Chapter 4 Determinants

The determinant, which has played a prominent role in the theory of linear algebra, is a special scalar-valued function defined on the set of square matrices. Although it still has a place in the study of linear algebra and its applications, its role is less central than in former times. The main use of determinants is to compute and establish the properties of eigenvalues.

行列式是定義在方矩陣集合上的特殊純量值函數。行列式在線性代數的應用與研究雖仍佔有一席之地，但重要性已經不如從前。

The determinant is not a linear transformation on $M_{n\times n}(F)$ for $n > 1$.

行列式不是一個定義在 $M_{n\times n}(F)$的線性轉換。

4-1 Determinants of Order 2

In this section, we define the determinant of a 2×2 matrix and investigate its geometric significance in terms of area and orientation.

定義 2×2 矩陣的行列式，並探討它在面積與定向的幾何意義。

DEFINTION 4.1 Determinant

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix with entries form a field F, then we define the determinant of A, denoted det(A) or $|A|$, to be the scalar ad – bc.

若 A 為 2×2 的矩陣，則 A 的行列式可註記為 det(A) 或$|A|$，其值為純量 ad – bc。

EXAMPLE 1

For the matrices $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & 2 \\ 6 & 4 \end{pmatrix}$ in $M_{2\times 2}(R)$, we have

$\det(A) = 1\times 4 - 3\times 2 = -2$ and $\det(B) = 3\times 4 - 2\times 6 = 0$.

Determinants

- The **determinant** of a 2 × 2 matrix A is denoted |A| and is given by

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

- Observe that the determinant of a 2 × 2 matrix is given by the different of the products of the two diagonals of the matrix.
- The notation **det(A)** is also used for the determinant of A.

Example

- Find the determinant of the matrix A

$$A = \begin{bmatrix} 2 & 4 \\ -3 & 1 \end{bmatrix}$$

$$\Rightarrow \det(A) = \begin{vmatrix} 2 & 4 \\ -3 & 1 \end{vmatrix} = (2\times 1) - (4\times(-3)) = 2 + 12 = 14$$

For the matrices A and B in Example 1, we have

$$A + B = \begin{pmatrix} 4 & 4 \\ 9 & 8 \end{pmatrix}$$

and so det(A + B) = 4x8-4x9 = -4.

Since det(A+B) ≠ det(A)+det(B), the function det: $M_{2\times 2}(R)$→R is not a linear transformation.

注意：行列式不是線性轉換函數。

Theorem 4.1

The function det: $M_{2\times 2}(F)$→F is a linear function of each row of a 2x2 matrix when the other row is held fixed. That is, if u, v, and w are in F^2 and k is a scalar, then

$$\det\begin{pmatrix} u + kv \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix} \quad \text{and} \quad \det\begin{pmatrix} w \\ u + kv \end{pmatrix} = \det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix}$$

當某一列（Row）被固定時，det: $M_{2\times 2}(F)$→F 為另一列的線性函數。

若 u、v 與 w 為 F^2 的元素，k 為純量，則

$$\det\begin{pmatrix} u + kv \\ w \end{pmatrix} = \det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix} \quad \text{且} \quad \det\begin{pmatrix} w \\ u + kv \end{pmatrix} = \det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix}$$

【Proof】

Let u = (a_1, a_2), v = (b_1, b_2), and w = (c_1, c_2) be in F^2 and k be a scalar. Then

$$\det\begin{pmatrix} u \\ w \end{pmatrix} + k\det\begin{pmatrix} v \\ w \end{pmatrix} = \det\begin{pmatrix} a_1 & a_2 \\ c_1 & c_2 \end{pmatrix} + k\det\begin{pmatrix} b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} = \ldots = \det\begin{pmatrix} a_1 + kb_1 & a_2 + kb_1 \\ c_1 & c_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} u + kv \\ w \end{pmatrix}$$

A similar calculation shows that

$$\det\begin{pmatrix} w \\ u \end{pmatrix} + k\det\begin{pmatrix} w \\ v \end{pmatrix} = \det\begin{pmatrix} w \\ u + kv \end{pmatrix}$$

Theorem 4.2

Let $A \in M_{2\times2}(F)$. Then the determinant of $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ is nonzero **if and only if** A is invertible. Moreover, if A is invertible, then $A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$

令 $A \in M_{2\times2}(F)$，則 A 的行列式不為零的『若且唯若』條件為 A 可逆。

再者，若 A 可逆，則 A 的反矩陣為

$$A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

【Proof】

If $\det(A) \neq 0$, then we can define a matrix $M = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$.

A straightforward calculation shows that AM = MA = I, and so A is invertible and M = A^{-1}.

Conversely, suppose that A is invertible, the rank of $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ must be 2. Hence $A_{11} \neq 0$ or $A_{21} \neq 0$.

If $\mathbf{A_{11} \neq 0}$, and add $-A_{21}/A_{11}$ times row 1 of A to row 2 to obtain the matrix $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \frac{A_{12}A_{21}}{A_{11}} \end{pmatrix}$.

Because elementary row operations are rank-preserving by the corollary to Theorem 3.4, it follows that $A_{22} - \frac{A_{12}A_{21}}{A_{11}} \neq 0$.

Therefore $\det(A) = A_{11}A_{22} - A_{21}A_{12} \neq 0$.

On the otherhand, if $A_{21} \neq 0$, we see that $\det(A) \neq 0$ by adding $-A_{11}/A_{21}$ times row 2 of A to row 1 and applying a similar argument. Thus, in either case, $\det(A) \neq 0$.

若 $\det(A) \neq 0$，定義矩陣 $M = \frac{1}{\det(A)}\begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$。

因 $AM = MA = I$ ，故 A 為可逆且 M 為 A 的反矩陣 $M = A^{-1}$。

反之，假設 A 為可逆，則 $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ 的 Rank 為 2。

因此 $A_{11} \neq 0$ 或 $A_{21} \neq 0$。

若 $A_{11} \neq 0$，則將 $-A_{21}/A_{11}$ 乘上第一列加上第二列，將 A 化成

$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} - \frac{A_{12}A_{21}}{A_{11}} \end{pmatrix}。$$

依據 **Corollary to Theorem 3.4**

$A_{22} - \frac{A_{12}A_{21}}{A_{11}} \neq 0$ 。(否則就違反 rank-preserving。)

因此，$\det(A) = A_{11} A_{22} - A_{21} A_{12} \neq 0$ 。

若 $A_{21} \neq 0$，則將 $-A_{11}/A_{21}$ 乘上第一列加上第二列，得知 $\det(A) \neq 0$ 。

不管哪一種情形，$\det(A) \neq 0$。

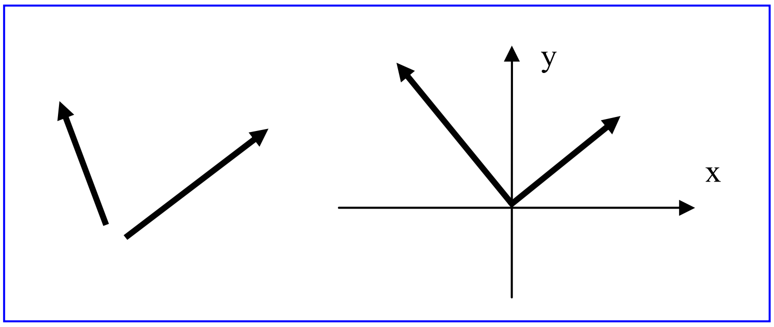
Corollary to Theorem 3.4 Elementary row and column operations on a matrix are rank-preserving. 矩陣的基本列與行運算不會改變矩陣的 Rank。

註：An $n \times n$ matrix is invertible if and only if its rank is n. 一個 $n \times n$ 的矩陣為可逆的『若且唯若』條件為該矩陣的 Rank 為 n。

The Aea of a Parallelogram

By the angle between two vectors in R^2, we mean the angle with measure $\theta (0 \leqq \theta < \pi)$ that is formed by the vectors having the same magnitude and direction as the given vectors but emanating from the origin.

