{v_1, v_2, \ldots, v_n\}$？可以使得 $[T]_\beta$ 成為對角矩陣。

若 $D = [T]_\beta$ 為對角矩陣，則對於每一向量 $v_j \in \beta$ 而言，$T(v_j) = \sum_{i=1}^{n} D_{ij} v_i = D_{jj} v_j = \lambda_j v_j$ where $\lambda_j = D_{jj}$（將 v_j 的運算結果 $T(v_j)$ 表達成有序基底 β 的線性組合）。

反之，若 $\beta = \{v_1, v_2, \ldots, v_n\}$ 為 V 的有序基底，滿足 $T(v_j) = \lambda_j v_j$，則

$$[T]_\beta = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & : \\ : & : & & : \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$ （以為基底 β 的運算子 T 的矩陣表達方式）。

D = $[T]_\beta$ = Definition 2.12 的 A，相當於 Definition 2.12 的 V = W 且 $\beta = \gamma$。

DEFINTION 5.2 Eigenvectors & Eigenvalues（固有向量與固有值）

Let T be a linear operator on a vector space V. A nonzero vector $v \in V$ is called an eigenvector of T if there exists a scalar λ such that T(v)= λv. The scalar λ is called the eigenvalues corresponding to the eigenvector v.

令 T 為向量空間 V 中的線性運算子，對 V 中任一非零向量 v 而言，若存在純量 λ 使得 T(v)= λv ，則稱 v 為 T 的 Eigenvector（固有向量），λ 為對應 v 的 Eigenvalue（固有值）。

Let A be in $M_{n\times n}(F)$. A nonzero vector $v \in F^n$ is called an eigenvector of A if v is an eigenvector of L_A; that is, if Av = λv for some scalar λ. The scalar λ is called the eigenvalue corresponding to the eigenvector v.

令 A $\in M_{n\times n}(F)$，對屬於 F^n 的非零向量 v 而言，若 v 是 L_A 的 Eigenvector，則 v 稱為 A 的 Eigenvector。若純量 λ，滿足 Av = λv，則 λ 稱為對應於 Eigenvector v 的 Eigenvalue。

Characteristic vector ≡ Proper vector ≡ Eigenvector.

Characteristic value ≡ Proper value ≡ Eigenvalue.

Theorem 5.1

A linear operator T on a finite-dimensional vector space V is diagonalizable if and only if there exits an ordered basis β for V consisting of eigenvectors of T. Furthermore, if T is diagonalizable, $\beta = \{v_1, v_2, \ldots, v_n\}$is an ordered basis of eigenvectors of T, and D = $[T]_\beta$, then D is a diagonal matrix and D_{jj} is the eigenvalue corresponding to v_j for $1 \leqq j \leqq n$.

在有限維度向量空間 V 的線性運算子 T 為可對角化，其「IF AND ONLY IF」條件為 V 內存在有序基底 β 且該基底係由 T 的 Eigenvectors 所組成。再者，若線性運算子 T 為可對角化，$\beta = \{v_1, v_2, \ldots, v_n\}$是 T 的 Eigenvectors 所組成的有序基底，且 D =

$[T]_\beta$，則 D 是一對角化矩陣且 D_{jj}（對角矩陣的元素）為對應 Eigenvectors v_j 的 Eigenvalues。

To diagonalize a matrix or a linear operator is to find a basis of eigenvectors and the corresponding eigenvalues.

要將矩陣或線性運算子對角化，則必須找出該矩陣或線性運算子的 Eigenvectors 與 Eigenvalues。

EXAMPLE 1

Let $A = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}$, $v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, and $v_2 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

Since $L_A(v_1) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} = -2\begin{pmatrix} 1 \\ -1 \end{pmatrix} = -2v_1$, v_1 is an eigenvector of L_A, and hence of A.

Here $\lambda_1 = -2$ is the eigenvalue corresponding to v_1.

v_1 是 L_A 的 eigenvector。

$\lambda_1 = -2$ 是對應 v_1 的 eigenvalue。

Furthermore, $L_A(v_2) = \begin{pmatrix} 1 & 3 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 15 \\ 20 \end{pmatrix} = 5\begin{pmatrix} 3 \\ 4 \end{pmatrix} = 5v_2$ and so v_2 is an eigenvector of L_A, and hence of A, with the corresponding to eigenvalue $\lambda_2 = 5$.

v_2 是 L_A 的 eigenvector。

$\lambda_2 = -2$ 是對應 v_2 的 eigenvalue。

Note that $\beta = \{v_1, v_2\}$ is an ordered basis of R^2 consisting of eigenvector of both of A and L_A, and therefore A and L_A are diagonalizable.

基底 β 由 Eigenvector v_1 與 v_2 組成。

Moreover, by Theorem 5.1,

$|L_A|_\beta = \begin{pmatrix} -2 & 0 \\ 0 & 5 \end{pmatrix}$（對角線元素為 Eigenvalues。）

EXAMPLE 2

Let T be the linear operator on R^2 that rotates each vector in the plane through an angle of $\pi/2$. It is clear geometrically that for any nonzero vector v, the vector v and T(v) are not collinear; hence T(v) is not a multiple of v. Therefore T has no eigenvectors and, consequently, no eigenvalues. Thus there exist operators (and matrices) with no eigenvalues or eigenvectors. Of course, such operators and matrices are not diagonalizable.

T 是 R^2 的線性運算子，該運算子係將平面上的每一個向量旋轉 $\pi/2$。對任何非零向量 v 而言，經過線性運算子 T 處理後 v 與 T(v)不共線，T(v)也不是 v 的倍數，因此 T 沒有 Eigenvector 也沒有 Eigenvalue。這種沒有 Eigenvector 與 Eigenvalue 的運算子或矩陣，當然也就不可對角化。

In order to obtain a basis of eigenvectors for a matrix (or a linear operator), we need to be able to determine its eigenvalues and eigenvectors. The following theorem gives us a method for computing eigenvalues.

爲了求矩陣或線性運算子的 Eigenvector 所組成的基底，必須求該矩陣或線性運算子的 Eigenvectors 與 Eigenvalues。

Theorem 5.2

Let $A \in M_{n\times n}(F)$. Then a scalar λ is an eigenvalue of A if and only if $\det(A-\lambda I_n) = 0$.

令 $A \in M_{n\times n}(F)$，則 λ 是 A 的 Eigenvalue 的「若且唯若」條件為 $\det(A-\lambda I_n) = 0$。

【Proof】

A scalar λ is an eigenvalue of A if and only if there exists a nonzero vector $v \in F$ such that $Av = \lambda v$, that is $(A - \lambda I_n)v = 0$. （存在非零的向量 v。）

This is true if and only if $A - \lambda I_n$ is not invertible.

→ Equivalent to say $\det(A-\lambda I_n) = 0$。

Corollary 矩陣可逆與行列式的關係

A matrix $A \in M_{n\times n}(F)$ is invertible if and only if $\det(A) \neq 0$. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

令 $A \in M_{n\times n}(F)$ 且為可逆的『若且唯若』條件為 $\det(A) \neq 0$。再者，若 A 可逆，則 $\det(A^{-1}) = \dfrac{1}{\det(A)}$。

DEFINTION 5.3 Characteristic polynomial

Let $A \in M_{n\times n}(F)$. The polynomial $f(t) = \det(A - tI_n)$ is called the characteristic polynomial of A.

