Chapter 2 Linear Transformations and Matrices

2-1 Linear Transformations, Null Spaces, and Ranges

A function T with domain V and codomain W is denoted by T: V→W

「定義域」為 V，「對應域」為 W 的函數註記為 T: V→W。

DEFINTION 2.1 Linear transformation

Let V and W be vector spaces (over F). We call a function T: V→W a linear transformation from V to W if, for all x,y ∈V and c∈F, we have

(a) T(x+y) = T(x)+T(y) and

(b) T(cx) = cT(x).

令 V 與 W 為佈於 F 的向量空間。若 T 要稱為「由 V 映至 W 的線性轉換」，則表示對所有屬於 V 的 x、y 與屬於 F 的 c 而言，必須滿足（a）T(x+y) = T(x)+T(y)；（b）T(cx) = cT(x)。

意即，轉換要能滿足這兩個條件才能稱為線性轉換。

We often simply call **T linear**. 常簡稱 T 為線性。

Verifying the following properties of a function T: V→W.

驗證線性轉換 T: V→W 所具備的性質：

1. If T is linear, then T(0) = 0.

   若 T 是線性轉換，則 T(0) = 0。

2. T is linear if and only if T(cx+y) = cT(x) + T(y) for all x, y∈V and c∈F.

   T 是線性「若且惟若」T(cx+y) = cT(x) + T(y) （對所有 x, y∈V 且 c∈F）。

3. If T is linear, then T(x-y) = T(x) - T(y) for all x, y∈V.

   若 T 是線性，則 T(x-y) = T(x) - T(y)（對所有 x, y∈V）。

4. T is linear if and only if, for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have

$$T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$$

   T 是線性轉換「若且惟若」對所有 $x_1, x_2, \ldots, x_n \in V$ 且 $a_1, a_2, \ldots,$ $a_n \in F \ldots T\left(\sum_{i=1}^{n} a_i x_i\right) = \sum_{i=1}^{n} a_i T(x_i)$。

EXAMPLE 1

Define T: $R^2 \to R^2$ by $T(a_1, a_2) = (2a_1+a_2, a_1)$.

To show that T is linear.

Let $c \in R$ and $x, y \in R^2$, where $x = (b_1, b_2)$ and $y = (d_1, d_2)$.

Since $cx+y = c(b_1, b_2) +(d_1, d_2) = (cb_1+d_1, cb_2+d_2)$

$T(cx+y) = T(cb_1+d_1, cb_2+d_2) = (2(cb_1 + d_1) + cb_2+ d_2, cb_1+d_1)$

Also $cT(x) + T(y) = cT(b_1,b_2) + T(d_1,d_2) = c(2b_1 + b_2, b_1) +(2d_1+d_2, d_1) = \ldots..$

$= (2(cb_1+d_1)+ cb_2+d_2, cb_1+d_1)$

So T is linear.

定義 T: $R^2 \to R^2$ 為 $T(a_1, a_2) = (2a_1+a_2, a_1)$，驗證 T 是線性。

利用 Property 2 得知 T 是線性轉換。

2. T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x,y \in V$ and $c \in F$.

DEFINITION 2.2 Rotation

For any angle θ, define T_θ: $R^2 \to R^2$ by the rule: $T_\theta(a_1, a_2)$ is the vector obtained by rotating (a_1, a_2) counterclockwise by θ if $(a_1, a_2) \neq (0, 0)$ and $T_\theta(0, 0) = (0, 0)$. Then T_θ:$R^2 \to R^2$ is a linear transformation that is called the rotation by θ.

定義 T_θ: $R^2 \to R^2$ 為：若$(a_1,a_2) \neq (0,0)$，則 $T_\theta(a_1, a_2)$為以逆時針方向將點(a_1, a_2)旋轉 θ 角後所得到的向量；若$(a_1, a_2) = (0, 0)$，則 $T_\theta(0,0) = (0,0)$ 。T_θ: $R^2 \to R^2$ 為線性轉換，且稱 T_θ 為將點(a_1, a_2)旋轉 θ 角。

導出 T_θ 的公式。

We determine an explicit formula for T_θ.

Fix a nonzero vector $(a_1, a_2) \in R^2$. Let α be the angle that (a_1, a_2) makes with the positive x-axis, and let $r = \sqrt{a_1^2 + a_2^2}$. Then

$a_1 = r \cos\alpha$ and $a_2 = r \sin\alpha$

Also, $T_\theta(a_1, a_2)$ has length r and makes an angle $\theta+\alpha$ with the posivive x-axis.

The explicit formula for T_θ 換轉公式：

$T_\theta(a_1, a_2) = (r\cos(\theta+\alpha), r\sin(\theta+\alpha)) =\ldots=(a_1\cos\theta - a_2\sin\theta, a_1\sin\theta + a_2\cos\theta)$

Finally, observe that this same formula is valid for $(a_1, a_2) = (0 ,0)$

r 是(a_1, a_2)與原點的距離，α 是(a_1, a_2)與 x 軸的起始夾角。

DEFINITION 2.3 Reflection

Define $T:R^2 \rightarrow R^2$ by the rule: $T(a_1, a_2) = (a_1, -a_2)$. T is called the reflection about the x-axis.

定義 $T:R^2 \rightarrow R^2$ 為 $T(a_1, a_2) = (a_1, -a_2)$。T 稱為繞 x 軸的鏡射。

DEFINTION 2.4 Projection

Define $T: R^2 \rightarrow R^2$ by the rule: $T(a_1, a_2) = (a_1, 0)$. T is called the projection on the x-axis.

定義 $T:R^2 \rightarrow R^2$ 為 $T(a_1, a_2) = (a_1, 0)$。T 稱為在 x 軸的投影。

EXAMPLE 2

Define $T: M_{m\times n}(F) \rightarrow M_{m\times n}(F)$ by the rule: $T(A) = A^t$, where A^t is called the transpose of A. To show that T is linear.

定義 $T: M_{m\times n}(F) \rightarrow M_{m\times n}(F)$為 $T(A) = A^t$ 。其中 A^t 為 A 的導置矩陣。證明 T 是線性轉換。

利用 Property 2 證明 T 是線性轉換：

令 A、B$\in M_{m\times n}(F)$且 $a\in F$，則 $T(aA+B) = (aA+B)^t = aA^t+B^t = aT(A)+T(B)$。

故 T 是線性轉換。

2. T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x,y \in V$ and $c\in F$.

EXAMPLE 3

Define $T: P_n(R) \rightarrow P_{n-1}(R)$ by $T(f(x)) = f'(x)$, where $f'(x)$ denotes the derivative of f(x). To show that T is linear.

定義 $T: P_n(R) \rightarrow P_{n-1}(R)$為 $T(f(x)) = f'(x)$。其中 $f'(x)$為 f(x)的導數，證明 T 是線性轉換。

利用 Property 2 得知 T 是線性轉換。

EXAMPLE 4

Let V = C(R), the vector space of continuous real-valued functions on R. Let $a, b\in R$, $a<b$.

Define $T: V \rightarrow R$ by $T(f) = \int_a^b f(t)dt$ for all $f\in V$. To show T is linear.

令 V = C(R) 是 R 上連續實數函數的向量空間，a 與 b 為 R 的元素且 a<b。

定義 T: V→R 為 $T(f)=\int_a^b f(t)dt$。證明 T 是線性轉換。

利用 Property 2 證明 T 是線性轉換：

函數 f(t)的線性組合的定積分等於函數定積分的線性組合：

$$T(cx+y)=\int_a^b (cx+y)dt=c\int_a^b xdt+\int_a^b ydt=cT(x)+T(y)$$

DEFINTION 2.5 Identity transformation

For vector space V and W, define identity transformation I_v: V→ V by $I_v(x) = x$ for all $x \in V$.

V 與 W 為向量空間，定義單位轉換 I_v: V→ V 為 $I_v(x) = x$；意即單位轉換為由 V 映至 V 的一種轉換，定義域內的所有元素 x，單位轉換後的「像」為本身，即單位轉換是一種「像」等於「前像」的轉換。

簡言之：I 是「自己對到自己」。

DEFINTION 2.6 Zero transformation

For vector space V and W, define zero transformation T_0: V→ W by $T_0(x) = 0$ for all $x \in V$.

V 與 W 為向量空間，定義零轉換 T_0: V→ W 為 $T_0(x) = 0$；意即零轉換是一種「像為 0」的轉換。

簡言之：T_0 是對到 0，即「像為 0」。

DEFINTION 2.7 Null space or Kernel

Let V and W be vector space, and let T: V→W be linear. We define the null space (or kernel) **N(T) of T to be the set of all vectors x in V such that T(x) = 0;** that is, $N(T) = \{x \in V: T(x) = 0\}$.

V 與 W 為向量空間，且 T: V→W 為線性轉換。定義線性轉換 T 的 Null space 為 V 內所有滿足 T(x) = 0 的向量 x 所形成的集合，註記為 N(T)。Null space 的元素 x 經線性轉換 T 轉換後所對應的「像」為 0，意即 T(x) = 0。Null space 內的元素的「像」皆為 0。

簡言之：T 的 NULL SPACE 是 V 內「T(x) = 0」的元素所形成的集合，「定義域 V」這

一端的子集合。

DEFINTION 2.8 Range or Image

Let V and W be vector space, and let T: V→W be linear. We define the range (or image) of T to be subset of W consisting of all images (under T) of vectors in V; that is, R(T) = {T(x): x∈V}.

V 與 W 為向量空間，且 T: V→W 為線性轉換。定義線性轉換 T 的 Range 為所有 V 內的元素 x 的像 T(x) 所形成的集合，是 W 的子集合，註記為 R(T)。Range 是對應域 W 的子集合，其所有元素係為 V 的所有元素 x 所對應的像 T(x)。意即 R(T) = {T(x): x∈V}。

簡言之：T 的 RANGE 是 W 內「T(x)」所形成的集合，「對應域 W」那這一端。

EXAMPLE 5

Let V and W be vector spaces, and let **I: V→V** and **T_0: V→W** be the identity and zero transformation, respectively. Then **N(I) = {0}, R(I) = V, N(T_0) = V, and R(T_0) = {0}**.

V 與 W 為向量空間。Identity transformation I: V→V 與 Zero transformation T_0: V→W 的 Null space 與 Range，分別為 N(I) = {0}、R(I) = V、N(T_0) = V、R(T_0) = {0}。

因 I(x) = x 是自己對到自己，會對到 0 者，當然就是 0，故 N(I) 當然就是{0}，Range 當然就是定義域 V。

因 T_0(x) = 0 是對應到 0，Range 當然就是 {0}，N(T_0) 當然就是定義域 V！

EXAMPLE 6

Let T: R^3→R^2 be linear transformation defined by T(a_1, a_2, a_3) = (a_1-a_2, 2a_3). To verify that N(T) = { (a, a, 0): a∈R} and R(T) = R^2.

定義線性轉換 T: R^3→R^2 為 T(a_1, a_2, a_3) = (a_1-a_2, 2a_3)。驗證 T 的 N(T) 與 R(T) 分別為 N(T) = {(a, a, 0): a∈R} 與 R(T) = R^2。

Null space：Image 為 0 者，其前像條件為 a_1 = a_2 且 a_3 = 0。

Theorem 2.1

Let V and W be vector spaces and T: V→W be linear. Then N(T) and R(T) are subspaces of V and W, respectively.

令 V 與 W 為向量空間且 T: V→W 為線性轉換。證明 N(T)與 R(T)分別為定義域 V 與對應域 W 的子空間。

子空間的條件？W 為 V 的一子空間，若且唯若下列三條件成立：(a) $0 \in W$。(b) 當 $x \in W$ 且 $y \in W$ 時，則 $x+y \in W$。(c) 當 $c \in F$ 且 $x \in W$ 時，$cx \in W$。

【Proof】

Using 0_V and 0_W to denote the zero vectors of V and W.

Since $T(0_V) = 0_W$, we have that $0_V \in N(T)$.

Let $x, y \in N(T)$ and $c \in F$.

Then $T(x+y) = T(x) + T(y) = 0_W + 0_W = 0_W$, and $T(cx) = cT(x) = c\, 0_W = 0_W$.

Hence $x+y \in N(T)$ and $cx \in N(T)$, so that N(T) is a subspace of V.

Because $T(0_V) = 0_W$, we have that $0_W \in R(T)$.

Let $x, y \in R(T)$ and $c \in F$.

Then there exist v and w in V such that $T(v) = x$ and $T(w) = y$.

So that $T(v+w) = T(v)+T(w) = x + y$, and $T(cv) = cT(v) = cx$.

Thus $x+y \in R(T)$ and $cx \in R(T)$, so that R(T) is a subspace of W.

令 0_V 與 0_W 分別為 V 與 W 的零向量。

依據線性轉換的特性：$T(0_V) = 0_W$，所以 $0_V \in N(T)$。

令 x、$y \in N(T)$且 $c \in F$，則 $T(x + y) = T(x) + T(y) = 0_W + 0_W = 0_W$ 且 $T(cx) = cT(x) = c0_W = 0_W$。

→ $x + y \in N(T)$ 且 $cx \in N(T)$【滿足封閉性】。

→N(T)是 V 的子空間。

依據線性轉換的特性：$T(0_V) = 0_W$，所以 $0_W \in R(T)$。

令 x、$y \in R(T)$ 且 $c \in F$，則 V 中存在 v 與 w 使得 $T(v) = x$ 且 $T(w) = y$ 且 $T(v+w) = T(v)+T(w) = x+y$ 且 $T(cv) = cT(v) = cx$。

→ $x+y \in R(T)$ 且 $cx \in R(T)$ 【滿足封閉性】。

→R(T)是 W 的子空間。

The next theorem provides a method for finding a spanning set for the range of a linear transformation. Theorem 2.2 用來找尋線性轉換的值域的 spanning set。

Theorem 2.2

Let V and W be vector spaces, and let T: V→W be linear. If $\beta = \{v_1, v_2,\ldots, v_n\}$ is basis for V, then $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$.