R^2 上兩個向量的夾角為一測量角度，該角度是由兩個由原點引出，且與二已知向量大小相同、方向相同的向量所形成，$0 \leqq \theta < \pi$。



If $\beta = \{u, v\}$ is an ordered basis for R^2, we define the orientation of β to be the real number.

若 $\beta = \{u, v\}$ 是 R^2 的有序基底，則 β 的定向（Orientation）可被定義為：

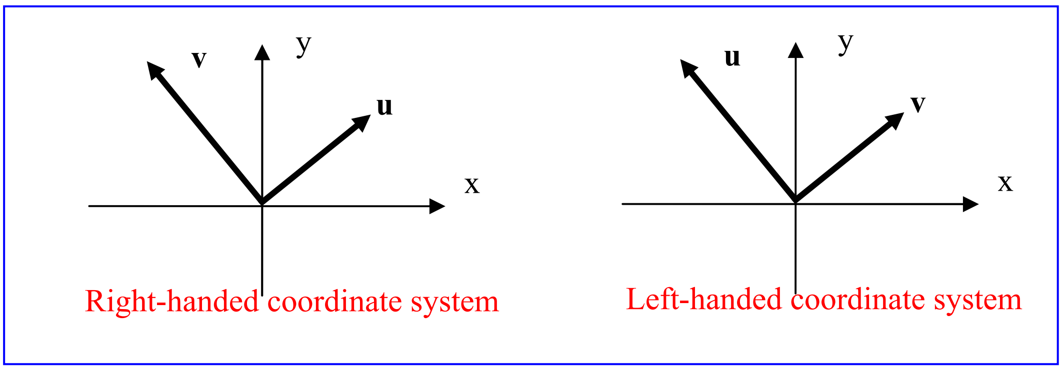
$$O\begin{pmatrix} u \\ v \end{pmatrix} = \frac{\det\begin{pmatrix} u \\ v \end{pmatrix}}{\left|\det\begin{pmatrix} u \\ v \end{pmatrix}\right|}$$ （β 的定向）（由 Theorem 4.2 得知分母不等於零。）

Clearly $O\begin{pmatrix} u \\ v \end{pmatrix} = \pm 1$.

Notice that $O\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = +1$ and $O\begin{pmatrix} e_1 \\ -e_2 \end{pmatrix} = -1$.

Recall that a coordinate system {u, v} is called **right-hand** if u can be rotated into a counterclockwise direction (CCW) through an angle θ $(0<\theta<\pi)$ to coincide with v。Otherwise, {u, v} is called a **left-hand** system.

一個座標系{u, v}被稱為右手定則，表示 u 可依逆時針方向經 θ（$0 < \theta < \pi$）角旋轉後與 v 方向一致；反之，座標系{u, v}則被稱為左手定則。



In general, $O\begin{pmatrix} u \\ v \end{pmatrix} = 1$ **if and only if** the ordered basis {u, v} forms a right-hand coordinate system.

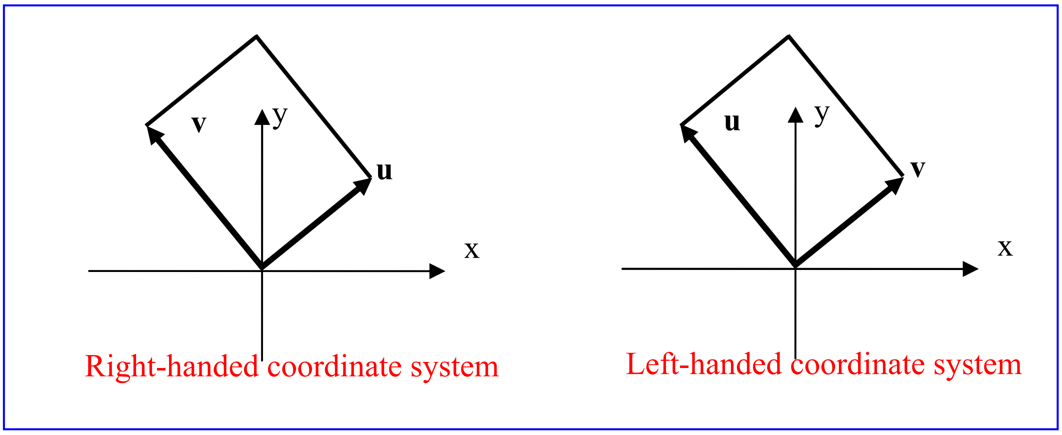
一般而言，$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$『若且唯若』有序基底{u, v}形成**右手座標系統**。

For convenience, we also define $O\begin{pmatrix} u \\ v \end{pmatrix} = 1$ if {u, v} is linearly dependent.

方便起見，若{u, v}為線性相依，也可定義為$O\begin{pmatrix} u \\ v \end{pmatrix} = 1$。

Any ordered set {u, v} in R^2. Regarding u and v as arrows emanating from the origin of R^2, we call the **parallelogram determined by u and v.**

{u, v}是 R^2 中任何一組有序集合。若將 u 與 v 看成是 R^2 中自原點引出的兩個箭號，並以 u 及 v 作為平行四邊形的鄰邊，則該平形四邊形可稱為「**由 u 及 v 所決定的平行四邊形**」。



當{u, v}為線性相依時，則以 u 及 v 作為鄰邊的平行四邊形，就變成了一線段，其面積也退化成 0。

以 u 及 v 作為鄰邊的平行四邊形的面積$A\begin{pmatrix} u \\ v \end{pmatrix}$與$\det\begin{pmatrix} u \\ v \end{pmatrix}$間，存在什麼關係？

由於$\det\begin{pmatrix} u \\ v \end{pmatrix}$可能為負數，因此平行四邊形的面積應該為

$$A\begin{pmatrix}u\\v\end{pmatrix}=O\begin{pmatrix}u\\v\end{pmatrix}\bullet\det\begin{pmatrix}u\\v\end{pmatrix}$$ 或 $$A\begin{pmatrix}u\\v\end{pmatrix}=\left|\det\begin{pmatrix}u\\v\end{pmatrix}\right|$$ 或 $$O\begin{pmatrix}u\\v\end{pmatrix}\bullet A\begin{pmatrix}u\\v\end{pmatrix}=\det\begin{pmatrix}u\\v\end{pmatrix}$$，

而非 $A\begin{pmatrix}u\\v\end{pmatrix}=\det\begin{pmatrix}u\\v\end{pmatrix}$

其中，$O\begin{pmatrix}u\\v\end{pmatrix}\bullet A\begin{pmatrix}u\\v\end{pmatrix}$ 可定義為 $\delta\begin{pmatrix}u\\v\end{pmatrix}$。