令 $A \in M_{n\times n}(F)$，則多項式 $f(t) = \det(A - tI_n)$稱為 A 的特徵多項式。

Theorem 5.2 states that the eigenvalues of a matrix are the zeros of its characteristic polynomial.

依據 Theorem 5.2 的敘述：矩陣的 Eigenvalues 為該矩陣特徵多項式的根。

EXAMPLE 4

To find the eigenvalues of

$$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(R)$$

We compute its characteristic polynomial:

$$\det(A - tI_2) = \det\begin{pmatrix} 1-t & 1 \\ 4 & 1-t \end{pmatrix} = t^2 - 2t + 3 = (t-3)(t+1)$$

The eigenvalues of A are 3 and -1.

DEFINTION 5.4 Characteristic polynomial

Let T be a linear operator on an n-dimensional vector space V with ordered basis β. We define the characteristic polynomial f(t) of T to be the characteristic polynomial of $A = [T]_\beta$. That is $f(t) = \det(A - tI_n)$.

令 T 為 n 維向量空間 V 的線性運算子，該向量空間的有序基底為 β。T 的特徵多項式即為 $A = [T]_\beta$ 的特徵多項式。即 $f(t) = \det(A - tI_n)$。

If T is a linear operator on a finite-dimensional vector space V and β is an ordered basis for V, then λ is eigenvalue of T if and only if λ is an eigenvalue of $[T]_\beta$. We often denote the characteristic polynomial of an operator T by det(T-tI).

若 T 為有限維度向量空間 V 的線性運算子，**β** 為 V 的有序基底，則 λ 為 T 的 Eigenvalue 的「若且唯若」條件為 λ 是$[T]_\beta$ 的 Eigenvalue。我們經常將 T 的特徵多項式表達成為 $\det(T - tI)$。

EXAMPLE 5

Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f(x)+(x+1)f'(x)$, let β be the standard ordered basis for $P_2(R)$, and let $A = [T]_\beta$. Then

有序基底 $\beta = \{1, x, x^2\}$，因

$T(1) = 1 = 1\cdot 1+0\cdot x+0\cdot x^2$

$T(x) = 1+2x = 1\cdot 1+2\cdot x+0\cdot x^2$

$T(x^2) = 0+2x+3x^2 = 0\cdot 1+2\cdot x+3\cdot x^2$

所以 $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix} = [T]_\beta$

The characteristic polynomial of T is

$$\det(A - tI_3) = \begin{pmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{pmatrix} = \ldots = -(t-1)(t-2)(t-3)$$

此為 T 的特徵多項式，解出特徵多項式的根，即為 T 的 Eigenvalues。

Hence **λ** is an eigenvalue of T (or A) if and only if **λ** = 1, 2, or 3.

Theorem 5.3

Let $A \in M_{n\times n}(F)$.

(a) The characteristic polynomial of A is a polynomial of degree n with leading coefficient $(-1)^n$.

(b) A has at most n distinct eigenvalues.

令 $A \in M_{n\times n}(F)$，則

（a）A 的特徵多項式是一個 n 階的多項式且其領導係數為$(-1)^n$。

（b）A 最多有 n 個相異的 Eigenvalues。

Theorem 5.4

Let T be the linear operator on a vector space V, and let λ be an eigenvalue of T. A vector $v \in F$ is an eigenvector of T corresponding to λ if and only if $v \neq 0$ and $v \in N(T - \lambda I)$.

令 T 為向量空間 V 的線性運算子，且 λ 是 T 的 Eigenvalue，有一個屬於 F 的向量 v（$v \in F$）是 T 對應於 λ 的 Eigenvector，則其「若且唯若」條件為 $v \neq 0$ 且 v

$\in$ N(T - λI)。

v $\neq$ 0 and v $\in$ N(T - λI): The eigenvectors of T corresponding to the eigenvalues λ are the nonzero vectors in the null space of T – λI.

「v $\neq$ 0 & v $\in$ N(T - λI)」表示：對應 Eigenvalue λ 的 Eigenvectors v 為 T – λI 的零核空間（Null space）內的非零向量。

N(T - λI) = {v$\in$V: T(v) = 0}。

DEFINTION 2.7 Null space or Kernel

Let V and W be vector space, and let T: V→W be linear. We define the null space (or kernel) **N(T) of T to be the set of all vectors x in V such that T(x) = 0**; that is, N(T) = {x$\in$V: T(x) = 0}.

V 與 W 為向量空間，且 T: V→W 為線性轉換。定義線性轉換 T 的 Null space 為 V 內所有滿足 T(x) = 0 的向量 x 所形成的集合，註記為 N(T)。Null space 的元素 x 經線性轉換 T 轉換後所對應的「像」為 0，意即 T(x) = 0。Null space 內的元素的「像」皆為 0。

註：T 的 NULL SPACE 是 V 內「T(x) = 0」的元素所形成的集合，「定義域 V」這一端的子集合。

EXAMPLE 6

To find all the eigenvectors of the matrix.

$A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \in M_{2\times 2}(R)$ in Example 4, recall that A has two eigenvalues, $\lambda_1 = 3$ and $\lambda_2 =$ -1.

由 EXAMPLE 4 得知，矩陣 A 的 Eigenvalues 為 $\lambda_1 = 3$、$\lambda_2 = -1$。

We begin by finding all the eigenvectors corresponding to $\lambda_1 = 3$.

Let $B_1 = A - \lambda_1 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}$.

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^2$ is an eigenvector corresponding to $\lambda_1 = 3$ if and only if x $\neq$ 0 and x $\in$ N(L_{B1}); that is , x $\neq$ 0 and

$$\begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -2x_1 + x_2 \\ 4x_1 - 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Clearly the set of all solutions to this equation is

$$\left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix}, t \in R \right\}.$$

Hence x is an eigenvector corresponding to $\lambda_1 = 3$ if and only if

$$x = t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{for some } t \neq 0.$$

x 是相對應 $\lambda_1 = 3$ 的 Eigenvector。

Now suppose that x is an eigenvector of A corresponding to$\lambda_2 = -1$. Let

$$B_2 = A - \lambda_2 I = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} - \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}.$$

Then $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in N(L_{B_2})$ if and only if x is a solution to the system

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2x_1 + x_2 \\ 4x_1 + 2x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Hence

$$N(L_{B_2}) = \left\{ t\begin{pmatrix} 1 \\ -2 \end{pmatrix}; t \in R \right\}$$

Thus x is an eigenvector corresponding to $\lambda_2 = -1$ if and only if

$$x = t\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{for some } t \neq 0.$$

x 是相對應 $\lambda_2 = -1$ 的 Eigenvector。

Observe that $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$ is a basis for R^2 consisting of eigenvectors of A.

Thus L_A, and hence A, is diagonalizable.

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$是 A 的 Eigenvector 組成的 R^2 的一組基底。

因此 L_A， 即 A 是可對角化的。

Suppose that β is a basis for F^n consisting of eigenvectors of A. The corollary to Theorem

2.23 assures us that if Q is the n×n matrix whose columns are the vectors in β, then $Q^{-1}AQ$ is a diagonal matrix.

設 β 是 F^n 的一組基底，由 A 的 Eigenvectors 所組成，由 Theorem 2.23 的 Collary 得知：若 Q 為 n×n 的矩陣，且 Q 矩陣的行（Column）為 β 的向量，則 $Q^{-1}AQ$ 為一對角矩陣。

In Example 6, for instance, if

$$Q=\begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}$$

Then

$$Q^{-1}AQ=\begin{pmatrix}3 & 0\\ 0 & -1\end{pmatrix}$$

The diagonal entries of this matrix are the eigenvalues of A that correspond to the respective column of Q.