令 V 與 W 為向量空間，且 T: V→W 為線性轉換。若 $\beta = \{v_1, v_2,\ldots, v_n\}$是定義域（Domain）V 的基底，則 T 的值域 Range R(T)可由 T(β)來生成。意即 $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2),\ldots, T(v_n)\})$。

提醒：令 V 是一空間向量且 $\beta = \{u_1, u_2,\ldots, u_n\}$為 V 的子集合，則 β 是 V 的基底的條件『若且惟若』「V 內每一個向量 v 皆能被唯一表達為 β 內向量 $u_1, u_2,\ldots, u_n$ 的線性組合：$v = a_1u_1 + a_2u_2 +\ldots+ a_nu_n$，且線性組合的係數 $a_1, a_2,\ldots, a_n$ 是唯一的。

提醒：令 S 是向量空間 V 的非空子集合，則 S 的生成集，註記為 Span(S)，是一個集合，該集合的元素是由 S 內向量經線性組合而成。

【Proof】

從 $\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})\subseteq R(T)$ 與 $R(T) \subseteq\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$兩個方向→$R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$。

Clearly $T(v_i)\in R(T)$ for each i.

Because R(T) is a subspace of W, R(T) contains $\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\}) = \text{span}(T(\beta))$ by Theorem 1.5.

Now suppose that $w\in R(T)$.

Then $w = T(v)$ for some $v\in V$.

Because β is basis for V, we have

$$v = \sum_{i=1}^{n} a_i v_i \quad \text{for some } a_1, a_2,\ldots, a_n\in F.$$

Since T is linear, it follows that

$$w = T(v) = \sum_{i=1}^{n} a_i T(v_i) \in \text{span}(T(\beta)).$$

So R(T) is contained in span(T(β)).

先證明 $\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})\subseteq R(T)$

很清楚地：$T(v_i) \in R(T)$。

由定理 1.5 得知：R(T)是 W 的一子空間， R(T)包含 $T(v_i)$的生成集，即 $\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\}) = \text{span}(T(\beta))\subseteq R(T)$。

再證明 $R(T) \subseteq\text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$

假設 $w\in R(T)$，則對某些 v 而言，$w = T(v)$。

因 $\beta = \{v_1, v_2,\ldots, v_n\}$ 是 V 的基底，故 V 中任意元素 v 可以寫成 $\{v_1, v_2,\ldots, v_n\}$ 的線性組合：$v = \sum_{i=1}^{n} a_i v_i$。

由於 T 是線性轉換，故 $w = T(v) = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i)$。

又依據 Span 的定義：$\sum_{i=1}^{n} a_i T(v_i) \in span(T(\beta))$。

因此 R(T)是被包含在 span(T(β))中，即 $R(T) \subseteq span(\{T(v_1), T(v_2),\ldots,T(v_n)\}) = span(T(\beta))$。

→ $R(T) = span(T(\beta)) = span(\{T(v_1), T(v_2),\ldots,T(v_n)\})$

Theorem 1.5 向量空間 V 的任意子集合 S 的生成集（Span of S）為 V 的子空間。意即 span(S)是 V 的子空間。進而言之，包含 S 的向量空間 V（S 為 V 的子集合），其任意子空間 W 也必然包含 S 的生成集。意即 $span(S) \subseteq W$。

簡言之：T 的值域是由對應域 V 的基底 $\beta = \{v_1, v_2,\ldots, v_n\}$的元素 $v_1, v_2,\ldots, v_n$ 的像 $T(v_1), T(v_2),\ldots,T(v_n)$所構成的集合$\{T(v_1), T(v_2),\ldots,T(v_n)\}$來生成。

It should be noted that Theorem 2.2 is true if β is infinite, that is, $R(T) = span(\{T(v):v\in\beta\})$. 若 β 是無限集合時，Theorem 2.2 亦為真。

The next example illustrates the usefulness of Theorem 2.2.

EXAMPLE 7

Define the linear transformation $T: P_2(R)\rightarrow M_{2\times2}(R)$ by

$$T(f(x)) = \begin{pmatrix} f(1)-f(2) & 0 \\ 0 & f(0) \end{pmatrix}$$

Since $\beta = \{v_1, v_2,\ldots, v_n\} = \{1, x, x^2\}$ is a basis for $P_2(R)$, we have

$R(T) = span(T(\beta)) = span(\{T(v_1), T(v_2),\ldots, T(v_n)\}) = span(\{T(1), T(x), T(x^2)\}) =$

$$= span\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right) = span\left(\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\right\}\right)$$

Thus we have found a basis for R(T), and so dim(R(T))= 2.

$P_2(R)$的基底為 $\beta = \{v_1, v_2,\ldots, v_n\} = \{1, x, x^2\}$

由 Theorem 2.2 得知 $R(T) = span(\{T(1), T(x), T(x^2)\})$

T(1)：f(x) = 1、f(1) - f(2) = 0、f(0) = 1。

T(x)：f(x) = x、f(1) - f(2) = -1、f(0) = 0。

$T(x^2)$：$f(x) = x^2$、f(1) - f(2) = -3、f(0) = 0。

故 $R(T) = \text{span}\left(\left\{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}, \begin{pmatrix}-3 & 0\\0 & 0\end{pmatrix}\right\}\right)$

因 $\begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}$ 與 $\begin{pmatrix}-3 & 0\\0 & 0\end{pmatrix}$ 線性相依，所以 $R(T) = \text{span}\left(\left\{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}\right\}\right)$

$\rightarrow \begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}$ 與 $\begin{pmatrix}-1 & 0\\0 & 0\end{pmatrix}$ 為 R(T)的基底。**dim(R(T) = 2。**

以維度來表示子空間的大小，至於 N(T)與 R(T)的大小？給予專有名詞。

DEFINITION 2.9

Let V and W be vector spaces, and let T: V→W be linear. If N(T) and R(T) are finite-dimensional, then we define the nullity of T, denoted nullity(T), and the rank of T, denoted rank(T), to be the dimensions of N(T) and R(T), respectively.

V 與 W 為向量空間，且 T: V→W 為線性轉換。

定義 T 的核次數（Nullity）：nullity(T) = dimensions of N(T)

定義 T 的秩（Rank）：rank(T) = dimension of R(T)

核次數（Nullity）與秩（Rank）分別代表 N(T)與 R(T)的維度。

註：T 的 NULL SPACE 是 V 內「T(x) = 0」的元素所形成的集合，「定義域 V」這一端的子集合。

註：T 的 RANGE 是 W 內「T(x)」所形成的集合，「對應域 W」那這一端。

Theorem 2.3 (Dimension Theorem)

Let V and W be vector spaces, and let T: V→W be linear. If V is finite-dimensional, then **Nullity(T)+rank(T) = dim(V)**

T 的核次數（Nullity）與秩（Rank）的和，等於定義域的維度 **dim(V)**。

【Proof】

Suppose that dim(V) = n, dim(N(T)) = k, and **$\{v_1, v_2,\ldots, v_k\}$ is a basis for N(T)**. By the corollary to Theorem 1.11, we may extend $\{v_1, v_2,\ldots, v_k\}$ to a basis **$\beta = \{v_1, v_2,\ldots, v_n\}$** for V.

We claim that **$S = \{T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)\}$ is a basis for R(T)**.

First we prove that S generates R(T). Using Theorem 2.2 and the fact that $T(v_i) = 0$ for $1 \leqq i \leqq k$, we have

$R(T) = \text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\}) = \text{span}(\{T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)\}) = \text{span}(S)$

Now we prove that S is linear independent. Suppose that

$\sum_{i=k+1}^{n} b_i T(v_i) = 0$ for $b_{k+1}, b_{k+2},\ldots, b_n \in F$.

Using the fact that T is linear, we have $T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$

So $\sum_{i=k+1}^{n} b_i v_i \in N(T)$

Hence there exist $c_1, c_2,\ldots, c_k \in F$ such that

$\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$ or $\sum_{i=k+1}^{n} b_i v_i + \sum_{i=1}^{k} (-c_i) v_i = 0$

Since β is a basis for V, we have $b_i = 0$ for all i. Hence S is linearly independent. Notice that this argument also shows that $T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)$ are distinct; therefore rank(T) = n-k.

假設 dim(V) = n、dim(N(T)) = k 且$\{v_1, v_2,\ldots, v_k\}$是 N(T)的基底。依據 Theorem 1.11 的推論得知，可將$\{v_1, v_2,\ldots, v_k\}$ 拓展至 $\beta = \{v_1, v_2,\ldots, v_n\}$，使得 **$\beta = \{v_1, v_2,\ldots, v_n\}$**成為 V 的基底。【N(T)是 V 的子空間。】

要證明者即為 $S = \{T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)\}$是否為 R(T)的基底？

首先證明 S 可生成 R(T)。

利用 Theorem 2.2 與「$T(v_i) = 0$ for $1 \leqq i \leqq k$」（$\{v_1, v_2,\ldots, v_k\}$是 N(T)的基底；故依 Null space 定義，$T(v_i) = 0$ for $1 \leqq i \leqq k$）的事實，得知：

$R(T) = \text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$（Theorem 2.2）

$= \text{span}(\{T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)\}) = \text{span}(S)$（$\{v_1, v_2,\ldots, v_k\}$是 N(T)的基底；故依 Null space 定義，$T(v_i) = 0$ for $1 \leqq i \leqq k$）

其次證明 $S = \{T(v_{k+1}), T(v_{k+2}),\ldots,T(v_n)\}$為線性獨立。

假設 $\sum_{i=k+1}^{n} b_i T(v_i) = 0$ for $b_{k+1}, b_{k+2}, \ldots, b_n \in F$。

並利用 T 是線性的事實，得知$T\left(\sum_{i=k+1}^{n} b_i v_i\right) = 0$。

依據 Null space N(T)定義：T(x) = 0，即 Null space 內的元素的像皆為 0。

故 $\sum_{i=k+1}^{n} b_i v_i \in N(T)$。

又因$\{v_1, v_2,\ldots, v_k\}$是 N(T)的基底。

因此，存在 $c_1, c_2, \ldots, c_k \in F$ 使得 $\sum_{i=k+1}^{n} b_i v_i = \sum_{i=1}^{k} c_i v_i$ 或 $\sum_{i=k+1}^{n} b_i v_i + \sum_{i=1}^{k} (-c_i) v_i = 0$ 。

由於 β 是 V 的基底，為線性獨立子集，故 $b_i = 0$，意即 $S = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ 為線性獨立。因此 rank(T) = n - k。

Corollary to Theorem 1.11 If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V. W 是 V 的子空間，則 W 的任一基底可以延伸為 V 的基底。

Theorem 2.2 Let V and W be vector spaces, and let T: V→W be linear. If $\beta = \{v_1, v_2,\ldots, v_n\}$ is basis for V, then $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2),\ldots,T(v_n)\})$. 令 V 與 W 為向量空間，且 T: V→W 為線性轉換。若 $\beta = \{v_1, v_2,\ldots, v_n\}$是定義域（Domain）V 的基底，則 T 的值域 Range R(T)可由 T(β)來生成。即 $R(T) = \text{span}(T(\beta)) = \text{span}(\{T(v_1), T(v_2),\ldots, T(v_n)\})$。

Theorem 2.4 (One-to-one)

Let V and W be vector spaces, and let T: V→W be linear. Then T is one-to-one if and only if N(T) = {0}.

V 與 W 為向量空間，且 T: V→W 為線性轉換。則 T 是一對一「若且唯若」N(T) = {0}。

【Proof】

T is one-to-one→N(T) = {0}

Suppose that T is one-to-one and $x \in N(T)$. Then T(x) = 0 = T(0).

Since T is one-to-one, we have x = 0. Hence N(T) = {0}.

T 是一對一→N(T) = {0}

依據 Null space 的定義：T(x) = 0，即 Null space 內的元素的像皆為 0。

再依據 T 是線性轉換的 Property：若 T 是線性轉換，則 T(0) = 0。

若 T 是線性且 $x \in N(T)$，則 T(x) = 0 = T(0)。

若 T 是一對一，則 x = 0，即 N(T) = {0}。

N(T) = {0}→T is one-to-one

Now assume that N(T) = {0}, and suppose that T(x) = T(y).

Then 0 = T(x) - T(y) = T(x - y).

Therefore $x,y \in N(T) = \{0\}$. So $x - y = 0$, or x = y.

This means that T is one-to-one.

N(T) = {0}→T 是一對一

依據 T 是線性轉換的 Property：若 T 是線性轉換，則 T(x - y) = T(x) - T(y) 。

假設 N(T) = {0}且 T(x) = T(y)，則 0 = T(x) - T(y) = T(x-y)。

若 x,y∈N(T) = {0}，則 x－y = 0 或 x = y，即 T 是一對一。

Theorem 2.5

Let V and W be vector spaces of **equal (finite) dimension**, and let T: V→W be linear. Then the following are equivalent:

V 與 W 為向量空間，具有相等的**維度**，且 T: V→W 為線性轉換，則下列敘述等價：

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

ONTO：當 f：A→B，若 f(A) = B，則稱 f 為映成（ONTO）函數，故 f 是映成若且唯若 f 的值域等於對應域。

【Proof】

From Theorem 2.3 (dimension theorem), we have **nullity(T)+rank(T) = dim(V).**

Now, with the use of Theorem 2.4, we have that T is one-to-one if and only if **N(T) = {0}**, if and only if **nullity(T) = 0**, if and only if **rank(T) = dim(V)**, if and only if **rank(T) = dim(W)**, and if and only if **dim(R(T)) = dim(W)**. By Theorem 1.11, this equality is equivalent to **R(T) = W**, the definition of **T being onto**.