即 $\delta\begin{pmatrix}u\\v\end{pmatrix}=O\begin{pmatrix}u\\v\end{pmatrix}\bullet A\begin{pmatrix}u\\v\end{pmatrix}$

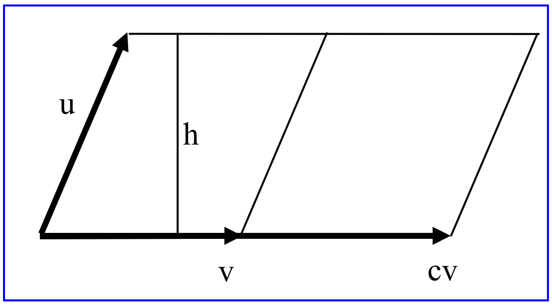
例如：由 u = (-1, 5) 及 v = (4, -2) 所形成的平行四邊形面積為

$$A\begin{pmatrix}u\\v\end{pmatrix}=\left|\det\begin{pmatrix}u\\v\end{pmatrix}\right|=\left|\det\begin{pmatrix}-1&5\\4&-2\end{pmatrix}\right|=18$$ 。

例如：以 u 及 v 作為鄰邊的平行四邊形，並將 v 放大 c 倍。

若 $c=0$，則 $\delta\begin{pmatrix}u\\cv\end{pmatrix}=O\begin{pmatrix}u\\0\end{pmatrix}\bullet A\begin{pmatrix}u\\0\end{pmatrix}=1\bullet 0=0$ 。

若 $c\neq 0$，則 cv 可視為將 v 放大 c 倍，新的平形四邊形的兩個鄰邊為 u 及 cv。



$$A\begin{pmatrix}u\\cv\end{pmatrix}=\text{base}\times\text{altitute}=|c|(\text{length of } v)(\text{altitude})=|c|\times A\begin{pmatrix}u\\v\end{pmatrix}$$ 。

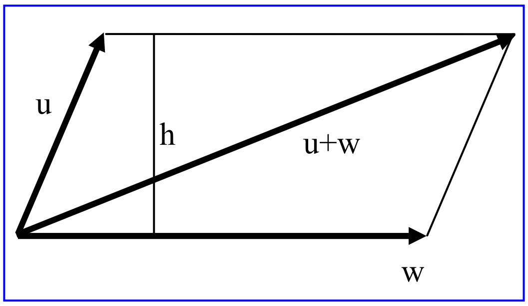
因為由 u 及 cv 所決定的平形四邊形的高 h 等於由 u 及 v 所決定的平形四邊形的高。

因此，$\delta\begin{pmatrix} u \\ cv \end{pmatrix} = O\begin{pmatrix} u \\ cv \end{pmatrix} \cdot A\begin{pmatrix} u \\ v \end{pmatrix} = \left[\frac{c}{|c|} \cdot O\begin{pmatrix} u \\ v \end{pmatrix}\right]\left[\frac{c}{|c|} \cdot A\begin{pmatrix} u \\ v \end{pmatrix}\right] = c \cdot O\begin{pmatrix} u \\ v \end{pmatrix} \cdot A\begin{pmatrix} u \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}$ 。

同理，$\delta\begin{pmatrix} cu \\ v \end{pmatrix} = c \cdot O\begin{pmatrix} u \\ v \end{pmatrix} \cdot A\begin{pmatrix} u \\ v \end{pmatrix} = c \cdot \delta\begin{pmatrix} u \\ v \end{pmatrix}$

例如：以 u 及 v 作為鄰邊的平行四邊形，並將 v 變成 au+bw。

$\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = b \cdot \delta\begin{pmatrix} u \\ w \end{pmatrix}$? For any u, $w \in R^2$ and any real numbers a and b.



由 u 及 w 所決定的平形四邊形，與由 u 及 u+w 所決定的平行四邊形具有共同的底及相同高，即 $A\begin{pmatrix} u \\ w \end{pmatrix} = A\begin{pmatrix} u \\ u + w \end{pmatrix}$ 。

若 $a = 0$，則 $\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = \delta\begin{pmatrix} u \\ bw \end{pmatrix} = b \cdot \delta\begin{pmatrix} u \\ w \end{pmatrix}$

若 $a \neq 0$，則 $\delta\begin{pmatrix} u \\ au + bw \end{pmatrix} = a \cdot \delta\begin{pmatrix} u \\ u + \frac{b}{a}w \end{pmatrix} = a \cdot \delta\begin{pmatrix} u \\ \frac{b}{a}w \end{pmatrix} = b \cdot \delta\begin{pmatrix} u \\ w \end{pmatrix}$

同理 $\delta\begin{pmatrix} u \\ v_1 + v_2 \end{pmatrix} = \delta\begin{pmatrix} u \\ v_1 \end{pmatrix} + \delta\begin{pmatrix} u \\ v_2 \end{pmatrix}$ 、$\delta\begin{pmatrix} u_1 + u_2 \\ v \end{pmatrix} = \delta\begin{pmatrix} u_1 \\ v \end{pmatrix} + \delta\begin{pmatrix} u_2 \\ v \end{pmatrix}$

4-2 Determinants of Order n

In this section, we extend the definition of the determinant to n×n matrices for n≧3.

Given $A \in M_{n\times n}(F)$, for $n \geqq 2$, denote the (n-1) ×(n-1) matrix obtained from A by deleting row i and column j by $\tilde{A}_{ij}$.

$\tilde{A}_{ij}$是(n-1) ×(n-1)矩陣，係將矩陣 A 的第 i 列第 j 行刪除後剩下來的矩陣。

$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \in M_{3\times 3}(R)$, we have $\tilde{A}_{11} = \begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix}$, $\tilde{A}_{13} = \begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix}$

and for $B = \begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix} \in M_{4\times 4}(R)$, we have $\tilde{B}_{23} = \begin{pmatrix} 1 & -1 & -1 \\ 2 & -5 & 8 \\ -2 & 6 & 1 \end{pmatrix}$

DEFINTION 4.2 Cofactor（餘因子） and Determinant

Let $A \in M_{n\times n}(F)$. If n = 1, so that $A = (A_{11})$, we define $\det(A) = A_{11}$.

For $n \geqq 2$, we define det(A) recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j})$$

The scalar det(A) is called the **determinant** of A and is also denoted by $|A|$.

The scalar $(-1)^{i+j}\det(\tilde{A}_{ij})$ is called the **cofactor** of the entry of A in row i, column j.

令 $A \in M_{n\times n}(F)$。若 $n = 1$，則 $A = (A_{11})$，行列式 $\det(A) = A_{11}$。

若 $n \geqq 2$，A 的行列式為 $\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \cdot \det(\tilde{A}_{1j}) = |A|$。

$(-1)^{i+j}\det(\tilde{A}_{ij})$：稱為 A 的第 i 列第 j 行元素的餘因子。

Letting $c_{ij} = (-1)^{i+j}\det(\tilde{A}_{ij})$ denote the cofactor of the row i, column j entry of A, we can express the formula for the determinant of A as

$$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \ldots.. + A_{1n}c_{1n}$$

令 $c_{ij} = (-1)^{i+j}\det(\tilde{A}_{ij})$ 表示 A 矩陣第 i 列第 j 行元素的餘因子，則 A 的行列式可寫

成：$\det(A) = A_{11}c_{11} + A_{12}c_{12} + \ldots\ldots + A_{1n}c_{1n}$

此公式稱為 Cofactor expansion along the first row of A（沿著 A 的第一列餘因子展開式）。

Thus the determinant of A equals the sume of the products of each entry in row 1 of A multiplied by its cofactor.