對角矩陣的對角線元素為 A 的 Eigenvalues，分別對應 Q 的行（為 A 的 Eigenvector）。

Theorem 2.23

Let T be a linear operator on a finite-dimensional vector space V, and let β and β' be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β'–coordinates into β-coordinates. Then

$$[T]_{\beta'}=Q^{-1}[T]_{\beta}Q.$$

T 為有限維度向量空間 V 的一個線性運算子，令 β 與 β'為有限維度空間向量 V 的兩個有序基底，且 Q 為由 β'座標系變換至 β 座標系的座標變換矩陣，則 $[T]_{\beta'}=Q^{-1}[T]_{\beta}Q$。

【Proof】

Let I be the identity transformation on V. Then T = IT = TI; hence,

by Theorem 2.11, $Q[T]_{\beta'}=[I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}=[IT]_{\beta'}^{\beta}=[TI]_{\beta'}^{\beta}=[T]_{\beta}^{\beta}[I]_{\beta'}^{\beta}=[T]_{\beta}Q$

Therefore $[T]_{\beta'}=Q^{-1}[T]_{\beta}Q$

令 I 是 V 上的單位轉換（Identity transformation I_V: V→ V by $I_V(x) = x$ for all $x \in V$），則 IT = TI = T。

因 **Q =** $[I_V]_{\beta'}^{\beta} \rightarrow Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}$

依據 Theorem 2.11 得知：

$Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [IT]_{\beta'}^{\beta} = [TI]_{\beta'}^{\beta} = [T]_{\beta}^{\beta}[I]_{\beta'}^{\beta} = [T]_{\beta}Q$

因此，$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$。

Theorem 2.11 Let V, W, and Z be finite-dimensional vector space with ordered bases α, β, and γ, respectively. Let T: V→W and U: W→Z be linear transformation. Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$.令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 T: V→W（先） 且 U: W→Z（後），則 $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$。

令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 T: V→W（先 α→β） 且 U: W→Z（後 β→ γ），則 $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$。

Corollary to Theorem 2.23

Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_{\gamma} = Q^{-1}AQ$, where Q is the n×n matrix whose jth column is the j^{th} vector of γ.

令 $A \in M_{n\times n}(F)$且 γ 為 F^n 的有序基底，則$[L_A]_{\gamma} = Q^{-1}AQ$；其中，Q 為 n×n 的矩陣，且其第 j 行為 γ 的第 j 個向量。

DEFINITION 2.20 Standard representation

Let β be an ordered basis for an n-dimensional vectors space V over the field. The standard representation of V with respect to β is the function Φ_β: V→F^n defined by $\Phi_\beta(x) = [x]_\beta$ for each $x \in V$.

令 β 是維度為 n 的向量空間 V 的有序基底，則 V 相對於 β 的標準表示式為由 V 映至 F^n 的函數 $\Phi_\beta(x)$（Φ_β: V→F^n），該函數定義為 $\Phi_\beta(x) = [x]_\beta$；其中，$x \in V$。

如何找出 T 的 Eigenvector

TO FIND the eigenvectors of a linear operator T on an n-dimensional vector space, select an ordered basis β for V and let $A = [T]_\beta$. Figure 5.1 is a special case of Section 2.4 in which V = W and β = γ.

Recall that for $v \in V$, $\Phi_\beta(v) = [v]_\beta$, the coordinate vector of v relative to β.

We show that $v \in V$ is an eigenvector of T corresponding to λ if and only if $\Phi_\beta(v)$ is an eigenvector of A cooresponding to λ.

Suppose that v is an eigenvector of T corresponding to λ. Then $T(v) = \lambda v$. Hence

$$A\Phi_\beta(v) = L_A\Phi_\beta(v) = \Phi_\beta T(v) = \Phi_\beta(\lambda v) = \lambda\Phi_\beta(v)$$

Now $\Phi_\beta(v) \neq 0$, since $\Phi_\beta(v)$ is an isomorphism; hence $\Phi_\beta(v)$ is an eigenvectir of A.

This argument is reversible. If $\Phi_\beta(v)$ is an eigenvectir of A corresponding to λ, then v is an eigenvector of T corresponding to λ.

爲找出 n 維向量空間內線性運算子 T 的 Eigenvector，我們選一個有序基底 β 並令 $A = [T]_\beta$ 。圖 5.1 爲 Section 2.4 的一個特例，$V = W$ 且 $\beta = \gamma$。

對於 $v \in V$，$\Phi_\beta(v) = [v]_\beta$ 爲 v 相對於 β 的座標向量。

證明：$v \in V$ 爲 T 對應 λ 的 Eigenvector，其「若且唯若」條件爲 $\Phi_\beta(v)$是 A 對應 λ 的 Eigenvector？

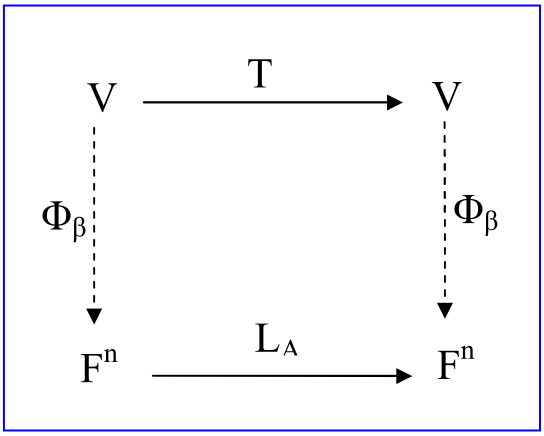
假設 v 是 T 對應 λ 的 Eigenvector，則 $T(v) = \lambda v$。因此

$$A\Phi_\beta(v) = L_A\Phi_\beta(v) = \Phi_\beta T(v) = \Phi_\beta(\lambda v) = \lambda\Phi_\beta(v)$$（參考 Figure 5.1）

$\rightarrow A\Phi_\beta(v) = \lambda\Phi_\beta(v)$

由於 $\Phi_\beta(v)$ 是一個同構轉換，所以 $\Phi_\beta(v) \neq 0$；因此，$\Phi_\beta(v)$ 是 A 的 Eigenvector。

反之，若 $\Phi_\beta(v)$ 是 A 對應 λ 的 Eigenvector，則 v 是 T 對應 λ 的 Eigenvector。



vs.

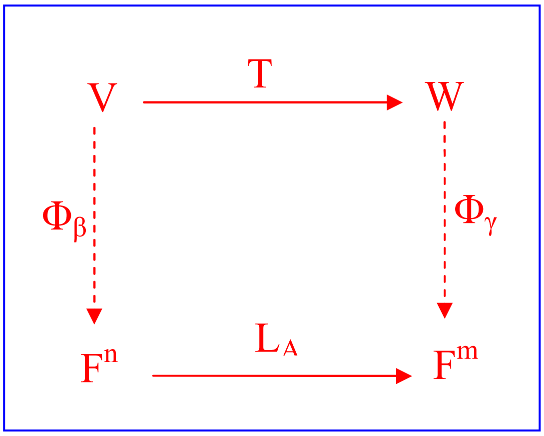


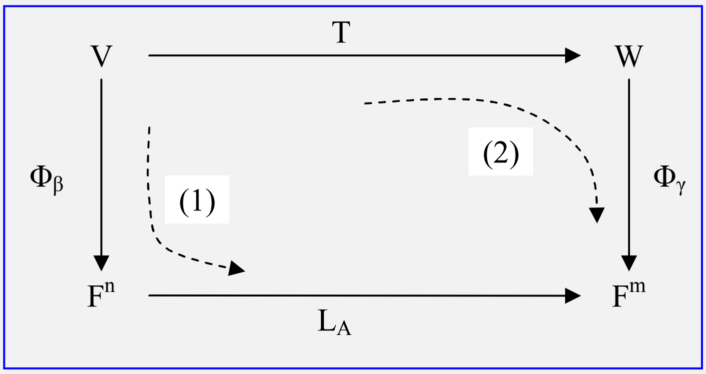
Figure 5.1 **Section 2.4**

CHAPTER 2

Let V and W be vector spaces of dimensions n and m, respectively, and let T: V→W be a linear transformation. Defined A = $[T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We are now able to use Φ_β and Φ_γ to study the relationship between the linear transformations T and L_A:F^n→F^m.