由 Theorem 2.3＋Theorem 2.4 → Rank(T) = dim (V)。由定理 1.11 dim(R(T)) = dim (W)→ R(T) = W。

Theorem 2.3 Let V and W be vector spaces, and let T: V→W be linear. If V is finite-dimensional, then Nullity(T)+rank(T) = dim(V). T 的核次數（Nullity）與秩（Rank）的和，等於定義域的維度 dim(V)。

Theorem 1.11 Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and dim(W)≦dim(V). Moreover, if dim(W) = dim(V), then V = W. 設 W 為有限維度的向量空間 V 內的子空間，則 W 為有限維度且 dim(W) ≦dim(V)。若 dim(W) = dim(V)，則 W = V。

If T is linear and one-to-one, then a subset S is linearly independent if and only if T(S) is linearly independent.

若 T 為線性且為一對一，則定義域內子集合 S 為線性獨立「若且唯若」條件為 T(S)是線性獨立。

We note that if V is not finite-dimensional and T: V→V is linear, then it does not follow that one-to-one and onto are equivalent.

若 V 不是有限且 T: V→V 為線性，則一對一與映成並非為等價。

The linearity of T in Theorem 2.4 and 2.5 is essential. For it is easy to construct examples of functions from R into R that are not one-to-one, but onto, and vice versa.

在 Theorem 2.4 與 2.5 中，「T 為線性」是必備條件，因為吾人可容易舉出一些由 R 映至 R 的函數，非「一對一」卻為「映成」者。反之亦然。

The next two examples make use of the proceeding theorems in determining whether a given linear transformation is one-to-one or onto.

以下兩個例子將利用前述定理來判斷已知線性轉換是「一對一」或「映成」。

EXAMPLE 8

Let T: $P_2(R)$→$P_3(R)$ be the linear transformation defined by

$T(f(x)) = 2f'(x)+\int_0^x 3f(t)dt$

Now R(T) = span({T(1), T(x), T(x^2) }) = span ({3x, 2+3x^2/2, 4x+x^3}).

Since {3x, 2+3x^2/2, 4x+x^3} is linearly independent, dim(R(T)) = 3= rank (T).

Since dim ($P_3(R)$) = 4, T is not noto.

From the dimension theorem, **nullity(T)+3 = dim(P_2) = 3**. So nullity(T) = 0, and therefore, N(T) = {0}. We conclude from Theorem 2.4 that T is one to one.

定義 T: $P_2(R)$→$P_3(R)$為 $T(f(x)) = 2f'(x)+\int_0^x 3f(t)dt$ 。

$P_2(R)$ 的基底為{1, x, x^2}，得知 R(T) = span({T(1), T(x), T(x^2) }) = span ({3x, 2+3x^2/2, 4x+x^3}).

因{3x, 2+3x^2/2, 4x+x^3}為線性獨立，故 dim(R(T)) = 3= rank (T)。【先交代 dim

(R(T)) = 3 再交代 rank(T) = 3。】

然因 dim (P_3(R)) = 4，故 T 非 ONTO。【T 為 ONTO 的條件是 dim(P_3(R)) = dim(R(T))。】

依據 Theorem 2.3 (Dimension Theorem)： nullity(T) + 3 = dim(P_2) = 3，故 nullity(T) = 0 且 N(T) = {0}，T 為一對一。【dim(V) = dim(P_2) = 3。】

Theorem 2.3 Let V and W be vector spaces, and let T: V→W be linear. If V is finite-dimensional, then Nullity(T) + rank(T) = dim(V). T 的核次數（Nullity）與秩（Rank）的和，等於定義域的維度 dim(V)。

註：P_2(R)與 P_3(R)的維度不同，不可引用 Theorem 2.5。

Theorem 2.4 T 是一對一「若且唯若」N(T) = {0}。

EXAMPLE 9

Let T: F^2→F^2 be the linear transformation defined by

T(a_1, a_2) = (a_1+a_2, a_1)

It is easy to see that N(T) = {0}; so T is one-to-one. Hence Theorem 2.5 tells us that T must be onto.

令 T: F^2→F^2 為線性轉換且 T(a_1, a_2) = (a_1+a_2, a_1) 。依據 Null space 的定義：N(T) = {x∈V: T(x) = 0} = {0}，所以 T 為 one-to-one（依據 Theorem 2.4）。再依據 Theorem 2.5，T 為 ONTO。

Theorem 2.4 Let V and W be vector spaces, and let T: V→W be linear. Then T is one-to-one if and only if N(T) = {0}. V 與 W 為向量空間，且 T: V→W 為線性轉換。則 T 是一對一「若且唯若」N(T) = {0}。

Theorem 2.5 V 與 W 為向量空間，具有相等的維度，且 T: V→W 為線性轉換，則下列敘述等價：

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

EXAMPLE 10

Let T: P_2(R)→R^3 be the linear transformation defined by

T(a_0 + a_1x + a_2x^2) = (a_0, a_1, a_2)

Clearly T is linear and one-to-one.

依據 Null space 的定義：$N(T) = \{x \in V: T(x) = 0\} = \{0\}$，故 T 是 one-to-one。

Theorem 2.4 T 是一對一「若且唯若」$N(T) = \{0\}$

Let $S = \{2-x+3x^2, x+x^2, 1-2x^2\}$. Then S is linearly independent in $P_2(R)$ because $T(S) = \{(2, -1, 3), (0, 1, 1), (1, 0, -2)\}$ is linearly independent in R^3.

令 $S = \{2-x+3x^2, x+x^2, 1-2x^2\}$，因 $T(S) = \{(2, -1, 3), (0, 1, 1), (1, 0, -2)\}$為線性獨立的集合，故 S 為 $P_2(R)$內線性獨立的子集合。

NOTE: If T is linear and one-to-one, then a subset S is linearly independent if and only if T(S) is linearly independent.

If T is linear and one-to-one, then a subset S is linearly independent if and only if T(S) is linearly independent.

若 T 為線性且為一對一，則定義域內子集合 S 為線性獨立「若且唯若」條件為 T(S)是線性獨立。

Theorem 2.6

Let V and W be vector spaces over F, and suppose that $\{v_1, v_2,\ldots, v_n\}$ is a basis for V. For $w_1, w_2,\ldots, w_n$ in W, there exists exactly one linear transformation T:V→W such that $T(v_i) = w_i$ for $i = 1, 2,\ldots, n$.

令 V 與 W 為佈於 F 的向量空間。設 $\{v_1, v_2, \ldots, v_n\}$為 V 的基底，對 W 內的 $w_1, w_2, \ldots, w_n$ 而言，存在一由 V 映至 W 的線性轉換 T，使得 $T(v_i) = w_i$。

【Proof】

Let $x \in V$ Then

$x = \sum_{i=1}^{n} a_i v_i$, where $a_1, a_2,\ldots, a_n$ are unique scalars.

V 內任一向量 x 可以表達為基底$\{v_1, v_2,\ldots, v_n\}$的線性組合。

Define T:V→W by $T(x) = \sum_{i=1}^{n} a_i w_i$

定義由 V 映至 W 的線性轉換 T 為$T(x) = \sum_{i=1}^{n} a_i w_i$

(a) T is linear （先證明 T 是線性轉換）

Suppose that $u, v \in V$ and $d \in F$.

設 u 與 v 為 V 任意向量。

Then we may have

$u = \sum_{i=1}^{n} b_i v_i$ and $v = \sum_{i=1}^{n} c_i v_i$ for some scalar $b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_n$.

u 與 v 可表達成基底的線性組合。

Thus $du + v = \sum_{i=1}^{n} (db_i + c_i) v_i$

So, T(du+v) = $T(du + v) = \sum_{i=1}^{n} (db_i + c_i) w_i = dT(u) + T(v)$

依據線性轉換的 Property 2，T 為線性轉換。

(b) Clearly $T(v_i) = w_i$ for i = 1,2,…,n.

(c) T is unique（再證明 T 是唯一）

Suppose that U: V→W is linear and $U(v_i) = w_i$ for i = 1, 2,…, n.

假設還有另一個線性轉換 U: V→W 且 $U(v_i) = w_i$ for i = 1, 2,…, n。

Then for $x \in V$ with $x = \sum_{i=1}^{n} a_i v_i$

we have $U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = T(x)$

Hence U = T.

Corollary

Let V and W be vector spaces over F, and suppose that V has a finite basis $\{v_1, v_2, \ldots, v_n\}$. If U, T: V→W are linear and $U(v_i) = T(v_i)$ for i = 1,2,…,n. then U = T.

令 V 與 W 為兩個佈於 F 的向量空間，設 V 具有一個有限的基底$\{v_1, v_2, \ldots, v_n\}$。若兩個由 U 映至 W 的線性轉換 U 與 T 相等且 $U(v_i) = T(v_i)$ for i = 1, 2,…, n，則 U = T 。

EXAMPLE 11

Let T: R^2→R^2 be the linear transformation defined by

$T(a_1, a_2) = (2a_2 - a_1, 3a_1)$

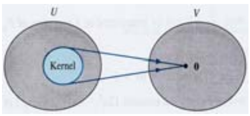
and suppose that U: R^2→R^2 is linear. If we know that U(1, 2) = (3, 3) and U(1, 1) = (1, 3), then U = T.

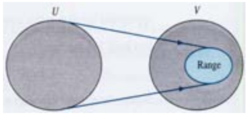
This follows from the corollary and from the fact that { (1, 2), (1, 1)} is a basis for R^2.

令 T: $R^2 \rightarrow R^2$ 為線性轉換並定義為 $T(a_1, a_2) = (2a_2 - a_1, 3a_1)$。另令 U: $R^2 \rightarrow R^2$ 為線性轉換。若已知 U(1, 2) = (3, 3) 且 U(1, 1) = (1, 3)，則 U = T。由推論與事實可知 {(1, 2), (1, 1)} 是 R^2 的一組基底。NOTE：T(1, 2) = (3, 3) 且 T(1, 1) = (1, 3)

Kernel and Range

- Let T: U → V be a linear transformation.
- The set of vectors in U that are mapped into the zero vector of V is called the **kernel** of T. The kernel is denoted ker(T).
- The set of vectors in V that are the images of vectors in U is called **range** of T. The image is denoted range(T).





Theorem

- Let T: U → V be a linear transformation. Let $\mathbf{0}_V$ and $\mathbf{0}_U$ be the zero vectors of U and V. Then

  $T(\mathbf{0}_U) = \mathbf{0}_V$

  That is, a linear transformation maps a zero vector into vector.

Proof

Let **u** be a vector in U and let T(**u**) = **v**.

Let **0** be the zero scalar. Since $0\mathbf{u} = \mathbf{0}_U$ and $0\mathbf{v} = \mathbf{0}_V$ and T is linear, we get

$T(\mathbf{0}_U) = T(0\mathbf{u}) = 0T(\mathbf{u}) = 0\mathbf{v} = \mathbf{0}_v$

Theorem

- Let T: U → V be a linear transformation.
  - ⇨The kernel of T is a subspace of U.
  - ⇨The range of T is a subspace of V.

Theorem

- Let T: U → V be a linear transformation. Let $\mathbf{0}_V$ and $\mathbf{0}_U$ be the zero vectors of U and V. Then

  $T(\mathbf{0}_U) = \mathbf{0}_V$

  That is, a linear transformation maps a zero vector into vector.

Proof

Let **u** be a vector in U and let T(**u**) = **v**.

Let **0** be the zero scalar. Since $0\mathbf{u} = \mathbf{0}_U$ and $0\mathbf{v} = \mathbf{0}_V$ and T is linear, we get

$T(\mathbf{0}_U) = T(0\mathbf{u}) = 0T(\mathbf{u}) = 0\mathbf{v} = \mathbf{0}_v$

Example 1/2

- Find the kernel and range of the linear operator $T(x, y, z) = (x, y, 0)$

Since the operator T maps $\mathbf{R}^3$ into $\mathbf{R}^3$, the kernel and range will both be subspaces of $\mathbf{R}^3$.

Kernel: ker(T) is the subset that is mapped into (0, 0, 0).

$T(x, y, z) = (x, y, 0) = (0, 0, 0)$, if x = 0, y = 0

Thus ker(T) is the set of all vectors of the form (0, 0, z). We express this as

$$\ker(T) = \{(0,0,z)\}$$

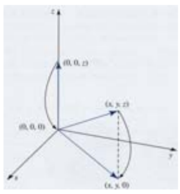
Geometrically, ker(T) is the set of all vectors that lie on the z axis.

Example 2/2

Range: The range of *T* is the set of all vectors of the form (*x*, *y*, 0).

$$\text{range}(T) = \{(x, y, 0)\}$$

Range(*T*) is the set of all vectors that lie in the *xy* plane.



Projection $f(x, y, z) = (x, y, 0)$

2.2 The Matrix Representation of a Linear Transformation

In this section, we embark on one of the most useful approaches to the analysis of a linear transformation on a finite-dimensional vector space: the representation of a linear transformation by a matrix. In fact, we develop a one-to-one correspondence between matrices and linear transformation that allows us to utilize properties of one to study properties of the other.

利用矩陣來表達線性轉換一是分析有限維度向量空間線性轉換的最有用方法。

DEFINITION 2.10

Let V be a finite-dimensional vector space. An ordered basis for V is a basis for V endowed with a specific order; that is, an ordered basis for V is a finite sequence of linearly independent vectors in V that generates V.

令 V 是一有限維度的空間向量。V 的有序基底為一組賦予一定次序的基底。V 的有序基底指 V 中可生成 V 的一組線性獨立、有限序列（Finite sequence）向量。

EXAMPLE 1

In R^3, $\beta = \{e_1, e_2, e_3\}$ can be considered an ordered basis. Also $\gamma = \{e_2, e_1, e_3\}$ is an ordered basis, but $\beta \neq \gamma$ as ordered bases.