因此，A 的行列式等於 A 的第一列上每一個元素乘上它的餘因子的乘積總和。

EXAMPLE 1

Let $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix} \in M_{3\times 3}(R)$. Using cofactor expansion along the first row of A, we obtain

$$\det(A) = (-1)^{1+1}A_{11}\cdot\det(\widetilde{A}_{11}) + (-1)^{1+2}A_{12}\cdot\det(\widetilde{A}_{12}) + (-1)^{1+3}A_{13}\cdot\det(\widetilde{A}_{13})$$
$$= (-1)^2(1)\cdot\det\begin{pmatrix} -5 & 2 \\ 4 & -6 \end{pmatrix} + (-1)^3(3)\cdot\cdot\det\begin{pmatrix} -3 & 2 \\ -4 & -6 \end{pmatrix} + (-1)^4(-3)\cdot\det\begin{pmatrix} -3 & -5 \\ -4 & 4 \end{pmatrix}$$
$$= 40.$$

利用沿著 A 的第一列餘因子展開式求 A 的行列式。

EXAMPLE 2

Let $B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix} \in M_{3\times 3}(R)$. Using cofactor expansion along the first row of B, we obtain

$$\det(B) = (-1)^{1+1}B_{11}\cdot\det(\widetilde{B}_{11}) + (-1)^{1+2}B_{12}\cdot\det(\widetilde{B}_{12}) + (-1)^{1+3}B_{13}\cdot\det(\widetilde{B}_{13})$$
$$= \ldots = 48.$$

利用沿著 B 的第一列餘因子展開式求 B 的行列式。

EXAMPLE 3

Let $C = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 1 & 3 & -3 \\ -2 & -3 & -5 & 2 \\ 4 & -4 & 4 & -6 \end{pmatrix} \in M_{4\times 4}(R)$. Using cofactor expansion along the first row of C, we obtain

$$\det(C) = (-1)^{1+1} C_{11} \cdot \det(\widetilde{C}_{11}) + (-1)^{1+2} C_{12} \cdot \det(\widetilde{C}_{12}) + (-1)^{1+3} C_{13} \cdot \det(\widetilde{C}_{13}) + (-1)^{1+4} C_{14} \cdot \det(\widetilde{C}_{14}) = \ldots = 32.$$

利用沿著 C 的第一列餘因子展開式求 C 的行列式。

EXAMPLE 4

The determinant of the n×n identity matrix is 1.

n×n 單位矩陣之行列式為 1。

The assertion is proved by mathematical induction on n.

The result is clearly true for the 1×1 identity matrix.

Assume that the determinant of the (n-1)×(n-1) identity matrix is 1 for some n≧2, and let I denote the n×n identity matrix.

Using cofactor expansion along the first row of I, we obtain

$$\det(I) = (-1)^{1+1}(1) \cdot \det(\widetilde{I}_{11}) + (-1)^{1+2}(0) \cdot \det(\widetilde{I}_{12}) + \ldots + (-1)^{1+n}(0) \cdot \det(\widetilde{I}_{1n}) = \ldots = 1.$$

利用數學歸納法來證明。

對 1x1 單位矩陣而言，行列式為 1 的結論甚為明顯。

對某些 n≧2 而言，假設(n-1)×(n-1)單位矩陣的行列式為 1，並令 I 代表 n×n 單位矩陣。

利用沿著 I 的第一列餘因子展開式，可得：

$$\det(I) = (-1)^{1+1}(1) \cdot \det(\widetilde{I}_{11}) + (-1)^{1+2}(0) \cdot \det(\widetilde{I}_{12}) + \ldots + (-1)^{1+n}(0) \cdot \det(\widetilde{I}_{1n}) = \ldots = 1.$$

其中，$\widetilde{I}_{11}$ 是 (n-1)×(n-1) 單位矩陣。

det(I)=1。

Theorem 4.3

The determinant of an n×n matrix is a linear function of each row when the remaining rows are held fixed. That is, for 1≦r≦n, we have

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix},$$

whenever k is scalar and u, v, and each a_i are row vectors in F^n.

當其他列（Row）被固定時，一個 n×n 矩陣的行列式是每一個列的線性函數。即對每一個 $1 \leqq r \leqq n$

$$\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u+kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$$ ，其中，k 為任一純量。

【Proof】

The proof is by mathematical induction on n.

The result is immediate if n = 1.

Assume that for some integer $n \geqq 2$ the determinant of any $(n-1)\times(n-1)$ matrix is a linear function of each row when the remaining rows are held fixed.

Let A be an n×n matrix with row $a_1, a_2, \ldots, a_n$, respectively, and suppose that for some r ($1 \leqq r \leqq n$), we have $a_r = u+kv$ for some $u, v \in F^n$ and some scalar k. Let $u = (b_1, b_2, \ldots, b_n)$ and $v = (c_1, c_2, \ldots, c_n)$ and let B and C be the matrices obtained from A by replacing row r of A by u and v, repectively. We must prove that det(A) = det(B) + kdet(C).

For r = 1 …

For r > 1 and $1 \leqq j \leqq n$, the rows of $\widetilde{A}_{1j}$, $\widetilde{B}_{1j}$, and $\widetilde{C}_{1j}$ are the same except for row r-1.

Moreover, row r-1 of $\widetilde{A}_{1j}$ is $(b_1+kc_1, \ldots, b_{j-1}+kc_{j-1}, b_{j+1}+kc_{j+1}, \ldots, b_n+kc_n)$ which is the sum of row r-1 of $\widetilde{B}_{1j}$ and k times r-1 row of $\widetilde{C}_{1j}$.

Since $\widetilde{B}_{1j}$ and $\widetilde{C}_{1j}$ are $(n-1)\times(n-1)$ matrices, we have $\det(\widetilde{A}_{1j}) = \det(\widetilde{B}_{1j}) + k\det(\widetilde{C}_{1j})$ by the induction hypothesis. Thus since $A_{1j} = B_{1j} = C_{1j}$, we have

$$\det(A)=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{A}_{1j})=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot[\det(\widetilde{B}_{1j})+k\det(\widetilde{C}_{1j})]$$

$$=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{B}_{1j})+k\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{C}_{1j})=\det(B)+k\det(C)$$

This shows that the theorem is true for n×n matrices, and so the theorem is true for all square matrices by mathematical induction.