令 V 與 W 分別為維度 n 與 m 的向量空間，T 為 V→W 的線性轉換。定義 A = $[T]_\beta^\gamma$，其中 β 與 γ 分別為 V 與 W 的有序基底。我們現在要利用 Φ_β 與 Φ_γ 來探討線性轉換 T 與 L_A:F^n→F^m 的關係。

Let us first consider the below figure. Notice that there are two composites of linear transformation that maps V into F^m.



1. Map V into F^n with Φ_β and follow this transformation with L_A; this yields the composite $L_A\Phi_\beta$.
   V→ F^n 的 Φ_β（先）與 L_A（後）合成為 $L_A\Phi_\beta$。
2. Map V into W with T and follow it by Φ_γ to obtain the composite $\Phi_\gamma T$.
   V→ W 的 T（先）與 Φ_γ（後）合成為 $\Phi_\gamma T$ 。

We conclude that $L_A\Phi_\beta = \Phi_\gamma T$.

An equivalent formulation of the result discussed in the preceding paragraph is that for an eigenvalue of λ of A (and hence of $[T]_\beta$), a vector $y \in F^n$ is an eigenvector of A corresponding to λ if and only if $\Phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

對 A（或 $[T]_\beta$）的任一 eigenvalue λ，若向量 $y \in F^n$ 是 A 相對應於 λ 的 Eigenvector 其「若且唯若」條件為 $\Phi_\beta^{-1}(y)$ 是 T 相對應於 λ 的 Eigenvector。

Thus we have reduced the problem of finding the eigenvectors of a linear operator on a finite-dimensional vector space to the problem of finding the eigenvectors of a matrix.

將找線性運算子 Eigenvectors 的問題簡化為找矩陣 Eigenvectors 的問題。

EXAMPLE 7

Let T be the linear operator on $P_2(R)$ defined in Example 5, and let β be the standard ordered basis for $P_2(R)$. Recall that T has eigenvalues 1, 2, 3 and that

參考 EXAMPLE 5，有序基底 $\beta = \{1, x, x^2\}$，

$\mathbf{T(1) = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2}$

$\mathbf{T(x) = 1+2x = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2}$

$\mathbf{T(x^2) = 0+2x+3x^2 = 0 \cdot 1 + 2 \cdot x + 3 \cdot x^2}$

所以 $A = [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$.

Eigenvalues $\boldsymbol{\lambda}$ = 1、2、3。

We consider each eigenvalue separately.

Let $\boldsymbol{\lambda_1} = 1$, and $B_1 = A - \lambda_1 I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

Then $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3$ is an eigenvector corresponding to $\boldsymbol{\lambda_1} = 1$ if and only if $x \neq 0$ and $x \in N(L_{B1})$; that is , x is a nonzero solution to the system

依據 Theorem 5.4，$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}\in R^3$ 是 B_1 相對應 $\boldsymbol{\lambda_1}=1$ 的 Eigenvector，其「若且唯若」條件為 $x\neq 0$ 且 $x\in N(L_{B1})$；即 x 是下列方程組的非零解。

$$\begin{aligned}x_2&=0\\x_2+2x_3&=0\\2x_3&=0\end{aligned}$$

Note that this system has three unknowns, x_1, x_2, and x_3, but one of these, x_1, does not actually appear in the system. Since the value of x_1 does not affect the system, we assign x_1 a parametric value, say $x_1=a$, and solve the system for x_2 and x_3. Clearly, $x_2=x_3=0$, and so the eigenvectors of A corresponding to $\boldsymbol{\lambda_1}=1$ are of the form

方程組本來有三個未知數，但其中的 x_1 並未出現，故指定 x_1 為一參數值 a，令 $x_1=a$，並得知 $x_2=x_3=0$。因此，A 相對應 $\boldsymbol{\lambda_1}=1$ 的 Eigenvector 形式為

$$a\begin{pmatrix}1\\0\\0\end{pmatrix}=a\cdot e_1 \quad \text{for } a\neq 0.$$

Consequently, the eigenvectors of T corresponding to $\boldsymbol{\lambda_1}=1$ are of the form

所以 T 相對應 $\boldsymbol{\lambda_1}=1$ 的 Eigenvector 形式為

$$\Phi_\beta{}^{-1}(ae_1)=a\Phi_\beta{}^{-1}(e_1)=a\cdot 1=a \quad \text{for any } a\neq 0.$$

Hence the nonzero constant polynomials are the eigenvectors of T corresponding to $\boldsymbol{\lambda_1}=1$.

因此 T 相對應 $\lambda_1=1$ 的 Eigenvector 是所有非零的常數多項式。

Next let $\boldsymbol{\lambda_2}=2$, and define $B_2=A-\lambda_2 I=\begin{pmatrix}-1&1&0\\0&0&2\\0&0&1\end{pmatrix}$

A 相對應 $\boldsymbol{\lambda_2}=2$ 的 Eigenvector 形式為

It is easily verified that $N(L_{B_2})=\left\{a\begin{pmatrix}1\\1\\0\end{pmatrix}; a\in R\right\}$, and hence the eigenvectors of T corresponding to $\lambda_2 = 2$ are of the form

T 相對應 $\lambda_2 = 2$ 的 Eigenvector 形式

$$\phi_\beta^{-1}\left(a\begin{pmatrix}1\\1\\0\end{pmatrix}\right) = a\phi_\beta^{-1}(e_1 + e_2) = a(1+x) \quad \text{for } a \neq 0.$$

Finally, consider $\lambda_3 = 3$ that $B_3 = A - \lambda_3 I = \begin{pmatrix}-2 & 1 & 0\\0 & -1 & 2\\0 & 0 & 0\end{pmatrix}$

A 相對應 $\lambda_3 = 3$ 的 Eigenvector 形式為

Since $N(L_{B_3}) = \left\{a\begin{pmatrix}1\\2\\1\end{pmatrix}; a \in R\right\}$,

the eigenvectors of T corresponding to $\lambda_3 = 3$ are of the form

T 相對應 $\lambda_3 = 3$ 的 Eigenvector 的形式

$$\phi_\beta^{-1}\left(a\begin{pmatrix}1\\2\\1\end{pmatrix}\right) = a\phi_\beta^{-1}(e_1 + 2e_2 + e_3) = a(1+2x+x^2) \quad \text{for } a \neq 0.$$

For each eigenvalue, select the corresponding eigenvector with **a = 1** in the preceding descriptions to obtain $\gamma = \{1,\ 1 + x,\ 1 + 2x + x^2\}$（$P_2(R)$的有序基底）, **which is an ordered basis for $P_2(R)$ consisting of eigenvectors of T. Thus T is diagonalizable, and**

$$[T]_\gamma = \begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 3\end{pmatrix}$$

Geometrical description of how a linear operator T acts on an eigenvector in the context of a vector space V over R.