$\beta = \{e_1, e_2, e_3\}$是有序基底。$\{e_2, e_1, e_3\}$也是有序基底。同樣是有序基底，但$\beta \neq \gamma$。

For the vector space F^n, we call $\{e_1, e_2, \ldots, e_n\}$ the standard ordered basis for F^n. Similarly, for the vectors space $P_n(F)$, we call $\{1, x, \ldots, x^n\}$ the standard ordered basis for $P_n(F)$.

$\{e_1, e_2, \ldots, e_n\}$是 F^n 的標準有序基底，$\{1, x, \ldots, x^n\}$也是 $P_n(F)$ 的標準有序基底。

DEFINITION 2.11

Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vectors space V. For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique scalars such that

令 $\beta = \{u_1, u_2, \ldots, u_n\}$為有限向量空間 V 的有序基底，則存在唯一的一組純量 $a_1, a_2, \ldots, a_n$，使得 V 內任意向量，均可表達成該組基底的線性組合：

$$x = \sum_{i=1}^{n} a_i u_i$$

we define the coordinate vector of x relative to β, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$ **稱為 x 相對於有序基底 β 的座標向量。**

EXAMPLE 2

Let V = $P_2(R)$, and let β = {1, x, x^2} be the standard ordered basis for V, If f(x) = 4+6x-7x^2, then

令 V = $P_2(R)$且 β = {1, x, x^2}為 V 的標準有序基底。將 f(x) = 4+6x-7x^2 表達成 β = {1, x, x^2}的線性組合 f(x) = 4 + 6x - 7x^2

$$[f]_\beta = \begin{pmatrix} 4 \\ 6 \\ -7 \end{pmatrix}$$ 稱為 **f(x)** 相對於有序基底 **β = {1, x, x^2}** 的座標向量。

DEFINITION 2.12

Suppose that V and W are finite-dimensional vector spaces with ordered bases β = {v_1, v_2,… , v_n} and γ = {w_1, w_2,… , w_n}, respectively. Let T: V→W be linear. Then for each j, 1 ≦j≦n, there exist unique scalars a_{ij}∈F, 1≦i≦m, such that

$$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i \quad \text{for } 1 \leqq j \leqq n$$

We call the m×n matrix A defined by $A_{ij} = a_{ij}$ the matrix representation of T in the ordered bases β and γ and write $A = [T]_\beta^\gamma$.

If V = W and β = γ, then we write $A = [T]_\beta$.

設 V 與 W 分別為有限維度的向量空間，β = {v_1, v_2,… , v_n}與 γ = {w_1, w_2,… , w_n}分別為 V 與 W 的有序基底，且 T: V→W 為 V 映至 W 的線性轉換。

對每一個 j（1≦j≦n）而言，存在唯一的純量 a_{ij} ∈F（1≦i≦m），使得 $T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ 。

將 m×n 的矩陣 A 定義為 $A_{ij} = a_{ij}$，並稱呼 A 為線性轉換 T 的矩陣表達方式。在 V 與 W 分別以 β 與 γ 作為有序基底下，A 可註記為 $A = [T]_\beta^\gamma$ 。

若 V = W 且 β = γ，則 $A = [T]_\beta$ 。

提示：$T(v_j)=\sum_{i=1}^{m} a_{ij}w_i$ 為「將 v_j 的像 $T(v_j)$表達成有序基底 γ 的線性組合」。

The jth column of A is simply $[T(v_j)]_\gamma$.

矩陣 A 的第 j 行，註記為$[T(v_j)]_\gamma$。

By corollary to Theorem 2.6: If U,T: V→W is a linear transformation such that $[T]_\beta^\gamma=[U]_\beta^\gamma$, then U = T.

依據 Theorem 2.6 的推論，若 U,T: V→W 為一線性轉換且$[T]_\beta^\gamma=[U]_\beta^\gamma$，則 U = T。

由 Example 3 即可看出 A = $[T]_\beta^\gamma$ 的第 jth Column 所對應者為 v_j。若只要第 j 行，則只須代入 v_j，求出 $T(v_j)$相對於 γ 的座標向量$[T(v_j)]_\gamma$。

EXAMPLE 3

Let T: $R^2 \to R^3$ be the linear transformation defined by

$T(a_1, a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$

Let β and γ be the standard ordered bases for R^2 and R^3, respectively. Now

$\beta=\{(1,0), (0,1)\}$ $\gamma = \{(e_1,e_2,e_3)\}$

令 T: $R^2 \to R^3$ 線性轉換且定義為 $T(a_1, a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$ 。

令 $\beta=\{v_1, v_2\} = \{(1,0), (0,1)\}$ 與 $\gamma = \{(e_1,e_2,e_3)\}$ 分別為 R^2 與 R^3 的標準有序基底。並將 $T(v_1)$與 $T(v_2)$ 表達成有序基底 $\gamma = \{(e_1, e_2, e_3)\}$的線性組合：

$T(v_1) = T(1, 0) = (1, 0, 2) = 1e_1 + 0e_2 + 2e_3$

$T(v_2) = T(0, 1) = (3, 0, -4) = 3e_1 + 0e_2 - 4e_3$

Hence $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix}$ （1^{st} 行代表 v_1，2^{nd} 行代表 v_2）

If we set $\gamma' = \{(e_3,e_2,e_1)\}$, then $[T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$

若 γ 變成 γ’， $[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix} \to [T]_\beta^{\gamma'} = \begin{pmatrix} 2 & -4 \\ 0 & 0 \\ 1 & 3 \end{pmatrix}$

EXAMPLE 4

Let T: $P_3(R) \rightarrow P_2(R)$ be the linear transformation defined by $T(f(x)) = f'(x)$. Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ be the standard ordered bases for $P_3(R)$ and $P_2(R)$, respectively. Then

令 T: $P_3(R) \rightarrow P_2(R)$為線性轉換且定義為 $T(f(x)) = f'(x)$。

令 $\beta = \{1, x, x^2, x^3\}$ 與 $\gamma = \{1, x, x^2\}$ 分別為 $P_3(R)$與 $P_2(R)$的標準有序基底。

將 T(1)、T(x)、$T(x^2)$與 $T(x^3)$ 表達成有序基底 $\gamma = \{1, x, x^2\}$的線性組合：

$T(1) = 0\cdot 1+0\cdot x+0\cdot x^2$

$T(x) = 1\cdot 1+0\cdot x+0\cdot x^2$

$T(x^2) = 0\cdot 1+2\cdot x+0\cdot x^2$

$T(x^3) = 0\cdot 1+0\cdot x+3\cdot x^2$

$[T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$（$1^{st}$ 行代表 $v_1 = 1$，2^{nd} 行代表 $v_2 = x$，3^{rd} 行代表 $v_3 = x^2$，4^{th} 行代表 $v_4 = x^3$）

DEFINITION 2.13 T+U & aT

Let T,U: V→W be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define

T+U: V→W by $(T+U)(x) = T(x)+U(x)$ for all $a \in F$, and

aT: V→W by $(aT)(x) = aT(x)$ for all $x \in V$.

Theorem 2.7

Let V and W be vector spaces over a field F, and let T, U: V→W be linear.

(a) For all $a \in F$, aT+U is linear

(b)Using the operations of addition and scalar multiplication in the preceding definition, the collection of all linear transformations from V to W is a vector space over F.

V 與 W 為佈於 F 的向量空間，且 T, U: V→W 為線性轉換。則

（a）對所有 $a \in F$ 而言，aT＋U 為線性轉換。

（b）利用 Definition 2.13 所定義的加法與純量乘積運算，所有由 U 到 W 的線性

轉換所形成的集合，為佈於 F 的空間向量。

【Proof】

依據 Defintion 2.13：T(cx+y) = cT(x) + T(y) for all x,y ∈V and c∈F。

aT+U 為線性？

Let x,y∈V and c∈F. Then

(aT + U)(cx + y) =a T(cx + y) + U(cx + y) =a[cT(x)+ T(y)] +cU(x) + U(y) = acT(x) + cU(x) +aT(y) + u(y)= c(aT+U)(x)+(aT+U)(y)

So aT + U is linear.

DEFINITION 2.14 ₤(V)

Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by ₤(V, W). In the case that V = W, we write ₤(V) instead of ₤(V, W).

令 V 與 W 為佈於 F 的向量空間。將所有 V 映至 W 的線性轉換所形成的向量空間註記為 ₤(V, W)。當 V = W，則將₤(V, W) 改寫成 ₤(V)。

由 Definition 2.13 得知「**all linear transformations from V to W is a vector space over F**」。

Theorem 2.8

Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively, and let T, U: V→W be linear transformations. Then

令 V 與 W 分別為有限維度的向量空間，β = {v_1, v_2,… , v_n}與 γ = {w_1, w_2,… , w_n}分別為 V 與 W 的有序基底，且令 T, U: V→W 為線性轉換，則

(a) $[T+U]_\beta^\gamma = [T]_\beta^\gamma + [U]_\beta^\gamma$ and

(b) $[aT]_\beta^\gamma = a[T]_\beta^\gamma$ for all scalars a.

【Proof】

Let β = {v_1, v_2,… , v_n} and γ = {w_1, w_2, …, w_n}.

There exist unique scalar a_{ij} and b_{ij} ($1 \leqq i \leqq m$, $1 \leqq j \leqq n$) such that

$T(v_j) = \sum_{i=1}^{m} a_{ij} w_i$ $U(v_j) = \sum_{i=1}^{m} b_{ij} w_i$ for $1 \leqq j \leqq n$ （Definition 2.12）

Hence $(T+U)(v_j) = T(v_j) + U(v_j) = \sum_{i=1}^{m} (a_{ij} + b_{ij}) w_i$ （Definition 2.13）

Thus $([T+U]_\beta^\gamma)_{ij} = a_{ij} + b_{ij} = ([T]_\beta^\gamma + [U]_\beta^\gamma)_{ij}$

Definition 2.13 Let T,U: V→W be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define

(a) T+U: V→W by (T+U)(x) = T(x)+U(x) for all $a \in F$, and

(b) aT: V→W by (aT)(x) = aT(x) for all $x \in V$.

EXAMPLE 5

Let T: R^2→R^3 and U: R^2→R^3 be the linear transformation respectively defined by

$T(a_1,a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$ and $U(a_1,a_2) = (a_1-a_2, 2a_1, 3a_1+2a_2)$.

Let β and γ be the standard ordered bases of R^2 and R^3, respectively. Then

令 T: R^2→R^3 與 U: R^2→R^3 皆為線性轉換且分別定義為：

$T(a_1,a_2) = (a_1+3a_2, 0, 2a_1-4a_2)$

$U(a_1,a_2) = (a_1-a_2, 2a_1, 3a_1+2a_2)$

令 β={(1,0), (0,1)} 與 γ = {(1, 0, 0),(0, 1, 0),(0, 0, 1)} 分別為 R^2 與 R^3 的有序基底。

T(1,0) = (1, 0, 2), T(0,1) = (3, 0, -4)

U(1,0) = (1, 2, 3), U(0,1) = (-1, 0, 2)

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 3 \\ 0 & 0 \\ 2 & -4 \end{pmatrix} \quad \text{and} \quad [U]_\beta^\gamma = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 2 \end{pmatrix}$$

If we compute T+U using the preceding definitions, we obtain

$(T+U)(a_1,a_2) = (a_1+3a_2, 0, 2a_1-4a_2) + (a_1-a_2, 2a_1, 3a_1+2a_2) = (2a_1+2a_2, 2a_1, 5a_1-2a_2)$

$$\text{So } [T+U]_\beta^\gamma = \begin{pmatrix} 2 & 2 \\ 2 & 0 \\ 5 & -2 \end{pmatrix}$$

Which is simply $[T]_\beta^\gamma + [U]_\beta^\gamma$

Definition 2.13 Let T,U: V→W be arbitrary functions, where V and W are vector spaces over F, and let $a \in F$. We define

(a) T+U: V→W by (T+U)(x) = T(x)+U(x) for all $a \in F$, and

(b) aT: V→W by (aT)(x) = aT(x) for all $x \in V$.

2.3 Composition of Linear Transformation and Matrix Multiplication

How the matrix representation of a composite of linear transformations is related to the matrix representation of each of the associated linear transformations. We use the more convenient notation of UT rather than U・T for the composite of linear transformations U and T.

之前已經介紹如何結合線性轉換與矩陣，如何以矩陣的和及純量乘積代表線性轉換的和及純量乘積。現在要介紹如何利用矩陣來代表線性轉換的合成。**利用 UT 來表示線性轉換 U 與 T 的合成 U・T。**

令 A、B、C 皆為集合且 f：A→B 且 g：B→C 皆為函數，則『g・f（Following f with g；g 接著 f 得到的函數，先 f 後 g）：A→C』稱為 g 與 f 的合成（Composition）函數。（g・f）(x) = g (f(x))，$\forall x \in$ A。

Theorem 2.9

Let V, W, and Z be vector spaces over the same field F, and let T: V→W and U: W→Z be linear. Then UT: V→Z is linear.

令 V、W 與 Z 為佈於 F 的向量空間，且 T: V→W、U:W→Z 分別為線性轉換，則 U 與 T 的合成 UT: V→Z 為線性轉換。

【Proof】

Let $x,y \in V$ and $a \in F$. Then

$UT(ax+y) = U(T(ax + y)) = U(aT(x)+T(y)) = aU(T(x)) + U(t(y)) = a(UT)(x) + UT(y)$

Theorem 2.10

Let V be a vector space. Let $T, U_1, U_2 \in £(V)$. Then

令 V 為向量空間，且 $T, U_1, U_2 \in £(V)$，即 T, U_1, U_2 都是由 V 映至 V 的線性轉換，則：

(a) $T(U_1 + U_2) = TU_1 + TU_2$ and $(U_1 + U_2)T = U_1T + U_2T$.

(b) $T(U_1U_2) = (TU_1)U_2$

(c) $TI = IT = T$.