利用數學歸納法來證明。

當 $n=1$ 時，結論顯然成立。

某個整數 $n \geqq 2$ 而言，可由 Theorem 4.1 延伸假設：當其餘列均被固定時，任意一個 (n-1)×(n-1) 矩陣的行列式是每一列的線性函數。**【後頭推論的依據】**

令 A 為 nxn 的矩陣，列向量分別為 $a_1, a_2,\ldots, a_n$，若將其中某一列 r（$1\leqq r\leqq n$）以 u + kv 替換，其餘各列均固定不變，即 $a_r = u + kv$，並將 A 的第 r 列換成 $u = (b_1, b_2,\ldots, b_n)$ 與 $v = (c_1, c_2,\ldots, c_n)$ 成為矩陣 B 與 C，

即 $B=\begin{pmatrix} a_1\\ \vdots\\ a_{r-1}\\ u\\ a_{r+1}\\ \vdots\\ a_n \end{pmatrix}$，$C=\begin{pmatrix} a_1\\ \vdots\\ a_{r-1}\\ v\\ a_{r+1}\\ \vdots\\ a_n \end{pmatrix}$。

證明 $\det(A) = \det(B) + k\det(C)$？

當 $r = 1$…（替換對象是第一列）

當 $r > 1$ 且 $1\leqq j\leqq n$ 時，刪除第一列（Row）第 j 行（Column）後(n-1)×(n-1)的餘因子 $\widetilde{A}_{1j}$、$\widetilde{B}_{1j}$與$\widetilde{C}_{1j}$除了第 r-1 列（原來在第 r 列，因刪除第一列後變成第 r – 1 列）外，其餘各列均相同。

$\widetilde{A}_{1j}$ 的第 r-1 列？

$a_r = (b_1+kc_1,\ldots, b_{j-1}+kc_{j-1}, b_{j+1}+kc_{j+1},\ldots, b_n+kc_n)$ = $\widetilde{B}_{1j}$ 的第 r-1 列＋k×（$\widetilde{C}_{1j}$ 的第 r-1 列）。（原來的第 j 行 b_j、c_j 被刪除了。）

因 $\widetilde{B}_{1j}$ 與 $\widetilde{C}_{1j}$ 均為 (n-1) × (n-1) 的矩陣，由數學歸納法假設得知 $\det(\widetilde{A}_{1j})=\det(\widetilde{B}_{1j})+k\det(\widetilde{C}_{1j})$。**【參考前面的假設】**

又因為 $A_{1j} = B_{1j} = C_{1j}$（$1\leqq j\leqq n$）（第一列沒有被更換），故

$$\det(A)=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{A}_{1j})=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot[\det(\widetilde{B}_{1j})+k\det(\widetilde{C}_{1j})]$$

$$=\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{B}_{1j})+k\sum_{j=1}^{n}(-1)^{1+j}A_{1j}\cdot\det(\widetilde{C}_{1j})=\det(B)+k\det(C)$$

因此對 nxn 矩陣而言，定理是成立的。

所以由數學歸納法可推知，對所有方矩陣而言，本定理為真。

Theorem 4.1。

Corollary

Let $A\in M_{n\times n}(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$.

令 $A\in M_{n\times n}(F)$。若 A 的某一列元素全為零，則行列式為零。

Lemma

Let $B\in M_{n\times n}(F)$, where $n\geqq 2$. If row i of B equals e_k for some k ($1\leqq k\leqq n$), then $\det(B)=(-1)^{i+k}\cdot\det(\widetilde{B}_{ik})$.

令 $B\in M_{n\times n}(F)$，$n\geqq 2$。若 B 的第 i 列等於 e_k，則 $\det(B)=(-1)^{i+k}\cdot\det(\widetilde{B}_{ik})$（因為該列僅有一個 1，其餘全為零）。

Theorem 4.4

The determinant of a square matrix can be evaluated by cofactor expansion along any row. That is, if $A\in M_{n\times n}(F)$, then for any integer i ($1\leqq i\leqq n$),

$$\det(A)=\sum_{j=1}^{n}(-1)^{i+j}A_{ij}\cdot\det(\widetilde{A}_{ij})$$

A 的行列式除了由第一列展開外，也可以由其他列展開。

【Proof】

Cofactor expansion along the first row of A gives the determinant of A by definition. So the result is true if $i = 1$.

Fix $i>1$. Row i of A can be written as $\sum_{j=1}^{n}A_{ij}e_j$.

For $1\leqq j\leqq n$, let B_j denote the matrix obtained from A by replaceing row i of A by e_j.

Then by Theorem 4.3 and the lemma, we have

$$\det(A)=\sum_{j=1}^{n}A_{ij}\det(B_j)=\sum_{j=1}^{n}(-1)^{i+j}A_{ij}\det(\widetilde{A}_{ij})$$

依據定義，矩陣的行列式可由矩陣第一列的餘因子展開求得，因此當 i = 1 時，定理為真。

若 $i > 1$，矩陣 A 的第 i 列可以寫成 $\sum_{j=1}^{n} A_{ij} e_j$。

對 $j = 1, 2, \ldots, n$ 而言，令 B_j 為將 A 的第 i 列換成 e_j 後所得到的矩陣。

$$\text{即 } A = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ a_i \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix} \rightarrow B_1 = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ e_1 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix},\ B_2 = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ e_2 \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix}, \ldots B_j = \begin{pmatrix} a_1 \\ \vdots \\ a_{i-1} \\ e_j \\ a_{i+1} \\ \vdots \\ a_n \end{pmatrix} 。$$

依據 Theorem 4.3 與 lemma 得知

$$\det(A) = \sum_{j=1}^{n} A_{ij} \det(B_j) = \sum_{j=1}^{n} (-1)^{i+j} A_{ij} \det(\tilde{A}_{ij})$$

【A_{ij} 相當於 Theorem 4.3 的 k，$\tilde{A}_{ij}$ 相當於 Lemma 的 $\tilde{B}_{ik}$】

Theorem 4.3 當其他列（Row）被固定時，一個 n×n 矩陣的行列式是每一個列的線性函數。即對每一個 $1 \leqq r \leqq n$ $\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u + kv \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ u \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix} + k\det\begin{pmatrix} a_1 \\ \vdots \\ a_{r-1} \\ v \\ a_{r+1} \\ \vdots \\ a_n \end{pmatrix}$，其中，k 為任一純量。

Lemma 令 $B \in M_{n\times n}(F)$，$n \geqq 2$。若 B 的第 i 列等於 e_k，則 $\det(B) = (-1)^{i+k} \cdot \det(\tilde{B}_{ik})$（因為該列僅有一個 1，其餘全為零）。

Corollary

Let $A \in M_{n\times n}(F)$ has two identical rows, then $\det(A) = 0$.

令 $A \in M_{n\times n}(F)$。若 A 有兩列（Row）相同，則行列式為零。

Theorem 4.5

If $A \in M_{n\times n}(F)$ and B is a matrix obtained from A by interchanging any two rows of A, then det(B) = - det(A).

令 $A\in M_{n\times n}(F)$ ，且 B 為將 A 的兩列（Row）互調後所得的矩陣，則

det(B) = - det(A)。

【**Proof**】

Let $A \in M_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix obtained from A by interchanging rows r and s, where r < s. Thus

$$A = \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix}$$

Consider the matrix obtained from A by replacing rows r and s by $a_r + a_s$. By the **corollary to Theorem 4.4** and **Theorem 4.3**, we have

$$0 = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r + a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r + a_s \\ \vdots \\ a_n \end{pmatrix} = \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_r \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_r \\ \vdots \\ a_n \end{pmatrix} + \det\begin{pmatrix} a_1 \\ \vdots \\ a_s \\ \vdots \\ a_s \\ \vdots \\ a_n \end{pmatrix}$$

$$= 0 + \det(A) + \det(B) + 0$$

Therefore det(B) = -det(A)

Theorem 4.6

Let $A\in M_{n\times n}(F)$, and let B be a matrix obtained by adding a multiple of one row of A to another row of A. Then det(B) = det(A).