在佈於 R 的向量空間 V 上，線性運算子 T 作用在 Eigenvector 的幾何意義？

Let v be an eigenvector of T and λ be the corresponding eigenvalue. We can think of W = span ({v}), the one-dimensional subspace of V spanned by v, as a line in V that passes through 0 and v. For any $w \in W$, w = cv for some scalar c, and hence

$T(w) = T(cv) = cT(v) = c\lambda v = \lambda w$;

So T acts on the vectors in W by multiplying each such vector by λ. There are several possible ways for T to act on the vectors in W, depending on the value of λ.

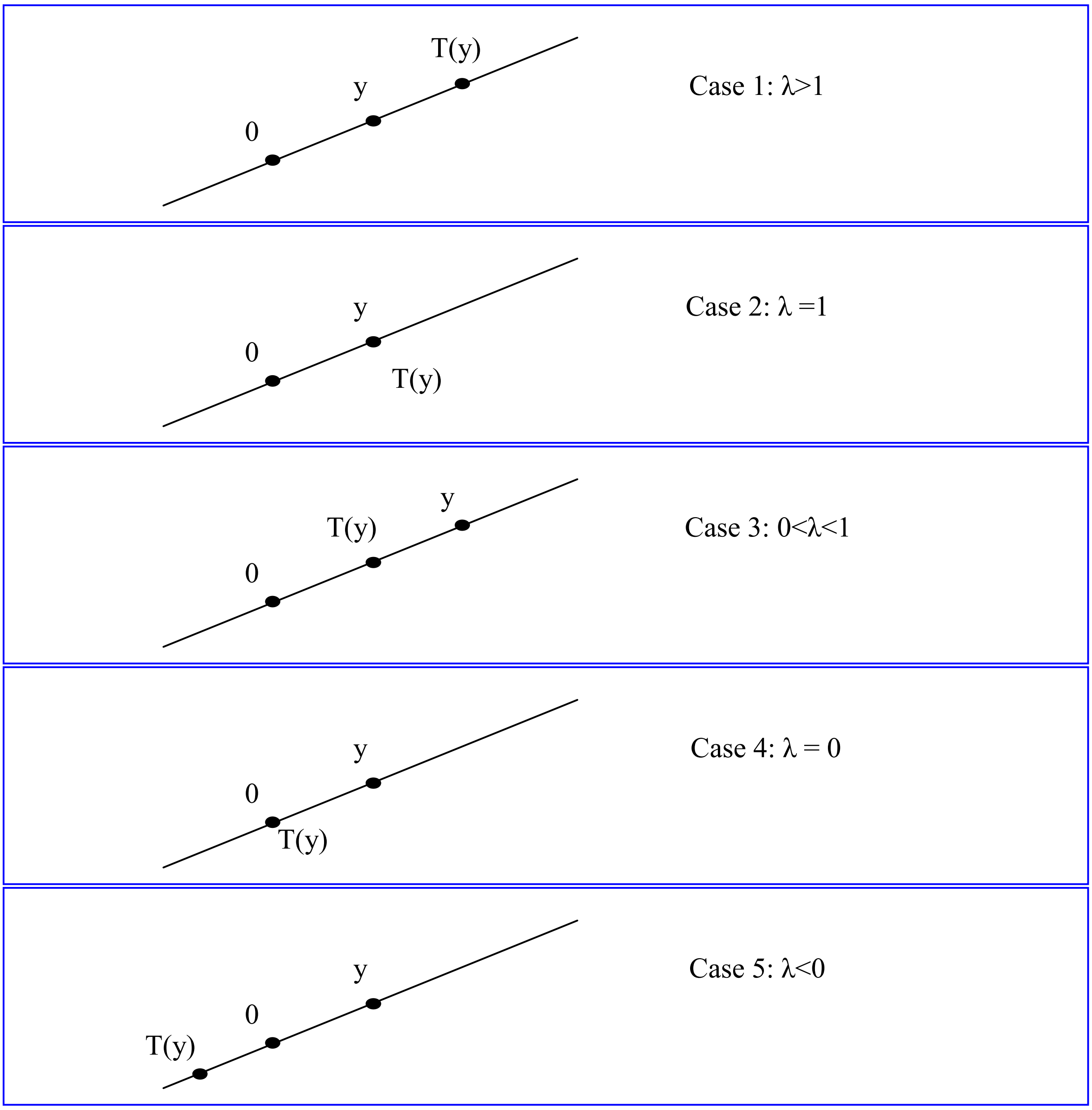
CASE 1. If $\lambda > 1$, then T moves vectors in W farther from 0 by a factor of λ.
CASE 2. If $\lambda = 1$, then T acts as the identity operator on W.
CASE 3. If $0 < \lambda < 1$, then T moves vectors in W closer to 0 by a factor of λ.
CASE 4. If $\lambda = 0$, then T acts as the zero transformation on W.
CASE 5. If $\lambda < 0$, then T reverses the orientation of W; that is, T moves vector in W from one side of 0 to the other.

令 v 是 T 的 Eigenvector，λ 是對應的 Eigenvalue，並令 W = span ({v})係由 v 生成的 V 內的一維子空間，W 可以想像成為 V 內通過 0 與 v 的直線：

$T(w) = T(cv) = cT(v) = c\lambda v = \lambda w$ （$w \in W$、$w = cv$ 、c 為純量。）

所以 T 作用在 W 上的向量係以純量 λ 乘上 W 的每一向量。T 的作用依純量 λ 值有下列數種可能：

CASE 1. 若 $\lambda > 1$, then T moves vectors in W farther from 0 by a factor of λ.
CASE 2. 若 $\lambda = 1$, then T acts as the identity operator on W.
CASE 3. 若 $0 < \lambda < 1$, then T moves vectors in W closer to 0 by a factor of λ.
CASE 4. 若 $\lambda = 0$, then T acts as the zero transformation on W.
CASE 5. 若 $\lambda < 0$, then T reverses the orientation of W; that is, T moves vector in W from one side of 0 to the other.



5-2 Diagonalizability

Not all linear operators or matrices are diagonalizable. Although we are able to diagonalize operators and matrices and even obtain a necessary and sufficient condition for diagnoalizability, we have not yet solved the diagonalization problem.

並非所有線性運算子或矩陣均可對角化。雖然已經可對運算子與矩陣對角化，甚至知道可對角化的充分與必要條件，但仍未能解決對角化問題。

What is still needed is a simple test to determine whether an operator or a matrix can be diagonalized, as well as a method for actually finding a basis of eigenvectors.

如何測試運算子或矩陣可被對角線化？找出 Eigenvectors 的方法？

→ Develop such a test and method.

Theorem 5.5

Let T be the linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. If $v_1, v_2, \ldots, v_k$ are eigenvectors of T such that λ_i corresponds to v_i ($1 \leqq i \leqq k$), then $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.

令 T 為向量空間 V 的線性運算子，且令 $\lambda_1, \lambda_2, \ldots, \lambda_k$ 為 T 的相異 Eigenvalues。若 $v_1, v_2, \ldots, v_k$ 為 T 的 Eigenvector，且 λ_i 與 v_i 相對應，則 $\{v_1, v_2, \ldots, v_k\}$ 為線性獨立。

Corollary

Let T be the linear operator on an n-dimensional vector space V. If T has n distinct eigenvalues, then T is diagonalizable.