(d) $a(U_1U_2) = (aU_1)U_2 = U_1(aU_2)$ for all scalars a.

註：令 V 與 W 為佈於 F 的向量空間。將所有 V 映至 W 的線性轉換所形成的向量空間註記為 $\pounds(V, W)$。

Let T: V→W and U: W→Z be linear transformations, and let $A = [U]_\beta^\gamma$ and $B = [T]_\alpha^\beta$, where $\alpha = \{v_1, v_2, \ldots, v_n\}$, $\beta = \{w_1, w_2, \ldots, w_m\}$, and $\gamma = \{z_1, z_2, \ldots, z_p\}$ are ordered bases for V, W, and Z, respectively.

令 T: V→W 與 U: W→Z 為線性轉換，且 $A = [U]_\beta^\gamma$ 、$B = [T]_\alpha^\beta$；其中，$\alpha = \{v_1, v_2, \ldots, v_n\}$、$\beta = \{w_1, w_2, \ldots, w_m\}$與 $\gamma = \{z_1, z_2, \ldots, z_p\}$分別為 V、W、Z 的有序基底。

We would like to define the product AB for two matrices so that $AB = [UT]_\alpha^\gamma$.

矩陣 A 與矩陣 B 的乘積為 AB，$AB = [UT]_\alpha^\gamma$。

Consider the matrix $[UT]_\alpha^\gamma$. For $1 \leqq j \leqq n$, we have

$$
\begin{aligned}
(UT)(v_j) = U(T(v_j)) &= U\left(\sum_{k=1}^{m} B_{kj} w_k\right) = \sum_{k=1}^{m} B_{kj} U(w_k) \\
&= \sum_{k=1}^{m} B_{kj}\left(\sum_{i=1}^{p} A_{ik} z_i\right) = \sum_{i=1}^{p}\left(\sum_{k=1}^{m} A_{ik} B_{kj}\right) z_i = \sum_{i=1}^{p} C_{ij} z_i
\end{aligned}
\quad \text{for } 1 \leqq j \leqq n
$$

where $C_{ij} = \sum_{k=1}^{m} A_{ik} B_{kj}$

DEFINITION 2.15 Product of A and B

Let A be an **m×n** matrix and B be an **n×p** matrix. We define the product of A and B, denoted AB, to be the **m×p** matrix such that

A 為 **m×n** 的矩陣與 B 為 **n×p** 的矩陣，則 AB 的乘積（Product of A and B）為 **m×p** 的矩陣。

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj} \quad \text{for } 1 \leqq i \leqq m,\ 1 \leqq j \leqq p$$

Recalling the definition of the transpose of a matrix. If A is an m×n matrix and B is an n×p matrix, then $(AB)^t = B^t A^t$.

轉置矩陣的定義：$(AB)^t = B^t A^t$

Since $(AB)^t_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki}$ and $(B^t A^t)_{ij} = \sum_{k=1}^{n} (B^t)_{ik} (A^t)_{kj} = \sum_{k=1}^{n} B_{ki} A_{jk}$

The transpose of a product is the product of the transpose in the opposite order.

→矩陣乘積的轉置為矩陣轉置並以相反次序相乘：$\mathbf{(AB)^t = B^t A^t}$。

Theorem 2.11

Let V, W, and Z be finite-dimensional vector space with ordered bases α, β, and γ, respectively. Let T: V→W and U: W→Z be linear transformation. Then $[UT]^\gamma_\alpha = [U]^\gamma_\beta [T]^\beta_\alpha$

令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 T: V→W（先 α→β） 且 U: W→Z（後 β→ γ），則 $[UT]^\gamma_\alpha = [U]^\gamma_\beta [T]^\beta_\alpha$。

提示：令 T: V_α→W_β（先） 且 U: W_β→Z_γ（後）。

Corollary

Let V be finite-dimensional vector space with ordered basis β. Let T, U∈£ (V). Then $[UT]_\beta = [U]_\beta [T]_\beta$

令 V 為有限維度的向量空間，β 為 V 的有序基底。令 T 與 U∈£ (V)，即 T 與 U 都是由 V 映至 V 的線性轉換，則 $[UT]_\beta = [U]_\beta [T]_\beta$（UT 合成後相對有序基底 β 的座標向量等於 U 與 T 分別相對有序基底 β 的座標向量的乘積）。

EXAMPLE 1

Let U: $P_3(R)$→ $P_2(R)$ and T: $P_2(R)$→ $P_3(R)$ be the linear transformations respectively defined by

U(f(x)) = f'(x) and $T(f(x)) = \int_0^x f(t)dt$

Let α and β be the standard ordered bases of $P_3(R)$ and $P_2(R)$, respectively. From calculus, it follows that UT = I, the identity transformation on $P_2(R)$. To illustrate Theorem 2.11, observe that

令 U: $P_3(R)$→ $P_2(R)$ 及 T: $P_2(R)$→ $P_3(R)$ 分別定義為

U(f(x)) = f'(x)

$T(f(x)) = \int_0^x f(t)dt$

令 $\alpha = \{1, x, x^2, x^3\}$ 與 $\beta = \{1, x, x^2\}$分別為 $P_3(R)$ 與 $P_2(R)$的標準有序基底。$UT = I$。

$$[UT]_\beta = [U]_\alpha^\beta [T]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = [I]_\beta$$

（ 3×3 diagonal matrix is called an identity matrix（單位矩陣））

DEFINITION 2.16

We define the kronecker delta δ_{ij} by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. The n×n identity matrix I_n is defined by $(I_n)_{ij} = \delta_{ij}$.

Kronecker delta 的定義及 identity matrix 的表達方式。

Theorem 2.12

Let A be an m×n matrix, B and C be n×p matrices, and D and E be q×m matrices. Then

(a) $A(B+C) = AB+AC$ and $(D+E)A = DA+EA$.

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a.

(c) $I_mA = A = AI_n$.

(d) If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.

（d）若 V 是一維度為 n 的向量空間，β 為其基底，則 $[I_V]_\beta = I_n$ 。

註：Identity transformation I_V: V→ V by $I_V(x) = x$ for all $x \in V$.

註：$[I_V]_\beta$ 為以 β 作為基底的單位轉換矩陣表示式。

Corollary

Let A be an m×n matrix,, $B_1, B_2, \ldots, B_k$ be n×p matrices, $C_1, C_2, \ldots, C_k$ be q×m matrices, and $a_1, a_2, \ldots, a_k$ be scalars. Then

$$A\left(\sum_{i=1}^{k} a_i B_i\right) = \sum_{i=1}^{k} a_i AB_i \quad \text{and} \quad \left(\sum_{i=1}^{k} a_i C_i\right)A = \sum_{i=1}^{k} a_i C_i A$$

For an n×n matrix A, we define $A^1 = A$, $A^2 = AA$, $A^3 = A^2A$, and in general, $A^k = A^{k-1}A$ for k = 2,3,...,n. We define $A^0 = I_n$.

With this notation, we see that if $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ then $A^2 = 0$ (the zero matrix) even though $A \neq 0$.

"From $A \cdot A = A^2 = 0 = A \cdot 0$, we would conclude that A = 0" is FALSE.

注意：即使 $A \neq 0$，也可能出現 $A^2 = 0$。

Theorem 2.13

Let A be an m×n matrix and B be n×p matrices. For each j ($1 \leqq j \leqq p$), let u_j and v_j denote the j^{th} columns of AB and B, respectively. Then

A 為 m×n 的矩陣與 B 為 n×p 的矩陣，u_j 與 v_j 分別為 AB 與 B 的第 j 行（**1≦j≦p**）。則

(a) $u_j = A v_j$.

(b) $v_j = B e_j$, where e_j is the j^{th} standard vector of F^p.

【Proof】

$$u_j = \begin{pmatrix} (AB)_{1j} \\ (AB)_{2j} \\ \vdots \\ (AB)_{mj} \end{pmatrix} = \begin{pmatrix} \sum_{k=1}^{n} A_{1k}B_{kj} \\ \sum_{k=1}^{n} A_{2k}B_{kj} \\ \vdots \\ \sum_{k=1}^{n} A_{mk}B_{kj} \end{pmatrix} = A \begin{pmatrix} B_{1j} \\ B_{2j} \\ \vdots \\ B_{nj} \end{pmatrix} = Av_j$$

From Theorem 2.13, the column j of AB is a linear combination of the column of A with the coefficients in the linear combination being the entries of column j of B. **An analogous result holds for rows; that is, row i of AB is a linear combination of the rows of B with the coefficients in the linear combination being the entries of row i of A.**

由定理 2.13 得知，AB 的第 j 行（Column）是 A 的所有行向量，以「B 的第 j 行元素作為係數」的一個線性組合。同理，AB 的第 i 列（Row）是 B 的所有列向量，以「A 的第 i 列元素作為係數」的一個線性組合。

Theorem 2.14

Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let T: V→W be linear. Then, for each u∈V, we have

$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$

V 與 W 為有限維度的向量空間，β 與 γ 分別 V 與 W 的有序基底。令 T: V→W 為線性轉換，則 V 中的 u 映至 W 的結果 T(u) 相對於有序基底 γ 的座標向量為

$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$。

【Proof】

Fix u∈V, and **define the linear transformations f: F→V by f(a) = au and g:F→W by g(a) = aT(u) for all a∈F.**

Let α = {1} be the standard ordered basis for F. Notice that g = Tf. Identifying column vectors as matrices and using Theorem 2.11, we obtain

$[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$

固定 u∈V 且**定義 f: F→V 為 f(a) = au 與 g:F→W 為 g(a) = aT(u)**。

令 α = {1} 為 F 的標準有序基底。

並注意 g（F→W）為 T（V→W，後）與 f（F→V，先）的合成 g = T・f 。

把行向量視為矩陣，並利用定理 2.11，得知

$[T(u)]_\gamma$ 為「V 中的 u 映至 W 的結果 T(u)」 相對於有序基底 γ 的座標向量。

由 g(a)與 f(a)定義中得知：g(1) = 1 T(u)，f(1) = u。

因此， $[T(u)]_\gamma = [g(1)]_\gamma = [g]_\alpha^\gamma = [Tf]_\alpha^\gamma = [T]_\beta^\gamma [f]_\alpha^\beta = [T]_\beta^\gamma [f(1)]_\beta = [T]_\beta^\gamma [u]_\beta$

Theorem 2.11 令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 T: V→W（先） 且 U: W→Z（後），則 $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$。

EXAMPLE 2

Let T: $P_3(R)$→ $P_2(R)$ be the linear transformations defined by T(f(x)) = f'(x), and let β and γ be the standard ordered bases for $P_3(R)$ and $P_2(R)$, respectively.

If A = $[T]_\beta^\gamma$, then from Example 4 of Section 2.2, we have

$$A = [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

We illustrate Theorem 2.14 by verifying that $[T(p(x))]_\gamma = [T]_\beta^\gamma [p(x)]_\beta$, where p(x)∈ $P_3(R)$ is the polynomial $p(x) = 2-4x+x^2+3x^3$. Let q(x) = T(p(x)), then q(x) = p'(x) = -

$4+2x+9x^2$. Hence

$$[T(p(x))]_\gamma = [q(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$, but also

$$[T]_\beta^\gamma [p(x)]_\beta = A[p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$

令 $T: P_3(R) \rightarrow P_2(R)$為線性轉換並定義為 $T(f(x)) = f'(x)$，且 $\beta = \{1, x, x^2, x^3\}$ 與 $\gamma = \{1, x, x^2\}$分別為 $P_3(R)$與 $P_2(R)$的有序基底。若 $A = [T]_\beta^\gamma$，則依據 Section 2.2 Example 4 的結果：

$$A = [T]_\beta^\gamma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

依據 Theorem 2.14 來驗證$[T(p(x))]_\gamma = [T]_\beta^\gamma [p(x)]_\beta$

其中，多項式 $p(x) \in P_3(R)$且 $p(x) = 2-4x+x^2+3x^3$（相當於 Theorem 2.14 的 u），令 $q(x) = T(p(x))$，則 $q(x) = p'(x) = -4+2x+9x^2$，因此 $q(x)$相對有序基底 $\gamma = \{1, x, x^2\}$的座標向量為

$$[T(p(x))]_\gamma = [q(x)]_\gamma = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$$

同時，$[T]_\beta^\gamma [p(x)]_\beta = A[p(x)]_\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -4 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 9 \end{pmatrix}$

DEFINITION 2.17

Let A be an m×n matrix with entries from a field F. We denote by L_A the mapping L_A: $F^n \rightarrow F^m$ defined by $L_A(x) = Ax$ (the matrix product A and x) for each column vector $x \in F^n$. We call L_A a **left-multiplication transformation**.

令 A 是 m×n 的矩陣，所有元素均來自 F。L_A 為 F^n 映至 F^m 的線性轉換並定義為 $L_A(x) = Ax$，其中 x 為行向量（Column vector）且 $x \in F^n$。L_A 被稱為左乘法轉換。

EXAMPLE 3

Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

Then $A \in M_{2\times3}(R)$ and L_A: $R^3 \rightarrow R^2$.

If $x = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$,

Then $L_A(x) = Ax = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 1 \end{pmatrix}$.

Theorem 2.15

Let A be an m×n matrix with entries from a field F. Then the left-multiplication transformation L_A: $F^n \rightarrow F^m$ is linear. Furthermore, if B is any other m×n matrix with entries from a field F and β and γ be the standard ordered bases for F^n and F^m, respectively, then we have the following properties.

令 A 為 m×n 的矩陣，**左乘法轉換** L_A: $F^n \rightarrow F^m$ 為線性。若 B 為另一個 m×n 的矩陣，$\beta = \{e_1, e_2, \ldots, e_n\}$ 與 $\gamma = \{e_1, e_2, \ldots, e_m\}$ 分別為 F^n 與 F^m 的標準有序基底，則左乘法轉換具有下列性質：

(a) $[L_A]_\beta^\gamma = A$.