令 $A\in M_{n\times n}(F)$ 且 B 為將 A 的某一列乘上某個倍數後加至 A 的另一列後所得到的矩陣，則 det(B) = det(A)。

【**Proof**】

Suppose that B is the n×n matrix obtained from A by adding k times row r to s, where $r\neq s$. Let the row of A be $a_1, a_2, \ldots, a_n$ and the row of B be $b_1, b_2, \ldots, b_n$. Then $b_i = a_i$ for $r\neq s$

and $b_s = a_s + ka_r$. Let C be the matrix obtained from A by replacing row s with a_r. Applying **Theorem 4.3** to row s of B, we obtain

det(B) = det(A) +k det(C) = det (A) because det(C) = 0 by the **corollary to Theorem 4.4**.

令 B 是將 A 的第 r 列乘上 k（k×a_r）加到第 s 列上（即 $a_s \rightarrow a_s + ka_r$）所得到的矩陣。

令 A 的列向量為 $a_1, a_2, \ldots, a_n$，B 的列向量為 $b_1, b_2, \ldots, b_n$，對所有 $r \neq s$ 的列而言，$b_i = a_i$，且 $b_s = a_s + ka_r$。

令 C 為將 A 的第 s 列改成 a_r 所得的矩陣。

將 Thoerem 4.3 應用到 B 的第 s 列，得知：

det(B) = det(A) +k det(C)

其中，C 因有兩列相同，故 det(C) = 0，det(B) = det(A) 。

Corollary

If $A \in M_{n \times n}(F)$ has rank less than n, then det(A) = 0.

令 $A \in M_{n \times n}(F)$ 且 rank < n，則 det(A) = 0。

The following rules summarize the effect of an elementary row operation on the determinant of a matrix $A \in M_{n \times n}(F)$.

令 $A \in M_{n \times n}(F)$。Elementary row operation $A \rightarrow B$ 對於矩陣 A 的行列式的影響：

1. If B is matrix obtained by interchanging any two rows of A, then det(B) = - det(A).
   任兩列互換，則 det(B) = - det(A)。
2. If B is a matrix obtained by multiplying a row of A by a nonzero scalar k, then det(B) = k×det(A).
   某一列乘上一非零的純量 k，則 det(B) = k×det(A)。
3. If B is a matrix obtained by adding a multiple of one row of A to another row of A, then det(B) = det(A).
   某一列乘上某倍數再加到另一列，則 det(B) =det(A）。

These facts can be used to simply the evaluation of a determinant.

EXAMPLE

Consider, for instance, the matrix $A = \begin{pmatrix} 1 & 3 & -3 \\ -3 & -5 & 2 \\ -4 & 4 & -6 \end{pmatrix}$.

Adding 3 times row 1 of A to row 2 and 4 times row 1 to row 3, we obtain

$$M = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 4 & -7 \\ 0 & 16 & -18 \end{pmatrix}$$

The cofactor expansion of M along the first row gives

$$\det(M) = (-1)^{1+1} M_{11} \cdot \det(\widetilde{M}_{11}) + (-1)^{1+2} M_{12} \cdot \det(\widetilde{M}_{12}) + (-1)^{1+3} M_{13} \cdot \det(M_{13}) = 40. \ldots$$

But we can do even better. If we add -4 times row 2 of M to row 3, we obtain

$$P = \begin{pmatrix} 1 & 4 & -3 \\ 0 & 4 & -7 \\ 0 & 0 & 10 \end{pmatrix} \qquad \det(P) = \det(M) = \det(A) = 1\times4\times10 = 40.$$

The determinant of an upper triangular matrix is the product of its diagonal entries.

EXAMPLE 5

To evaluate the determinant of the matrix $B = \begin{pmatrix} 0 & 1 & 3 \\ -2 & -3 & -5 \\ 4 & -4 & 4 \end{pmatrix}$.

Interchanging rows 1 and 2 of B produces

$$C = \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix}$$

By means of a sequence of elementary row operations.

$$\begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 4 & -4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & -10 & -6 \end{pmatrix} \rightarrow \begin{pmatrix} -2 & -3 & -5 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{pmatrix}$$

Thus det(C) = -2×1×24= -48.

Since C was obtained from B by an interchanging of rows, it follows that det(B) = -det(C) = 48.

Determinants

- Let A be a square matrix.
  - ⇨ The **minor** of the element a_{ij} is denoted M_{ij} and is the determinant of the matrix that remains after deleting row i and column j of A.
  - ⇨ The **cofactor** of a_{ij} is denoted C_{ij} and is given by $C_{ij} = (-1)^{i+j} M_{ij}$
- Note that $C_{ij} = M_{ij}$ or $-M_{ij}$.

Example

- Determine the minors and cofactors of the elements a_{11} and a_{32} of the following matrix A.

$$A=\begin{bmatrix}1 & 0 & 3\\ 4 & -1 & 2\\ 0 & -2 & 1\end{bmatrix}$$

$$\text{Minor of } a_{11}: M_{11}=\begin{vmatrix}1 & 0 & 3\\ 4 & -1 & 2\\ 0 & -2 & 1\end{vmatrix}=\begin{vmatrix}-1 & 2\\ -2 & 1\end{vmatrix}=(-1\times 1)-(2\times(-2))=3$$

$$\text{Cofactor of } a_{11}: C_{11}=(-1)^{1+1}M_{11}=(-1)^2(3)=3$$

$$\text{Minor of } a_{32}: M_{32}=\begin{vmatrix}1 & 0 & 3\\ 4 & -1 & 2\\ 0 & -2 & 1\end{vmatrix}=\begin{vmatrix}1 & 3\\ 4 & 2\end{vmatrix}=(1\times 2)-(3\times 4)=-10$$

$$\text{Cofactor of } a_{32}: C_{32}=(-1)^{3+2}M_{32}=(-1)^5(-10)=10$$

Determinants

- The **determinant of a square matrix** is the sum of the products of the elements of the first row and their cofactors.

If A is 3×3, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$

If A is 4×4, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$

⋮

If A is $n\times n$, $|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + \cdots + a_{1n}C_{1n}$

- These equations are called **cofactor expansions** of $|A|$.

Example

- Evaluate the determinant of the matrix A.

$$A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 1\\ 4 & 2 & 1\end{bmatrix}$$

$$|A| = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$= 1(-1)^2\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} + 2(-1)^3\begin{vmatrix}3 & 1\\ 4 & 1\end{vmatrix} + (-1)(-1)^4\begin{vmatrix}3 & 0\\ 4 & 2\end{vmatrix}$$

$$= [(0\times 1)-(1\times 2)] - 2[(3\times 1)-(1\times 4)] - [(3\times 2)-(0\times 4)]$$

$$= -2+2-6$$

$$= -6$$

Determinants

- The determinant of a square matrix is the sum of the products of the elements of any row or column and their cofactors.

ith row expansion: $|A| = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$

jth column expansion: $|A| = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$

- There is a useful rule that can be used to give the sign part, (-1)i+j, of the cofactors in these expansions. The rule is summarized in the following array.

$$\begin{bmatrix}+ & - & + & \cdots\\ - & + & - & \cdots\\ + & - & + & \\ \vdots & \vdots & & \end{bmatrix}$$

Example

- Find the determinant of the matrix A using the second row.