令 T 為 n 維向量空間 V 的線性運算子，若 T 有 n 個相異 Eigenvalues，則 T 可對角化。

【Proof】

Suppose that T has n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For each i choose an eigenvector v_i corresponding to λ_i. By Theorem 5.5, $\{v_1, v_2, \ldots, v_k\}$ is linearly independent, and since dim(V) = n, this set $\{v_1, v_2, \ldots, v_k\}$ is a basis for V. This by Theorem 5.1, T is diagonalizable.

EXAMPLE 1

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in M_{2\times 2}(R)$.

The characteristic polynomial of A (and hence of L_A) is

$$\det(A - tI) = \begin{pmatrix} 1-t & 1 \\ 1 & 1-t \end{pmatrix} = t(t-2),$$

and thus the eigenvalues of L_A are 0 and 2. Since L_A is a linear operator on the two-dimensional vector space R^2, we conclude from the preceding corollary that L_A (and hence of A) is diagonalizable.

L_A（或 A）可對角化。

The converse of Theorem 5.5 is false. That is, it is not true that if T is diagonalizable, then it has n distincet eigenvalues. For example, the identity operator is diagonalizable even though it has only one eigenvalue, namely, $\lambda = 1$.

Theorem 5.5 的逆定理不成立。意即：「若 T 可對角化，則 T 有 n 個相異 Eigenvalues。」不成立。例如，單位運算子為可對角化且只有一個 eigenvalue，即 $\lambda = 1$。

DEFINTION 5.5

A polynomial f(t) in P(F) splits over F if there are scalars $c, a_1, a_2, \ldots, a_n$ (not necessarily distinct) in F such that

$$f(t) = c(t-a_1)(t-a_2)\ldots(t-a_n).$$

P(F)中的多項式 f(t)可以分解成 F 中的一次因式（split over F），意指存在 F 中有純量 $c, a_1, a_2, \ldots, a_n$（不見得相異）使得 $f(t) = c(t-a_1)(t-a_2)\ldots(t-a_n)$。

For example, $t^2-1 = (t-1)(t+1)$ splits over R, but $(t^2+1)(t-2)$ does not split over R because t^2+1 cannot be factored into a product of linear factors. However, $(t^2+1)(t-2)$ does split over C because it factors into the product $(t+i)(t-i)$.

可不可分解要看所屬範圍。$(t^2+1)(t-2)$在 R 內不可分解，但在 F 內可分解。

Theorem 5.6

The characteristic polynomial of any diagonalizable linear operator splits.

任一可對角化線性運算子 T 的特徵多項式皆可分解一次因式。

From Theorem 5.6, it is clear that if T is a diagonalizable linear operator on an n-dimensional vector space that fails to have distinct eigenvalues, then the characteristic polynomial of T must have repeated zeros.

由 Theorem 5.6 可知，若 T 為 n 維向量空間可對角化的線性運算子，卻不具有 n 個相異 Eigenvalues，則 T 的特徵多項式必然具有重根（Repeated zeros）。

The converse of Theorem 5.6 is false; that is, the characteristic polynomial of T may split, but T need not be diagonalizable.

Theorem 5.6 的逆定理不成立：T 的特徵多項式可以分解成一次因式，未必意謂 T 可對角化。

DEFINTION 5.6

Let λ be an eigenvalue of a linear operator or matrix with characteristic polynomial f(t). The (algebraic) multiplicity of λ is the largest positive integer k for which $(t-\lambda)^k$ is a factor of f(t).

若 λ 為某一線性運算子或矩陣的特徵多項式 f(t) 的 Eigenvalue，λ 的相重數 k，為讓 $(t-\lambda)^k$ 成為 f(t) 的 factor 的最大正整數。

EXAMPLE 2

Let $A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$.

Which has characteristic polynomial $f(t) = -(t-3)^2(t-4)$. Hence λ = 3 is an eigenvalue of A with multiplicity 2, and λ = 4 is an eigenvalue of A with multiplicity 1.

λ = 3 的相重數為 2。

If T is a diagonalizable linear operator on a finite-dimensional vector space V, then there is an ordered basis β for V consisting of eigenvectors of T. We know from Theorem 5.1 that $[T]_\beta$ is a dignoal matrix in which the diagonal entries are the eigenvalues of T. Since the characteristic polynomial of T is $\det([T]_\beta - tI)$, it is easily seen that each eigenvalue of T must occur as a diagonal entry of $[T]_\beta$ exactly as many times as its multiplicity.

若 T 為有限維度向量空間 C 內一個可對角化的線性運算子，則存在一有序基底 β，且該有序基底係由 T 的 Eigenvectors 所組成。由 Theorem 5.1 得知，$[T]_\beta$ 為對角矩陣，且對角線元素為 T 的 Eigenvalues。由於 T 的特徵多項式為 $\det([T]_\beta - tI)$，故 T 的每一個 Eigenvalues 必定出現在$[T]_\beta$ 的對角線上，且出現的次數等於該 Eigenvalue 的相重數（as many times as its multiplicity）。

DEFINTION 5.7 Eigenspace（固有空間） E_λ

Let T be a linear operator on a vector space V, and λ be an eigenvalue of T. Define $E_\lambda = \{x \in V;\ T(x) = \lambda x\} = N(T - \lambda I_V)$. The set E_λ is called the eigenspace of T corresponding to the eigenvalue λ. Analogously, we define the eigenspace of a square matrix A to be the eigenspace of L_A.

若 T 為向量空間 V 的線性運算子，且 λ 為 T 的 Eigenvalue。定義 $E_\lambda = \{x \in V;\ T(x) = \lambda x\} = N(T - \lambda I_V)$，並稱 E_λ 為 T 對應於 Eigenvalue λ 的 Eigenspace（固有空間）。同理，A 的 Eigenspace 即為 L_A 的 Eigenspace。

If A is an n×n matrix with an eigenvalue λ, then the set of all eigenvectors of λ together with the zero vector is a subspace of V. This subspace is called the eigenspace of λ .

v_1 and v_2 are eigenvectors corresponding to λ

$Av_1=\lambda v_1$, $Av_2=\lambda v_2$

$A(v_1 + v_2) = Av_1 + Av_2 = \lambda v_1 + \lambda v_2 = \lambda(v_1 + v_2)$

i.e. $x_1 + x_2$ is an eigenvector corresponding to λ

$A(cv_1) = c(Av_1) = c(\lambda v_1) = \lambda(cv_1)$

i.e. cx_1 is an eigenvector corresponding to λ

Dimension of E_λ

Clearly, E_λ is a subspace of V consisting of the zero vector and the eigenvectors of T corresponding to the eigenvalues λ. The maximum number of linearly independent eigenvectors of T corresponding to the eigenvalues λ is therefore the **dimension of E_λ**.

顯然，E_λ 是 V 的子空間，係由零向量與 T 內相對應 Eigenvalue λ 的 Eigenvectors 所組成。因此 T 中對應 Eigenvalues λ 的線性獨立 Eigenvectors 最多個數即為 E_λ 的維度。

Theorem 5.7 Dimension of E_λ vs. Multiplicity of λ

Let T be a linear operator on a finite-dimensional vector space V, and let λ be an eigenvalue of T having multiplicity m. Then $1 \leqq \dim(E_\lambda) \leqq m$.