(b) $L_A = L_B$ if and only if $A = B$.

(c) $L_{A+B} = L_A + L_B$ and $L_{aA} = a L_A$ for all $a \in F$.

(d) If T: $F^n \rightarrow F^m$ is linear, then there exists a unique m×n matrix C such that $T = L_C$. In fact $C = [T]_\beta^\gamma$.

(e) If E is an n×p matrix, then $L_{AE} = L_A L_E$.

(f) If m = n, then $L_{I_n} = I_{F^n}$.

Theorem 2.13 A 為 m×n 矩陣與 B 為 n×p 矩陣，u_j 與 v_j 分別為 AB 與 B 的第 j 行（**1≦j≦p**）。則

(a) $u_j = A v_j$.

(b) $v_j = B e_j$, where e_j is the j^{th} standard vector of F^p.

Theorem 2.14 V 與 W 為有限維度的向量空間，β 與 γ 分別 V 與 W 的有序基底。令 T: V→W 為線性轉換，則 T(u) 相對有序基底 γ 的座標向量 $[T(u)]_\gamma$ 為

$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.

Theorem 2.16 Associative

Let A, B, and C be matrices such that A(BC) is defined. Then (AB)C is also defined and A(BC) = (AB)C; that is, matrix multiplication is associative.

矩陣乘法具有結合性。

【Proof】

Using Theorem 2.15 (e)

$L_{A(BC)} = L_A L_{BC} = L_A (L_B L_C) = (L_A L_B) L_C = L_{AB} L_C = L_{(AB)C}$.

So from theorem 2.15 (b), it follows that A(BC) = (AB)C.

Composition of Linear Transformations

- Let U,V, and W be vector spaces and T_1:U→V, T_2:V→W be linear transformations between these spaces. These two transformations can be combined into a single composite transformation T:U→W .
- Let u be a vector in U. Then $T(\mathbf{x}) = T_2(T_1(\mathbf{x}))$
- This composition transformation is defined by $T=T_2 \circ T_1$.

Theorem

- The composite of two linear transformation is itself a linear transformation.

Let U, V and W be vector spaces and T_1: U → V, T_2: V → W be linear transformations. Let **u** and **v** be vectors in U and c be a scalar. We use the linearity of T_1 and T_2 to get

$$T_2 \circ T_1(u+v) = T_2(T_1(u+v)) = T_2(T_1(u)+T_1(v))$$
$$= T_2(T_1(u)) + T_2(T_1(v)) = T_2 \circ T_1(u) + T_2 \circ T_1(v)$$
$$T_2 \circ T_1(cu) = T_2(T_1(cu)) = T_2(cT_1(u)) = cT_2(T_1(u)) = cT_2 \circ T_1(u)$$

Thus $T_2 \circ T_1$ is linear.

Example

- Find the $T_2 \circ T_1$ of the transformation $T_1(x, y) = (3x, x + y)$ and $T_2(x, y) = (2x, -y)$. Determine the image of (2, –3).

We get

$$T_2 \circ T_1(x, y) = T_2(T_1(x, y)) = T_2(3x, x+y) = (6x, -x-y)$$

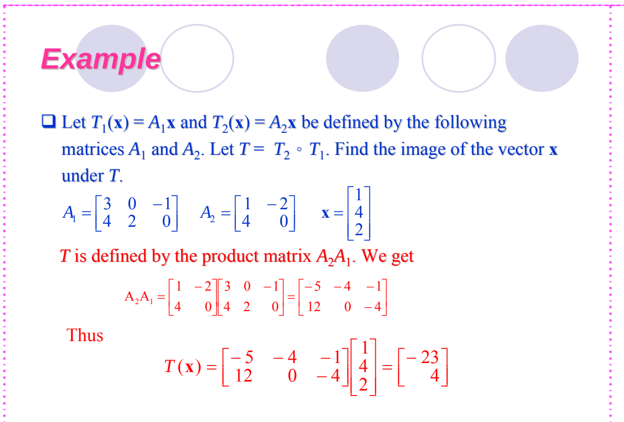
The image of $(2,-3)$ is $T_2 \circ T_1(2,-3) = (12, 1)$.

Composition of Matrix Transformations

- Linear transformations defined by matrices are particularly important. Let T_1: $\mathbf{R}^n \to \mathbf{R}^m$ and T_2: $\mathbf{R}^m \to \mathbf{R}^n$ be matrix transformations defined by $T_1(\mathbf{x}) = A_1\mathbf{x}$ **and** $T_2(\mathbf{x}) = A_2\mathbf{x}$. The composite transformation is defined by the the product matrix A_2A_1.

$$T_2 \circ T_1(x) = T_2(T_1(x)) = T_2(A_1x) = A_2A_1x$$

If $T_1(x) = A_1x$ and $T_2(x) = A_2x$, then $T_2 \circ T_1(x) = A_2A_1x$



2.4 Invertibility and Isomorphisms 可逆性與同構轉換

DEFINITION 2.18

Let V and W be vector spaces, and let T:V→W be linear. A function U:W→V is said to be an **inverse** of T if TU = I_W and UT = I_V. If T has an inverse, then T is said to be invertible. If T is invertible, then the inverse of T is unique and is denoted by T^{-1}.

令 V 與 W 為向量空間，且 T: V→W 為線性轉換。若 U: W→V 可稱為 T 的逆轉換，其條件條件為 TU = I_W 且 UT = I_V。 TU: W → W，UT: V→V。

若 T 具有逆轉換，則 T 稱為可逆轉換。若 T 為可逆轉換，則 T 的逆轉換為唯一，並註記為 T^{-1}。

The following facts hold for invertible functions T and U.

可逆轉換函數 T 與 U 具有下列性質：

1. $(TU)^{-1} = U^{-1}T^{-1}$.
2. $(T^{-1})^{-1} = T$; in particular, T^{-1} is invertible.
3. Let T: V→W be a linear transformation, where V and W are finite-dimensional spaces of equal dimension. Then T is invertible if and only if rank(T) = dim(V).

   令 T 為線性轉換且 V、W 為有限且維度相等的向量空間，則 T 為可逆的「若且唯若」條件為 rank(T) = dim(V)。

Nullity(T) + rank(T) = dim (V) ； rank(T) = dim(R(T))；Nullity(T) = dim(N(T))

EXAMPLE 1

Let T: $P_1(R)$→R^2 be the linear transformartion defined by T(a+bx) = (a, a+b). Then T^{-1}:

$R^2 \rightarrow P_1(R)$ is defined by $T^{-1}(c,d) = c+(d-c)x$. T^{-1} is also linear.

由 $T: P_1(R) \rightarrow R^2$ 及定義 $T(a+bx) = (a, a+b)$，得知 $T^{-1}: R^2 \rightarrow P_1(R)$及其定義 $T^{-1}(c,d) = c+(d-c)x$。T 為線性，T^{-1} 亦為線性。

Theorem 2.17

Let V and W be vector spaces, and let $T: V \rightarrow W$ be linear and invertible.

Then $T^{-1}: W \rightarrow V$ is linear.

令 V 與 W 為向量空間，且 $T: V \rightarrow W$ 為線性且為可逆，則 $T^{-1}: W \rightarrow V$ 亦為線性。

【Proof】

Let $y_1, y_2 \in W$ and $c \in F$. Since T is onto and one-to-one, there exist unique vector x_1 and x_2 such that $T(x_1) = y_1$ and $T(x_2) = y_2$.

Thus $x_1 = T^{-1}(y_1)$ and $x_2 = T^{-1}(y_2)$;

So $T^{-1}(cy_1+y_2) = T^{-1}[cT(x_1)+T(x_2)] = T^{-1}[T(cx_1+x_2)] = cx_1+x_2 = cT^{-1}(y_1)+T^{-1}(y_2)$

利用線性轉換應具備的 Property：T 是線性轉換「若且惟若」$T(cx+y) = cT(x)+T(y)$ for all $x,y \in V$ and $c \in F$。

由於 T 為映成且一對一，故存在唯一的向量 x_1 與 x_2，使得 $T(x_1) = y_1$ 與 $T(x_2) = y_2$。

因此 $x_1 = T^{-1}(y_1)$ 且 $x_2 = T^{-1}(y_2)$。

所以 $T^{-1}(cy_1+y_2) = T^{-1}[cT(x_1)+T(x_2)] = T^{-1}[T(cx_1+x_2)] = cx_1+x_2 = cT^{-1}(y_1)+T^{-1}(y_2)$

DEFINITION 2.19

Let A be an $n \times n$ matrix. Then A is invertible if there exists and $n \times n$ matrix B such that $AB = BA = I$.

If A is invertible, then the matrix B such that $AB = BA = I$ is unique. The matrix B is called the inverse of A and is denoted by A^{-1}.

令 A 為 $n \times n$ 的矩陣，則 A 為可逆的條件為「存在一 $n \times n$ 的矩陣 B，使得 $AB = BA = I$」。

若 A 為可逆，則使得 $AB = BA = I$ 的矩陣 B 為唯一，B 可註記為 A^{-1}。

Lemma 引理

Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case, dim(V) = dim(W).

令 T 為 V→W 為可逆且為線性，則 V 是有限維度「若且唯若」W 是有限維度。在此情況，dim(V) = dim(W)。

Theorem 2.18

Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let T: V→W be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$.

令 V 與 W 是有限維度的向量空間，β 與 γ 分別為 V 與 W 的有序基底。令 T: V→W 為線性。T 可逆的『若且唯若』條件為$[T]_\beta^\gamma$是可逆。再者，$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$ 。

【Proof】

先證明 T 是可逆→$[T]_\beta^\gamma$ 可逆且$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$ 。

Suppose that T is invertible. By the lemma, we have dim(V) = dim(W).
Let n = dim(V). So $[T]_\beta^\gamma$ is an n×n matrix.

Now T^{-1}:W→V satisfies $TT^{-1} = I_W$ and $T^{-1}T = I_V$.
Thus $I_n = [I_V]_\beta = [T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$
Similarly, $[T]_\beta^\gamma [T^{-1}]_\gamma^\beta = I_n$.
So $[T]_\beta^\gamma$ is invertible and $([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$.

假設 T 為可逆，故依 Lemma 可知 dim(V) = dim(W)。
令 n = dim(V)，所以 $[T]_\beta^\gamma$ 為一 n×n 的矩陣。

已知 T 為可逆，其逆轉換 T^{-1}:W→V 滿足 $TT^{-1} = I_W$ 且 $T^{-1}T = I_V$。
依據 Theorem 2.12 (d) $I_n = [I_V]_\beta \rightarrow [I_V]_\beta = [T^{-1}T]_\beta$ 。（先 T、後 T^{-1}；先 β→γ、後 γ→β）
依據 Theorem 2.11 $[T^{-1}T]_\beta = [T^{-1}]_\gamma^\beta [T]_\beta^\gamma$（先 T、後 T^{-1}；先 β→γ、後 γ→β）。
依據 Theorem 2.12 (d) $I_n = [I_W]_\gamma \rightarrow [I_W]_\gamma = [TT^{-1}]_\gamma$（先 T^{-1}、後 T；先 γ→β、後 β→γ）
依據 Theorem 2.11 $\rightarrow [TT^{-1}]_\gamma = [T]_\beta^\gamma [T^{-1}]_\gamma^\beta$（先 T^{-1}、後 T；先 γ→β、後 β→γ）
故$[T]_\beta^\gamma$為可逆且$([T]_\beta^\gamma)^{-1} = [T^{-1}]_\gamma^\beta$ 。

再證明$[T]_\beta^\gamma$ 可逆且$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$→T 是可逆。

Now suppose that A = $[T]_\beta^\gamma$ is invertible.

Then there exists an n×n matrix B such that AB = BA = I_n.

By Theorem 2.6, there exists $U \in \pounds(W,V)$ such that

$$U(w_j) = \sum_{k=1}^{n} b_{ij} v_j \quad \text{for} \quad j = 1,2,\ldots n.$$

Where $\gamma = \{w_1,w_2,\ldots,w_n\}$ and $\beta = \{v_1,v_2,\ldots,v_n\}$.

It follows that $[U]_\gamma^\beta = B$. To show that $U = T^{-1}$, observe that

$$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_v]_\beta$$

So $UT = I_V$, and similarly, $TU = I_W$.

假設 $A = [T]_\beta^\gamma$ 是可逆，則存有另一個 $n\times n$ 的矩陣 B，使得 $AB = BA = I_n$。

依據 Theorem 2.6，存在由 W 映至 V 的轉換 U（$U \in \pounds(W,V)$），使得

$U(w_j) = v_j$ for $j = 1,2,\ldots,n$

$$\rightarrow U(w_j) = \sum_{k=1}^{n} b_{ij} v_j \quad \text{for} \quad j = 1,2,\ldots n。$$

其中，$\gamma = \{w_1,w_2,\ldots,w_n\}$，$\beta = \{v_1,v_2,\ldots,v_n\}$分別為 W 與 V 的有序基底。

於是，$[U]_\gamma^\beta = B$。至於 U 是否為 T^{-1}？

$$[UT]_\beta = [U]_\gamma^\beta [T]_\beta^\gamma = BA = I_n = [I_v]_\beta$$

$\rightarrow UT = I_v$。

同理 $TU = I_W$。

U 當然為 T^{-1}。

DEFINITION 2.18 令 V 與 W 為向量空間，且 $T: V\rightarrow W$ 為線性轉換。若 $U: W\rightarrow V$ 可稱為 T 的逆轉換，其條件條件為 $TU = I_W$ 且 $UT = I_V$。

Theorem 2.6 Let V and W be vector spaces over F, and suppose that $\{v_1, v_2,\ldots, v_n\}$ is a basis for V. For $w_1, w_2,\ldots, w_n$ in W, there exists exactly one linear transformation $T:V\rightarrow W$ such that $T(v_i) = w_i$ for $i = 1, 2,\ldots, n$. 令 V 與 W 為佈於 F 的向量空間。設 $\{v_1, v_2, \ldots, v_n\}$為 V 的基底，對 W 內的 $w_1, w_2, \ldots, w_n$ 而言，存在一由 V 映至 W 的線性轉換 T，使得 $T(v_i) = w_i$。

Theorem 2.11 Let V, W, and Z be finite-dimensional vector space with ordered bases α, β, and γ, respectively. Let $T: V\rightarrow W$ and $U: W\rightarrow Z$ be linear transformation. Then $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$. 令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 $T: V\rightarrow W$（先） 且 $U: W\rightarrow Z$（後），則 $[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta$。

Theorem 2.12 Let A be an $m\times n$ matrix, B and C be $n\times p$ matrices, and D and E be $q\times m$ matrices. Then

(a) $A(B+C) = AB+AC$ and $(D+E)A = DA+EA$.