$$A=\begin{bmatrix}1 & 2 & -1\\ 3 & 0 & 1\\ 4 & 2 & 1\end{bmatrix}$$

$$|A| = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$= -3\begin{vmatrix}2 & -1\\ 2 & 1\end{vmatrix} + 0\begin{vmatrix}1 & -1\\ 4 & 1\end{vmatrix} - 1\begin{vmatrix}1 & 2\\ 4 & 2\end{vmatrix}$$

$$= -3[(2\times 1)-(-1\times 2)] + 0[(1\times 1)-(-1\times 4)] - 1[(1\times 2)-(2\times 4)]$$

$$= -12+0+6 = -6$$

Example

- Evaluate the determinant of the matrix.

$$\begin{bmatrix}2 & 1 & 0 & 4\\ 0 & -1 & 0 & 2\\ 7 & -2 & 3 & 5\\ 0 & 1 & 0 & -3\end{bmatrix}$$

$$|A| = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33} + a_{43}C_{43}$$

$$= 0(C_{13}) + 0(C_{23}) + 3(C_{33}) + 0(C_{43})$$

$$= 3\begin{vmatrix}2 & 1 & 4\\ 0 & -1 & 2\\ 0 & 1 & -3\end{vmatrix}$$

$$= 3(2)\begin{vmatrix}-1 & 2\\ 1 & -3\end{vmatrix} = 6(3-2) = 6$$

Computing Determinants

$$A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix} \Rightarrow |A| = a_{11}a_{22} - a_{12}a_{21}$$

$$A=\begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \Rightarrow \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{matrix}a_{11} & a_{12}\\ a_{21} & a_{22}\\ a_{31} & a_{32}\end{matrix}$$

$$\Rightarrow |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

(diagonal products from left to right)

$$- a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33}$$

(diagonal products from right to left)

4-3 Properties of Determinants

In Theorem 3.1, we saw that performing an elementary row operation on a matrix can be accomplished by multiplying the matrix by an elementary matrix. This result is very useful in studying the effects on the determinant of applying a sequence of elementary row operations.

依據 Theorem 3.1，對一矩陣執行基本列運算等同於對該矩陣乘上一個基本矩陣。此結果用來研究一矩陣經過一系列的基本列運算後，對其行列式的影響，是非常有用的。

For an n×n identity matrix I, the following facts are about the determinants of elementary matrices.【Identity matrix I → elementary row operation→ E】

1. If E is an elementary matrix obtained by interchanging any two rows of I, the det(E) = -1.
   E 是將 I 的任兩列互換，則 det(E) = -1。
2. If E is an elementary matrix obtained by multiplying some row of U by the zonzero scalar k, then det(E) = k.
   E 是將 I 的某一列乘上非零純量，則 det(E) = k。
3. If E is an elementary matrix obtained by adding a multiply of some row of I to another row, then det(E) = 1.
   E 是將 I 的某一列乘上非零純量再加到另一列，則 det(E) = 1。

Theorem 4.7 矩陣相乘的行列式

For any A, B $\in M_{n\times n}(F)$, det(AB) = det(A)・det(B).

Corollary 矩陣可逆與行列式的關係

A matrix A $\in M_{n\times n}(F)$ is invertible if and only if det(A) $\neq$ 0. Furthermore, if A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

令 A $\in M_{n\times n}(F)$ 且為可逆的『若且唯若』條件為 det(A) $\neq$ 0。再者，若 A 可逆，則 $\det(A^{-1}) = \dfrac{1}{\det(A)}$。

Theorem 4.8 矩陣倒置的行列式

For any $A \in M_{n\times n}(F)$, $\det(A^t) = \det(A)$.

Theorem 4.9 Cramer's Rule

Let $Ax = b$ be the matrix form of a system of n linear equations in n unknowns, where $x = (x_1, x_2, \ldots, x_n)^t$. If $\det(A) \neq 0$, then this system has a unique solution, and for each k (k = 1,2,…,n),

$x_k = \dfrac{\det(M_k)}{\det(A)}$, where M_k is the n×n matrix obtained from A by replacing column k of A by b.

令 Ax = b 代表含 n 個未知數、n 個線性方程式的方程組，若 $\det(A) \neq 0$，則此方程組具有唯一解，且該組解的每一元素為 $x_k = \dfrac{\det(M_k)}{\det(A)}$。其中 M_k 為將 A 矩陣的第 k 行換成 b 後所得到的矩陣，也是一個 n×n 的矩陣。

【Proof】

If $\det(A) \neq 0$, then the system $Ax = b$ has a unique solution by the corollary to Theorem 4.7 and Theorem 3.10.

For each integer k ($1 \leqq k \leqq n$), let a_k denote the kth column of A and X_k denote the matrix obtained from the n×n identity matrix by replacing column k by x.

Then by Theorem 2.13, AX_k is the n×n matrix whose ith column is

$Ae_i = a_i$ if $i \neq k$ and $Ax = b$ if $i = k$.

Thus $AX_k = M_k$.

Evaluating X_k by cofactor expansion along row k produces

$$\det(X_k) = x_k \cdot \det(I_{n-1}) = x_k$$

Hence by Theorem 4.7

$$\det(M_k) = \det(AX_k) = \det(A) \cdot \det(X_k) = \det(A) \cdot x_k$$

Therefore

$$x_k = |\det(A)|^{-1} \cdot \det(M_k).$$

依據 **Corollary to Theorem 4.7** 與 **Theorem 3.10**，

若 $\det(A) \neq 0$，則 $Ax = b$ 具有唯一解。

對每一個正整數 k（$1 \leqq k \leqq n$）而言，令 a_k 為 A 的第 k 行（Column），X_k 為將

單位矩陣的第 k 行換成 x。

即 $A=[a_1\ \ a_2\ \ ..\ \ a_k\ \ ..\ \ a_{n-1}\ \ a_n]$（$a_1, a_2,\ldots a_k,.., a_{n-1}, a_n$ 均為行向量）

$X_1=[x\ \ 0\ \ ..\ \ 0\ \ ..\ \ 0\ \ 0]$, $X_2=[1\ \ x\ \ ..\ \ 0\ \ ..\ \ 0\ \ 0],\ldots$

$X_k=[1\ \ 0\ \ ..\ \ x\ \ ..\ \ 0\ \ 0]$ （x 為行向量，X_k 為 n×n 矩陣）

依據 Theorem 2.13，AX_k 是 n×n 的矩陣，其第 i 行為

$Ae_i=a_i$（若 i≠k）及 $Ax=b$（若 i=k）。

因此 $AX_k = M_k$。（當 i≠k，第 i 行為 a_i，當 i=k，第 i 行為 b，與 M_k 的定義相符，故 AX_k 的結果即為 M_k。）

利用沿著第 k 列的餘因子展開計算行列式

$\det(X_k)=x_k\cdot\det(I_{n-1})=x_k$。

依據 **Theorem 4.7**，

$\det(M_k)=\det(AX_k)=\det(A)\cdot\det(X_k)=\det(A)\cdot x_k$。

因此 $x_k=|\det(A)|^{-1}\cdot\det(M_k)$。

Theorem 2.13 A 為 m×n 的矩陣與 B 為 n×p 的矩陣，u_j 與 v_j 分別為 AB 與 B 的第 j 行（$1\leqq j\leqq p$）。則 (a) $u_j = A v_j$. (b) $v_j = B e_j$, where e_j is the j^{th} standard vector of F^p.