令 T 是有限維度向量空間 V 的線性運算子，λ 為 T 的 Eigenvalue，相重數為 m，則 $1 \leqq \dim(E_\lambda) \leqq m$。

EXAMPLE 3

Let T be the linear operator on $P_2(R)$ defined by $T(f(x)) = f'(x)$. The matrix representation of T with respect to the standard ordered basis $\beta = \{1, x, x^2\}$ for $P_2(R)$ is

$$[T]_\beta = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}.\ (=A)$$

Consequently, the characteristic polynomial of T is

$$\det([T]_\beta - tI) = \det\begin{pmatrix} -t & 1 & 0 \\ 0 & -t & 2 \\ 0 & 0 & -t \end{pmatrix} = -t^3$$

Thus T has only one eigenvalue with multiplicity 3.

Solving $T(f(x)) = f'(x) = 0$ shows that $E_\lambda = N(T- \lambda I) = N(T)$ is the subspace of $P_2(R)$ consisting of the constant polynomials.

$$(A - \lambda I)x = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\rightarrow x_1 = a,\ x_2 = x_3 = 0$

So {1} is a basis for E_λ, and therefore $\dim(E_\lambda) = 1$.

Consequently, there is no basis for $P_2(R)$ consisting of eigenvectors of T, and therefore T is not diagonalizable.

T 只有一個 Eigenvalue（$\lambda = 0$），其相重數 3，解 $T(f(x)) = f'(x) = 0$ 證明 $E_\lambda = N(T- \lambda I) = N(T)$是 $P_2(R)$的子空間，是常數多項式所組成。{1}是 E_λ 的基底，$\dim(E_\lambda) = 1$。

EXAMPLE 4

Let T be the linear operator on R^3 defined by

$$T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 4a_1 + a_3 \\ 2a_1 + 3a_2 + 2a_3 \\ a_1 + 4a_3 \end{pmatrix}.$$

We determine the eigenspace of T corresponding to each eigenvalue. Let β be the standard ordered basis for R^3. Then

$$[T]_\beta = \begin{pmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{pmatrix} = A$$

and hence the characteristic polynomial of T is

$$\det([T]_\beta - tI) = \det\begin{pmatrix} 4-t & 0 & 1 \\ 2 & 3-t & 2 \\ 1 & 0 & 4-t \end{pmatrix} = -(t-5)(t-3)^2$$

So the eigenvalues of T are $\lambda_1 = 5$ and $\lambda_2 = 3$ with multiplicities 1 and 2 respectively.

Since

$$E_{\lambda_1} = N(T - \lambda_1 I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 : \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 2 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

E_λ is the solution space of the system of linear equations

$$-x_1 + x_3 = 0$$
$$2x_1 - 2x_2 + 2x_3 = 0$$
$$x_1 - x_3 = 0$$

It is easily seen that

$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right\}$ is a basis for $E_{\lambda 1}$. Hence $\dim(E_{\lambda 1}) = 1$

Similaryly, $E_{\lambda_2} = N(T - \lambda_2 I)$ is the solution space of the system of linear equations

$$x_1 + x_3 = 0$$
$$2x_1 + 2x_3 = 0$$
$$x_1 + x_3 = 0$$

The general solution to the system

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \text{for } s, t \in R.$$

It follows that

$\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for $E_{\lambda 2}$. Hence $\dim(E_{\lambda 2}) = 2$

EXAMPLE

Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

If $v = (x, y)$ then $Av = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$

For a vector on the *x*-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} -x \\ 0 \end{bmatrix} = -1\begin{bmatrix} x \\ 0 \end{bmatrix}$$ Eigenvalue $\lambda_1 = -1$

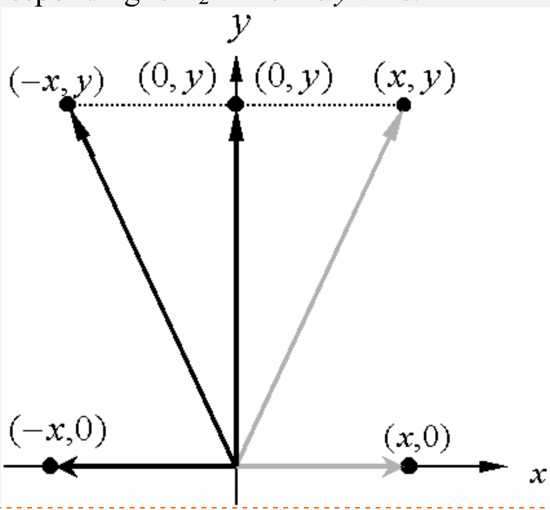
For a vector on the *x*-axis

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix} = 1\begin{bmatrix} 0 \\ y \end{bmatrix}$$ Eigenvalue $\lambda_2 = +1$

Geometrically, multiplying a vector (x, y) in R^2 by the matrix *A* corresponds to a reflection in the *y*-axis.

The eigenspace corresponding to $\lambda_1 = -1$ is the *x*-axis.

The eigenspace corresponding to $\lambda_2 = 1$ is the *y*-axis.



EXAMPLE

Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$$|A-\lambda I| = \begin{bmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (\lambda+2)^2(\lambda-4)$$

Eigenvalues $\lambda_1 = 4$, $\lambda_2 = -2$

The eigenspaces for these two eigenvalues are as follows.

$E_{\lambda 1} = \{ (1, 1, 0) \}$ corresponding to $\lambda_1 = 4$

$E_{\lambda 2} = \{ (1, -1, 0), (0, 0, 1) \}$ corresponding to $\lambda_2 = -2$

EXAMPLE

Find the eigenvalues and corresponding eigenspaces of $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 5 & -10 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 0 & 3 \end{bmatrix}$

Characteristic equation

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 5 & -10 \\ 1 & 0 & 2-\lambda & 0 \\ 1 & 0 & 0 & 3-\lambda \end{vmatrix} = (\lambda-1)^2(\lambda-2)(\lambda-3) = 0$$

Eigenvalues $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 3$

For $\lambda_1 = 1$

$$(A-\lambda_1 I)x = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -10 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ s \\ 2t \\ t \end{bmatrix} = s\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad s, t \neq 0$$

$\rightarrow \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda_1 = 1$

For $\lambda_2 = 2$

$\rightarrow \left\{ \begin{bmatrix} 0 \\ 5 \\ 1 \\ 0 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda_2 = 2$

For $\lambda_3 = 3$

$\rightarrow \left\{ \begin{bmatrix} 0 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for the eigenspace of A corresponding to $\lambda_3 = 3$

Lemma

Let T be a linear operator on a finite-dimensional vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each i = 1, 2, …,k, let $v_i \in E_\lambda$, the eigenspace corresponding to λ. If $v_1 + v_2 + \ldots + v_k = 0$, then $v_i = 0$ for all i.

令 T 是有限為度向量空間 V 的線性運算子，且 $\lambda_1, \lambda_2, \ldots, \lambda_k$ 為 k 個相異 Eigenvalues。$v_1, v_2, \ldots, v_k$ 為相對應 λ 的 eigenvectors。若 $v_1 + v_2 + \ldots + v_k = 0$，則 $v_i = 0$。

【Proof】

Suppose otherwise. By renumbering if necessary, suppose that, for 1≦m≦k, we have $v_i \neq 0$ for 1≦i≦m, and $v_i = 0$ for i > m. Then, for each i≦m, v_i is an eigenvector of T corresponding to λ_i and $v_1 + v_2 + \ldots + v_m = 0$. According to Theorem 5.5, these v_i's are linearly independent. We conclude that, therefore, that $v_i = 0$ for all i.