(b) $a(AB) = (aA)B = A(aB)$ for any scalar a.

(c) $I_m A = A = AI_n$.

(d) If V is an n-dimensional vector space with an ordered basis β, then $[I_V]_\beta = I_n$.

EXAMPLE 2

Let β and γ be the standard ordered bases of $P_1(R)$ and R^2, respectively. Let T: $P_1(R) \rightarrow R^2$ be the linear transformartion defined by $T(a + bx) = (a, a + b)$. $T^{-1}: R^2 \rightarrow P_1(R)$ defined by $T^{-1}(c, d) = c + (d - c)x$ is also linear. We have

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \text{ and } [T^{-1}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},$$

where β = {1, x} and γ = {(1, 0), (0, 1)}

β = {1, x}與 γ = {(1, 0), (0, 1)}分別為 $P_1(R)$ 與 R^2 的標準有序基底。T: $P_1(R) \rightarrow R^2$ 為線性轉換並定義為 $T(a + bx) = (a, a + b)$。$T^{-1}: R^2 \rightarrow P_1(R)$為線性並定義為 $T^{-1}(c, d) = c + (d - c)x$，則

$$[T]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} 、 [T^{-1}]_\beta^\gamma = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Corollary 1

Let V be finite-dimensional vector space with an ordered basis β , and let T: V→V be linear. Then T is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

V 為有限維度的向量空間，β 是 V 的有序基底，令 T: V→V 為線性轉換。T 為可逆的『若且唯若』條件為$[T]_\beta$可逆。再者，$[T^{-1}]_\beta = ([T]_\beta)^{-1}$ 。

Corollary 2

Let A be an n×n matrix. Then A is invertible if and only if L_A is invertible. Furthermore, $(L_A)^{-1} = L_A{}^{-1}$.

令 A 為 n×n 的矩陣。A 為可逆的『若且唯若』條件為 L_A 可逆。再者，$(L_A)^{-1} = L_A{}^{-1}$ 。

DEFINITION 2.19 Isomorphism

Let V and W be vector spaces. We say that V is isomorphic to W if there exists a linear transformation T: V→W that is invertible. Such a linear transformation is called an isomorphism from V onto W.

V 與 W 為向量空間，稱 V 同構於 W，表示存在一線性轉換 T: V→W 為可逆。這種線性轉換稱為「由 V 映成至 W 的同構轉換」。

"Is isomorphic to" is an equivalence relation. So we need ony say that V and W are isomorphic.

由於「**同構於**」是一種等價關係，故「V **同構於 B**」可簡稱「AB **同構**」。

EXAMPLE 3

Define T: F^2→$P_1(F)$ by $T(a_1,a_2) = a_1+a_2x$. It is easily checked that T is an isomorphism; so F^2 is isomorphic to $P_1(F)$.

定義 T: F^2→$P_1(F)$為 $T(a_1, a_2) = a_1 + a_2x$。T 為同構轉換？若 T 為同構轉換，則稱 F^2 同構於 $P_1(F)$。

Theorem 2.19

Let V and W be finite-dimensional vector spaces. Then V is isomorphic to W if and only if dim(V) = dim(W).

V 與 W 為有限向量空間。V 同構於 W 的「若且唯若」條件為 dim(V) = dim(W)。

【Proof】

先證明 V 同構於 W→ dim(V) = dim(W)。

Suppose that V is isomorphic to W and that T: V→W is an isomorphism from V to W. By the lemma preceeding Theroem 2.18, we have that dim(V) = dim(W).

若 V 同構於 W，表示由 V 到 W 的線性轉換 T 為可逆且為 ONTO。故依據 Lemma to Theorem 2.18，可知 dim(V) = dim(W)。

再證明 dim(V) = dim(W)→V 同構於 W。

Now suppose that dim(V) = dim(W), and let $\beta = \{v_1, v_2,\ldots, v_n\}$ and $\gamma = \{w_1, w_2,\ldots, w_n\}$ be bases for V and W, respectively.

By Theorem 2.6, there exists T:V→W such that T is linear and $T(v_i) = w_i$ for i =

1,2,….,n.

Using Theorem 2.2, we have R(T) = span(T(β)) = span(γ) = W.

So T is onto. From Theorem 2.5, we have that T is also one-to-one. Hence T is an isomorphis.

By the lemma to Theorem 2.18, if V and W are isomorphic, then either both of V and W are finite-dimensional or both are infinite-dimensional.

假設 dim(V) = dim(W)，並令 β = $\{v_1, v_2, \ldots, v_n\}$與$\{w_1, w_2, \ldots, w_n\}$ 分別為 V 與 W 的基底。

依據 Theorem 2.6 得知：存在線性轉換 T:V→W，使得 $T(v_i) = w_i$ for i = 1,2,….,n 。

依據 Theorem 2.2 得知：T 的值域 Range R(T)可由 T(β)來生成，意即 R(T) = span(T(β)) = span(γ) = W。

所以 T 是映成（R(T) = W，即值域等於對應域。）。

依據 Theorem 2.5 得知 T 也是一對一。

故 T 為 V 映至 W 的同構轉換。

Theorem 2.2 Let V and W be vector spaces, and let T: V→W be linear. If β = $\{v_1, v_2, \ldots, v_n\}$ is basis for V, then R(T) = span(T(β)) = span($\{T(v_1), T(v_2), \ldots, T(v_n)\}$). 令 V 與 W 為向量空間，且 T: V→W 為線性轉換。若 β = $\{v_1, v_2, \ldots, v_n\}$是定義域（Domain）V 的基底，則 T 的值域 Range R(T)可由 T(β)來生成。意即 R(T) = span(T(β)) = span($\{T(v_1), T(v_2), \ldots, T(v_n)\}$)。

Theorem 2.5 V 與 W 為向量空間，具有相等的維度，且 T: V→W 為線性轉換，則下列敘述等價：

(a) T is one-to-one.

(b) T is onto.

(c) rank(T) = dim(V).

Theorem 2.6 Let V and W be vector spaces over F, and suppose that $\{v_1, v_2, \ldots, v_n\}$ is a basis for V. For $w_1, w_2, \ldots, w_n$ in W, there exists exactly one linear transformation T:V→W such that $T(v_i) = w_i$ for i = 1, 2,…, n. 令 V 與 W 為佈於 F 的向量空間。設 $\{v_1, v_2, \ldots, v_n\}$為 V 的基底，對 W 內的 $w_1, w_2, \ldots, w_n$ 而言，存在一由 V 映至 W 的線性轉換 T，使得 $T(v_i) = w_i$。

Lemma to Theorem 2.18 Let T be an invertible linear transformation from V to W. Then V is finite-dimensional if and only if W is finite-dimensional. In this case, dim(V) = dim(W). 令 T 為 V→W 為可逆且為線性，則 V 是有限維度「若且唯若」W 是有限維度。在此情況，dim(V) = dim(W)。

Corollary

Let V be a vector space over F. Then V is isomorphic to F^n if and only if dim(V) = n.
令 V 為佈於 F 的向量空間，則 V 同構於 F^n『若且唯若』條件為 dim(V) = n。

Theorem 2.20

Let V and W be finite-dimensional vector spaces over F of dimensions n and m, respectively, and let β and γ be ordered bases for V and W, respectively. Then the function Φ: £(V, W) → $M_{m\times n}(F)$, defined by $\Phi(T) = [T]_\beta^\gamma$ for T∈£(V, W), is an isomorphism.

令 V 與 W 為佈於 F，維度分別為 n 與 m 的向量空間，且 β 與 γ 分別為 V 與 W 的基底。函數 Φ: £(V, W) → $M_{m\times n}(F)$是同構轉換，其中，$\Phi(T) = [T]_\beta^\gamma$。

Corollary

Let V and W be finite-dimensional vector spaces of dimensions n and m, respectively. Then £(V, W) is finite-dimensional of dimension mn.

令 V 與 W 為佈於 F，維度分別為 n 與 m 的向量空間，則£(V,W)為有限維度且維度為 m×n。

DEFINITION 2.14 Let V and W be vector spaces over F. We denote the vector space of all linear transformations from V into W by £(V, W). In the case that V = W, we write £(V) instead of £(V, W). 令 V 與 W 為佈於 F 的向量空間。將所有 V 映至 W 的線性轉換所形成的向量空間註記為 £(V, W)。當 V = W，則將£(V, W) 改寫成 £(V)。

DEFINITION 2.20 Standard representation

Let β be an ordered basis for an n-dimensional vectors space V over the field. The standard representation of V with respect to β is the function Φ_β: V→F^n defined by $\Phi_\beta(x) = [x]_\beta$ for each x∈V.

令 β 是維度為 n 的向量空間 V 的有序基底，則 V 相對於 β 的標準表示式為由 V 映至 F^n 的函數 $\Phi_\beta(x)$（Φ_β: V→F^n），該函數定義為 $\Phi_\beta(x) = [x]_\beta$；其中，x∈V。

EXAMPLE 4

Let $\beta = \{(1, 0), (0, 1)\}$ and $\gamma = \{(1, 2), (3, 4)\}$. It is easily observed that β and γ are ordered bases for R^2. For $x = (1, -2)$, we have

$$\Phi_\beta(x) = [x]_\beta = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \text{and } \Phi_\gamma(x) = [x]_\gamma = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$$

Theorem 2.21

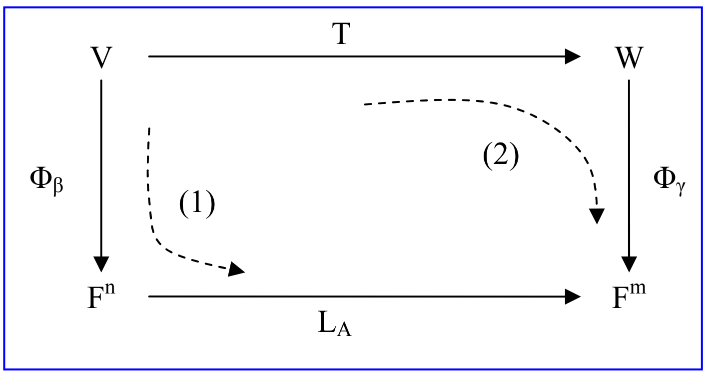
For any finite-dimensional vector space V with ordered basis β, Φ_β is isomorphism.

對任一有限維度、具有有序基底 β 的向量空間 V 而言，Φ_β 為一同構轉換。

Let V and W be vector spaces of dimensions n and m, respectively, and let T: V→W be a linear transformation. Defined $A = [T]_\beta^\gamma$, where β and γ are arbitrary ordered bases of V and W, respectively. We are now able to use Φ_β and Φ_γ to study the relationship between the linear transformations T and $L_A:F^n \rightarrow F^m$.

令 V 與 W 分別為維度 n 與 m 的向量空間，T 為 V→W 的線性轉換。定義 $A = [T]_\beta^\gamma$，其中 β 與 γ 分別為 V 與 W 的有序基底。我們現在要利用 Φ_β 與 Φ_γ 來探討線性轉換 T 與 $L_A:F^n \rightarrow F^m$ 的關係。

Let us first consider the below figure. Notice that there are two composites of linear transformation that maps V into F^m.



1. Map V into F^n with Φ_β and follow this transformation with L_A; this yields the

composite $L_A\Phi_\beta$.

V→ F^n 的 Φ_β（先）與 L_A（後）合成為 $L_A\Phi_\beta$。

2. Map V into W with T and follow it by Φ_γ to obtain the composite $\Phi_\gamma T$.

V→ W 的 T（先）與 Φ_γ（後）合成為 $\Phi_\gamma T$ 。

We conclude that $L_A\Phi_\beta = \Phi_\gamma T$.

EXAMPLE 5

The linear transformation T: $P_3(R)$→ $P_2(R)$ by $T(f(x)) = f'(x)$. Let $\beta = \{1, x, x^2, x^3\}$ and $\gamma = \{1, x, x^2\}$ be the standard ordered bases for $P_3(R)$ and $P_2(R)$, respectively, and letΦ_β: $P_3(R)$→ R^4 and Φ_γ: $P_2(R)$→R^3 be the corresponding standard representation of $P_3(R)$ and $P_2(R)$. If A = $[T]_\beta^\gamma$, then

令 T: $P_3(R)$→ $P_2(R)$為線性轉換且定義為 $T(f(x)) = f'(x)$，$\beta =\{1, x, x^2, x^3\}$ 與 $\gamma = \{1, x, x^2\}$ 分別為 $P_3(R)$與 $P_2(R)$的標準有序基底。令 Φ_β: $P_3(R)$→ R^4 及 Φ_γ: $P_2(R)$→R^3 分別為 $P_3(R)$ 與 $P_2(R)$的對應標準表示式。若 A = $[T]_\beta^\gamma$，則

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Consider the polynomial $p(x) = 2+x-3x^2+5x^3$. We show that $L_A\Phi_\beta(p(x)) = \Phi_\gamma T(p(x))$.