Corollary to Theorem 4.7 令 $A\in M_{n\times n}(F)$ 且為可逆的『若且唯若』條件為 $\det(A) \neq 0$。再者，若 A 可逆，則 $\det(A^{-1}) = \dfrac{1}{\det(A)}$。

Theorem 3.10 令 $Ax = b$ 為含 n 個未知數、n 個線性方程式的方程組，若 A 為可逆，則方程組恰有一個解，解為 $A^{-1}b$。反之，若該方程組恰有一個解，則 A 為可逆。

EXAMPLE 1

We illustrate Theorem 4.9 by using Cramer's rule to solve the following system of linear equations:

利用 Cramer's rule 來解下列方程式：

$x_1 + 2x_2 + 3x_3 = 2$

$x_1 \qquad + x_3 = 2$

$x_1 + x_2 - x_3 = 1.$

The matrix form of this system of linear equations is Ax = b, where

$$A=\begin{pmatrix}1&2&3\\1&0&1\\1&1&-1\end{pmatrix} \text{ and } b=\begin{pmatrix}2\\3\\1\end{pmatrix}$$

Because det(A) = 6, Cramer's rule applies. Using the notation of Therorem 4.9, we have

$$x_1 = \frac{\det(M_1)}{\det(A)} = \frac{\det\begin{pmatrix} 2 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{15}{6} = \frac{5}{2}$$

$$x_2 = \frac{\det(M_2)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 1 & -1 \end{pmatrix}}{\det(A)} = \frac{-6}{6} = -1$$

$$x_3 = \frac{\det(M_3)}{\det(A)} = \frac{\det\begin{pmatrix} 1 & 2 & 2 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}}{\det(A)} = \frac{3}{6} = \frac{1}{2}$$

Thus the unique solution to the given system of linear equations is $(x_1, x_2,\ldots,x_n)$ = (5/2, -1, 1/2)

EXAMPLE 2

The volume of the parallelpiped having the vectors a_1 = (1, -2, 1), a_2 = (1, 0, -1), and a_3 = (1, 1, 1) as adjacent sides is

利用六面體三個鄰邊向量來決定平行六面體的體積：

$$\left|\det\begin{pmatrix} 1 & -2 & 1 \\ 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix}\right| = 6.$$

Properties of Determinants

- Let A be an n × n matrix and c be a nonzero scalar.
  - If a matrix B is obtained from A by multiplying the elements of a row (column) by c then |B| = c|A|.
  - If a matrix B is obtained from A by interchanging two rows (columns) then |B| = –|A|.
  - If a matrix B is obtained from A by adding a multiple of a row (column) to another row (column), then |B| = |A|.

Example

- If $A = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ -2 & -4 & 10 \end{bmatrix}$, $|A| = 12$ is known. Evaluate the determinants of the following matrices.

(a) $B_1 = \begin{bmatrix} 1 & 12 & 3 \\ 0 & 6 & 5 \\ -2 & -12 & 10 \end{bmatrix}$ (b) $B_2 = \begin{bmatrix} 1 & 4 & 3 \\ -2 & -4 & 10 \\ 0 & 2 & 5 \end{bmatrix}$ (c) $B_3 = \begin{bmatrix} 1 & 4 & 3 \\ 0 & 2 & 5 \\ 0 & 4 & 16 \end{bmatrix}$

(a) $A \underset{3C2}{\approx} B_1$ Thus $|B_1| = 3|A| = 36$.

(b) $A \underset{R2 \leftrightarrow R3}{\approx} B_2$ Thus $|B_2| = -|A| = -12$.

(c) $A \underset{R3+2R1}{\approx} B_3$ Thus $|B_3| = |A| = 12$.

Cramer's Rule

- Let AX = B be a system of n linear equations in n variables such that |A| ≠ 0. The system has a unique solution given by

$$x_1 = \frac{|A_1|}{|A|},\ x_2 = \frac{|A_2|}{|A|},\ \dots,\ x_n = \frac{|A_n|}{|A|}$$

Where A_i is the matrix obtained by replacing column i of A with B.

Proof of Cramer's Rule

- Since |A| ≠ 0 ⇒ the solution to AX = B is unique and is given by

$$X = A^{-1}B = \frac{1}{|A|}\mathrm{adj}(A)B$$

- xi, the ith element of X, is given by

$$x_i = \frac{1}{|A|}[\text{row i of adj}(A)] \times B = \frac{1}{|A|}[C_{1i} C_{2i} \cdots C_{ni}]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= \frac{1}{|A|}(b_1C_{1i} + b_2C_{2i} + \cdots + b_nC_{ni}) \quad \Rightarrow \quad \text{Thus } x_i = \frac{|A_i|}{|A|}$$

the cofactor expansion of |*Ai*| in terms of the *i*th column

Example 1/2

- Solving the following system of equations using Cramer's rule.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= -2 \\ 2x_1 + 5x_2 + x_3 &= -5 \\ x_1 + 2x_2 + 3x_3 &= 6 \end{aligned}$$

⇨ The matrix of coefficients *A* and column matrix of constants *B* are

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 5 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}$$

Example 2/2

⇨ It is found that |*A*| = –3 ≠ 0. Thus Cramer's rule be applied. We get

$$A_1 = \begin{bmatrix} -2 & 3 & 1 \\ -5 & 5 & 1 \\ 6 & 2 & 3 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 1 & 6 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 3 & -2 \\ 2 & 5 & -5 \\ 1 & 2 & 6 \end{bmatrix}$$

Giving $|A_1| = -3, |A_2| = 6, |A_3| = -9$

Cramer's rule now gives

$$x_1 = \frac{|A_1|}{|A|} = \frac{-3}{-3} = 1,\ x_2 = \frac{|A_2|}{|A|} = \frac{6}{-3} = -2,\ x_3 = \frac{|A_3|}{|A|} = \frac{-9}{-3} = 3$$

The unique solution is $x_1 = 1, x_2 = -2, x_3 = 3$

4-4 SUMMARY – Important Facts about Determinants

The determinant of an n×n matrix A having entries from a field F is a scalar in F, denoted by det(A) or $|A|$, and can be computed in the following manner:

佈於 Field 的矩陣 A 的行列式可以下列方式求得：

1. If A is 1×1, then det(A) = A_{11}, the single entry of A.
2. If A is 2×2, then det(A) = $A_{11}A_{22} - A_{12}A_{21}$.
3. If A is n×n for n>2, then $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}A_{ij}\cdot\det(\tilde{A}_{ij})$ (if the determinant is evaluated by the entries of row i of A)or $\det(A) = \sum_{i=1}^{n}(-1)^{i+j}A_{ij}\cdot\det(\tilde{A}_{ij})$ (if the determinant is evaluated by the entries of column j of A), **where $\tilde{A}_{ij}$ is the is (n-1)× (n-1) matrix obtained by deleting row u and column j from A.**

Properties of the determinants

1. If B is matrix obtained by interchanging any two rows or interchanging any two columns of an n×n matrix A, then det(B) = - det(A).
2. If B is a matrix obtained by multiplying each entry of sonme row or column of an n×n matrix A by a scalar k, then det(B) = k・det(A).
3. If B is a matrix obtained from an n×n matrix A by adding a multiple of row i to ro j or a multiple of column i to column j for i≠j, then det(B) = det(A).