Theorem 5.8

Let T be a linear operator on a vector space V, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. For each i = 1, 2, …, k, let S_i be a finite linearly independent subset of eigenspace E_λ. Then $S = S_1 \cup S_2 \cup \ldots \cup S_k$ is a linear independent subset of V.

令 T 是有限為度向量空間 V 的線性運算子，且 $\lambda_1, \lambda_2, \ldots, \lambda_k$ 為 k 個相異 Eigenvalues。S_i 是 Eigensapce E_λ 的線性獨立子集合，則 $S = S_1 \cup S_2 \cup \ldots \cup S_k$ 是 V 的線性獨立子集合。

【Proof】

Suppose that for each i

$S_i = \{v_{i1}, v_{i2}, ..., v_{in}\}$.

Then $S = \{v_{ij} : 1 \le j \le n_i, \text{and } 1 \le i \le k\}$.

Consider any $\{a_{ij}\}$ such that

$$\sum_{i=1}^{k} \sum_{j=1}^{n_i} a_{ij} v_{ij} = 0.$$

For each i, let $w_i = \sum_{j=1}^{n_i} a_{ij} v_{ij}$.

Then $w_i \in E_{\lambda i}$ for each i, and $w_1 + \ldots + w_k = 0$. Therefore, by the lemma, $w_i = 0$ for all i. But each S_i is linearly independent, and hence $a_{ij} = 0$ for all j.

We conclude that S is linear independent.

Theorem 5.8 tells us how to construct a linearly independent subset of eigenvectors, namely, by collecting bases for the individual eigenspaces. The next theorem tells us when the resulting set is a basis for the entire space.

Theorem 5.8 告訴我們透過個別 eigenspace 的收集找出 eigenvectors 的線性獨立子集合。

Theorem 5.9

Let T be a linear operator on a finite-dimensional vector space V such that the characteristic polynomial of T splits. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be distinct eigenvalues of T. Then

(a) T is diagonalizable if and only if the multiplicity of λ_i is equal to $\dim(E_{\lambda i})$ for all i.

(b) If T is diagonalizable and β_i is an ordered basis for $E_{\lambda i}$, for each i, then $\beta = \beta_1 \cup \beta_2 \cup ... \cup \beta_k$ is an ordered basis for V consisting of eigenvectors of T.

令 T 是有限為度向量空間 V 的線性運算子，且 $\lambda_1, \lambda_2, \ldots, \lambda_k$ 為 k 個相異 Eigenvalues。

(a) T 可對角化的「若且唯若」條件為 λ_i 的**相重數等於** $E_{\lambda i}$ 的維度。

(b)若 T 可對角化且 β_i 是 $E_{\lambda i}$ 的有序基底，則 $\beta = \beta_1 \cup \beta_2 \cup ... \cup \beta_k$ 為 V 的有序基底，並由 T 的 eigenvectors 所組成。

TEST for Diagonalization

Let T be a linear operator on an n-dimensional vector space V. Then T is diagonalizable if and only if both of the following conditions hold.

令 T 是 n 維向量空間 V 的線性運算子，則 T 為可對角化的「若且唯若」條件為：

1. The characteristic polynomial of T splits. T 的特徵多項式可分解成一次因式。
2. For each eigenvalue λ of T, the multiplicity of λ equals n – rank(T-λI).對 T 的每一個 Eigenvalue λ，λ 的相重數等於 n – rank(T-λI)。

EXAMPLE 5

We test the matrix

$$A = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix} \in M_{3\times 3}(R)$$ for diagonalizability.

The characteristic polynomial of A is det(A-tI) = $-(t-4)(t-3)^2$, which splits, and so condition 1 of the test for diagnoalization is satisified. Also A has eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 3$ with multiplicity 1 and 2, respectively. Since λ_1 has multiplicity 1, condition 2 is satisified for λ_1. Thus we need only test condition 2 for λ_2.

Because $A - \lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ has rank 2, we see that 3 – rank(T-λI) = 1, which is not the multiplicity of λ_2. Thus condition 2 fails for λ_2, and A is therefore not diagonalizable.

EXAMPLE 6

Let T be the linear operator on $P_2(R)$ defined by

$T(f(x)) = f(1) + f'(0)x + (f'(0)+f''(0))x^2$.

We first test T for diagonalizability. Let α denote the standard ordered basis for $P_2(R)$ and $B = [T]_\alpha$. Then

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix}.$$

The characteristic polynomial of B, and hence of T, is $-(t-1)^2(t-2)$, which splits.

Hence condition 1 of the test for diagnoalization is satisified. Also B has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 2$ with multiplicity 2 and 1, respectively. Condition 2 is satisfied for λ_2 because it has multiplicity 1. So we need only verify condtion 2 for λ_1. For this case,

$3 - \text{rank}(B - \lambda_1 I) = 3 - \text{rank}\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = 3 - 1 = 2$, which is equal to the multiplicity of λ_1. Therefore T is diagonalizable.

→ Find an ordered basis γ for R^3 of eigenvectors of B. We consider each eigenvalue separately.

The eigenspace corresponding to $\lambda_1 = 1$ is

$$E_{\lambda_1} = N(T - \lambda_1 I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 : \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

Which is the solution space for the system

$x_2 + x_3 = 0$,

and thus $\gamma_1 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ as a basis.

The eigenspace corresponding to $\lambda_2 = 2$ is

$$E_{\lambda_2} = N(T - \lambda_2 I) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in R^3 : \begin{pmatrix} -1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

Which is the solution space for the system

$$\begin{aligned} -x_1 + x_2 + x_3 &= 0 \\ x_2 &= 0 \end{aligned},$$

and thus $\gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ as a basis.

Let $\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Then γ is an ordered basis for R^3 consisting of eigenvectors of B.

Finally, observe that the vectors in γ are the coordinate vectors relative to α of the vector in the set

$\beta = \{1, -x + x^2, 1 + x^2\}$, which is an ordered basis for $P_2(R)$ consisting of eigenvectors of T. Thus

$$[T]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

EXAMPLE 7

Let $A = \begin{pmatrix} 0 & -2 \\ 1 & 3 \end{pmatrix}$.

We show that A is diagonalizable and find a 2×2 matrix Q such that $Q^{-1}AQ$ is a diagonal matrix. We then show how to use this result to compute An for any positive integer n.

First observe that the characteristic polynomial of A is (t - 1)(t - 2), and hence A has two distinct eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 2$. By applying the corollary to Theorem 5.5 to the operator L_A, we see that A is diagonalizable. Moreover

$$\gamma_1 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\} \text{ and } \gamma_2 = \left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

are bases for the eigenspaces $E_{\lambda 1}$ and $E_{\lambda 2}$, respectively. Therefore

$\gamma = \gamma_1 \cup \gamma_2 = \left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is an ordered basis for R^2 consisting of eigenvectors of A.

Let $D = Q^{-1}AQ = [L_A]_\beta = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

To find A^n for any positive integer n, observe that $A = QDQ^{-1}$. Therefore

$$A^n = (QDQ^{-1})^n = \ldots = QD^nQ^{-1} = Q\begin{pmatrix} 1^n & 0 \\ 0 & 2^n \end{pmatrix}Q^{-1} = \ldots = \begin{pmatrix} 2 - 2^n & 2 - 2^{n+1} \\ -1 + 2^n & -1 + 2^{n+1} \end{pmatrix}$$