考慮多項式 $p(x) = 2+x-3x^2+5x^3$，證明 $L_A\Phi_\beta(p(x)) = \Phi_\gamma T(p(x))$。

Now

$$\Phi_\beta(p(x)) = \begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix}$$ （相當於 $p(x) = 2+x-3x^2+5x^3$ 相對於 $\beta = \{1, x, x^2, x^3\}$的座標向量。）

$$L_A\Phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}$$ （先 Φ_β 後 L_A）

But since $T(p(x)) = p'(x) = 1-6x+15x^2$, we have

$$\Phi_\gamma T(p(x)) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}$$ （先 T 後 Φ_γ）（相當於 $T(p(x)) = p'(x) = 1-6x+15x^2$ 相對於 γ =

$\{1, x, x^2\}$的座標向量。）

So $L_A\Phi_\beta(p(x)) = \Phi_\gamma T(p(x))$.

NOTE: $L_A: R^4 \rightarrow R^3$

2.5 The Change of Coordinate Matrix

In many areas of mathematics, a change of variable is used to simplify the appearance of an expression. For example,

利用變數轉換來化簡式子。

$\int 2xe^{x^2}dx = \int e^u du = e^u + c = e^{x^2} + c$ by making the change of variable $u = x^2$

在微積分中利用變數變換 $u = x^2$。

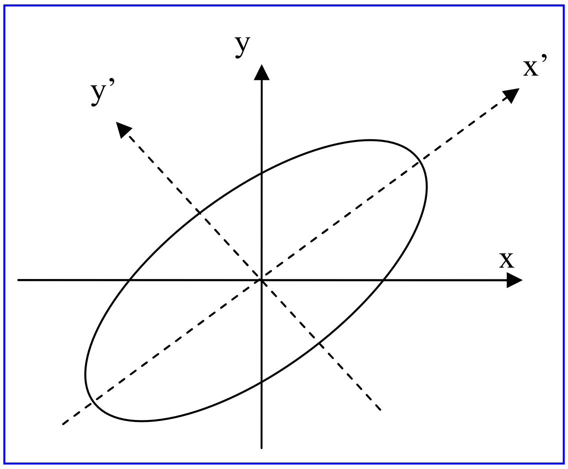
Similary, in geometry the change of variable

在平面幾何中利用變數轉換

$$x = \frac{2}{\sqrt{5}}x' - \frac{1}{\sqrt{5}}y' \qquad y = \frac{1}{\sqrt{5}}x' + \frac{2}{\sqrt{5}}y'$$

can be used to transform the equation $2x^2-4xy+5y^2 = 1$ into the simpler equation $(x')^2+6(y')^2 = 1$, in which it is easily seen to be the equation of an ellipse.

將 $2x^2-4xy+5y^2 = 1$ 變成一個橢圓方程式$(x')^2+6(y')^2 = 1$。



Geometrically, the change of variables $\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix}$ is a change in the way that the position of a point P in the plane is described. This is done by introducing a new frame of reference, and x'y'-coordinate system with coordinate axes rotated from the original xy-coordinate axes. In this case, the new coordinate axes are chosen to lie in the direction of the axes of the ellipse. The unit vectors along the x'-axis and the y'-axis form an ordered basis.

引進另一個參考座標系統 x'y'座標系，新座標系統位於橢圓的雙軸方向上，沿 x'軸與 y'軸，成為一新的有序基底 β'。

$$\beta' = \left\{ \frac{1}{5}\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\}$$

For R^2, and the change of variable is actually a change from $[P]_\beta = \begin{pmatrix} x \\ y \end{pmatrix}$, the coordinate vector of P relative to the standard ordered basis $\beta = \{e_1, e_2\}$, to $[P]_{\beta'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$, the coordinate vector of P relative to the new rotated basis β'.

這種變數轉換，使得相對標準有序基底 $\beta = \{e_1, e_2\}$的 P 座標向量$[P]_\beta = \begin{pmatrix} x \\ y \end{pmatrix}$，轉換至相對旋轉基底 β'的座標向量$[P]_{\beta'} = \begin{pmatrix} x' \\ y' \end{pmatrix}$。

How can a coordinate vector relative to one basis be changed into a coordinate vector relative to the other? Notice that the system of equations relating the new and old coordinates can be represented by the matrix equation

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix}$$ 新舊座標以矩陣方程式表示

Notice also that the matrix $Q = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ equal $[I]_{\beta'}^{\beta}$, where I denotes the identity transformation（單位轉換 I: $R^2 \to R^2$）on R^2. Thus $[v]_\beta = Q[v]_{\beta'}$ for all $v \in R^2$.

單位轉換 I_V: V→ V 為 $I_V(x) = x$。

Theorem 2.22

Let β and β' be two ordered bases for a finite-dimensional vector space V, and let Q = $[I_v]_{\beta'}^{\beta}$ Then

令 β 與 β'為有限維度空間向量 V 的兩個有序基底，且 Q = $[I_v]_{\beta'}^{\beta}$（由 I: V→V），則

(a) Q is invertible.

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$

【Proof】

(a) Since I_V is invertible, Q is invertible by Theorem 2.18.

因為 I_V 為可逆，依據 Theorem 2.18 得知$[I_v]_{\beta'}^{\beta}$ 為可逆。

(b) For any $v \in V$, $[v]_\beta = [I_v(v)]_\beta = [I_v]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$ by Theorem 2.14.

對於任一屬於 V 中的 v 而言（**v∈V**），依據 Definition 2.5：$[v]_\beta = [I_v(v)]_\beta$。

依據 Theorem 2.14：$[v]_\beta = [I_v(v)]_\beta = [I_v]_{\beta'}^{\beta}[v]_{\beta'} = Q[v]_{\beta'}$。

DEFINITION 2.5 For vector space V and W, define identity transformation I_V: V→ V by $I_V(x)$ = x for all $x \in V$. V 與 W 為向量空間，定義單位轉換 I_V: V→ V 為 $I_V(x)$ = x；意即單位轉換為由 V 映至 V 的一種轉換，定義域內的所有元素 x，轉換後所對應的「像」為本身，即「像」等於「前像」的一種轉換。

Theorem 2.14 Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let T: V→W be linear. Then, for each $u \in V$, we have $[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$.

V 與 W 為有限維度的向量空間，β 與 γ 分別 V 與 W 的有序基底。令 T: V→W 為線性轉換，則 T(u) 相對有序基底 γ 的座標向量$[T(u)]_\gamma$ 為$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta$。

Theorem 2.18 Let V and W be finite-dimensional vector spaces with ordered bases β and γ, respectively. Let T: V→W be linear. Then T is invertible if and only if $[T]_\beta^\gamma$ is invertible. Furthermore, $[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$. 令 V 與 W 是有限維度的向量空間，β 與 γ 分別為 V 與 W 的有序基底。令 T: V→W 為線性。T 可逆的『若且唯若』條件為$[T]_\beta^\gamma$是可逆。再者，$[T^{-1}]_\gamma^\beta = ([T]_\beta^\gamma)^{-1}$。。

The matrix Q = $[I_v]_{\beta'}^{\beta}$ defined by Theorem 2.22 is called a change of coordinate matrix. Because of part (b) of the Theorem, we say that Q changes β'–coordinates into β-coordinates. Observe that if β = $\{x_1, x_2, \ldots, x_n\}$ and β' = $\{x'_1, x'_2, \ldots, x'_n\}$. Then

$x'_j = \sum_{i=1}^{n} Q_{ij} x_i$ for j = 1,2,...,n; that is, the jth column of A is $[x'_j]_\beta$【相當於探討 β’中的 x'_j 相對於 β 的座標向量】

Theorem 2.22 所定義的 Q = $[I_V]_{\beta'}^{\beta}$ 稱為座標轉換矩陣（Change of coordinate matrix）。依 Theorem 2.22(b)稱 Q 為由 β’座標系變換至 β 座標系的座標變換矩陣。若 β = {x_1, x_2,…, x_n}且 β’ = {x'_1, x'_2,…, x'_n}，則 $x'_j = \sum_{i=1}^{n} Q_{ij} x_i$ 。$[x'_j]_{\beta'}$ 是 Q 矩陣的第 j 行。

反之，由 β 座標系變換至 β'座標系者為 Q^{-1}。

EXAMPLE 1

In R^2, β = {(1, 1), (1, -1)} and β’ = {(2, 4), (3, 1)}. Since

(2, 4) = 3(1, 1)-1(1, -1) and (3, 1) = 2(1, 1)+1(1, -1),

the matrix that changes β’-coordinates into β-coordinates is

$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$ （由 β’座標系變換至 β 座標系）

Thus, for instance $[(2, 4)]_\beta = Q\,[(2, 4)]_{\beta'} = Q\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

例如：β’內的(2,4) **相對於 β 的座標向量。**

Theorem 2.23

Let T be a linear operator on a finite-dimensional vector space V, and let β and β’ be ordered bases for V. Suppose that Q is the change of coordinate matrix that changes β’–coordinates into β-coordinates. Then

$[T]_{\beta'} = Q^{-1}[T]_\beta Q$.

T 為有限維度的向量空間 V 上的一個線性運算子，令 β 與 β’為有限維度空間向量 V 的兩個有序基底，且 Q 為由 β’座標系變換至 β 座標系的座標變換矩陣，則 $[T]_{\beta'} = Q^{-1}[T]_\beta Q$ 。

【Proof】

Let I be the identity transformation on V. Then T = IT = TI; hence,

by Theorem 2.11, $Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [IT]_{\beta'}^{\beta} = [TI]_{\beta'}^{\beta} = [T]_{\beta}^{\beta}[I]_{\beta'}^{\beta} = [T]_\beta Q$

Therefore $[T]_{\beta'} = Q^{-1}[T]_\beta Q$

令 I 是 V 上的單位轉換（Identity transformation I_V: V→ V by $I_V(x) = x$ for all $x \in V$）

，則 $IT = TI = T$。

因 $\mathbf{Q} = [I_V]_{\beta'}^{\beta} \rightarrow Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'}$

依據 Theorem 2.11 得知：

$Q[T]_{\beta'} = [I]_{\beta'}^{\beta}[T]_{\beta'}^{\beta'} = [IT]_{\beta'}^{\beta} = [TI]_{\beta'}^{\beta} = [T]_{\beta}^{\beta}[I]_{\beta'}^{\beta} = [T]_{\beta}Q$

因此，$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q$。

Theorem 2.11 Let V, W, and Z be finite-dimensional vector space with ordered bases α, β, and γ, respectively. Let T: V→W and U: W→Z be linear transformation. Then $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$.令 V、W 與 Z 是有限維度的向量空間，α、β 與 γ 分別為 V、W 與 Z 的有序基底。令 T: V→W（先） 且 U: W→Z（後），則 $[UT]_{\alpha}^{\gamma} = [U]_{\beta}^{\gamma}[T]_{\alpha}^{\beta}$。

EXAMPLE 2

Let T be the linear operator in R^2 defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 3a-b \\ a+3b \end{pmatrix}$$

and let β = {(1, 1), (1, -1)} and β' = {(2, 4), (3, 1)} be the ordered bases.

$$[T]_{\beta} = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}^{\text{註}}$$

The change of coordinate matrix that changes β'–coordinates into β-coordinates is

$$Q = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$$ 【(2, 4) = 3(1, 1)-1(1, -1) and (3, 1) = 2(1, 1)+1(1, -1)】

and it is easily verifed that $Q^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}$

Hence, by Theorem 2.23

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q = \begin{pmatrix} 4 & 1 \\ -2 & 2 \end{pmatrix}$$

To show that this is the correct matrix, we can verify that the image under T of each vector of β' is the linear combination of the vectors of β' with the image of the second vector in β' is

為驗證所求出來的矩陣，可以 β'的每一向量經 T 映射的像是 β'中諸向量的線性組合，而對應行的元素是該線性組合的係數。

$$T\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix} = 1\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 3 \\ 1 \end{pmatrix}$$ (3, 1)是 β'中第 2 個向量，(1, 2)是$[T]_{\beta'}$的第二個行。

註：

Let T: $R^2 \rightarrow R^2$ be the linear transformation defined by

T(a, b) = (3a - b, a + 3b)

Let β and β' be the ordered bases for R^2 and R^2, respectively. Now

β = {(1, 1), (1, -1)} and β' = {(2, 4), (3, 1)}

$T(v_1)$ = T(1, 1) = (2, 4) = 3 (1, 1) - 1(1, -1)

$T(v_2)$ = T(1, -1) = (4, -2) =1 (1, 1) + 3 (1, -1)

→ $[T]_\beta = \begin{pmatrix} 3 & 1 \\ -1 & 3 \end{pmatrix}$

Corollary

Let $A \in M_{n\times n}(F)$, and let γ be an ordered basis for F^n. Then $[L_A]_\gamma = Q^{-1}AQ$, where Q is the n×n matrix whose jth column is the j^{th} vector of γ.

令 $A \in M_{n\times n}(F)$且 γ 為 F^n 的有序基底，則$[L_A]_\gamma = Q^{-1}AQ$；其中，Q 為 n×n 的矩陣，且其第 j 行為 γ 的第 j 個向量。

EXAMPLE 3

Let $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 3 \\ 0 & -1 & 0 \end{pmatrix}$

and let $\gamma = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$,

which is an ordered basis for R^3. Let Q be the 3x3 matrix whose jth column is the jth vector of γ. Then

$Q = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ and $Q^{-1} = \begin{pmatrix} -1 & 2 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

So by the preceeding corollary,

$[L_A]_\gamma = Q^{-1}AQ = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}$

DEFINITION 2.21

Let A and B be matrices in $M_{n \times n}(F)$. We say that B is similar to A if there exists an invertible matrix Q such that $B = Q^{-1}AQ$.

令 A 與 B $\in M_{n \times n}(F)$，若稱 B 相似於 A，則存在一可逆矩陣 Q 使得 $B = Q^{-1}AQ$。