線性代數

Linear Algebra

A set is a collection of objects, called elements of the set. If x is an element of the set A, then we write $x \in A$; otherwise, we write $x \notin A$.

「集合（Set）」係物體的「集合（Collection）」；構成「集合（Set）」的物體稱為「集合（Set）」的元素（Elements）或成員（Members）。

若 x 是集合 A 的元素，則寫成 $x \in A$。若 x 不是 A 的元素，則寫成 $x \notin A$。

集合 A 與 B 相等（Equal），則寫成 A ＝ B。A ＝ B 意指集合 A 與 B 含有相同的元素。

集合可用下列兩種方式描述：

1.於集合符號｛｝間列出集合的元素。
2.以某特性來描述集合的元素。

元素 1, 2, 3,及 4 所形成的集合，可寫成｛1, 2, 3, 4｝或｛x：x 是小於 5 的正整數｝。

EXAMPLE

Let A denote the set of real number between 1 and 2. Then A may be written as
$A = \{x \in R : 1 < x < 2\}$
若 A 表示 1 與 2 間的實數所形成的集合，則 A 可寫成 $A = \{x \in R : 1 < x < 2\}$。

Relations

A *relation* from *A* to *B* is a rule that assigns elements of *A* to elements of *B*. An example of a relation would be a function but not all relations are functions. Indeed there are two elementary examples to show this. The first is the relation that assigns to each element of *A* every element of *B*. The second is the *empty relation* that assigns no element of *A* to any in *B*.

集合 A 與 B，由 A 到 B 的『關係（Relation）』是一種規則（Rule），一種「Assigns elements of *A* to elements of *B*」的規則。『關係』是一種「函數」，但非所有關

係都是「函數」。有兩種很基本的關係，一是 A 中每一元素均與 B 中每一元素有關係，另一是 A 中找不到任何元素與 B 中任一元素有關。

Relations on a set

Reflexivity 反身性

A relation ~ on A is said to be reflexive if for each a∈A a is related to a. If we let R denote the relation then we have aRa for each a∈A. An example of a non reflexive relation is the relation "is the father of" on a set of people. As no person is the father of themself the relation is not reflexive. As another example consider the relation ≈ on { 0, 1, 3 } defined by a≈b if a×b is odd. Then 1≈1 and 3≈3 but 0≉0 and so the relation is not reflexive. A relation on A is said to be irreflexive if for each a∈A a is not related to a. This is not the negation of the definition of reflexive. The relation "is the father of " is irreflexive.

若集合 A 內每一個 a，都與 a 有『關係~』，則稱在『關係~』下 A 具有反身性。若關係為『是誰…的爸爸』，而集合為「一群人」，則對「一群人」所組成的集合而言，因沒有人會是自己的爸爸，所以在『是誰…的爸爸』關係下這「一群人」所組成的集合並不具反身性。若集合{ 0, 1, 3 } 上的『關係≈』定義為「若 axb is odd，則 a≈b」，因「1≈1 and 3≈3, but 0≉0」，故在『關係≈』下 集合{ 0, 1, 3 } 不具反身性。

Symmetry 對稱性

A relation R on A is symmetric if given a~b then b~a.

集合 A 的任意元素 a 與 b，若存在「若 a~b，則 b~a」，則稱在『關係（~）』下 A 具有對稱性。

Transitivity 遞延性

A relation R on *A* is *transitive* if given *a*R*b* and *b*R*c* then *a*R*c*.

集合 A 的任意元素 a、b 與 c，若存在「若 a~b 且 b~c，則 a~c」，則稱在『關係~』下 A 具有遞延性。

Equivalence relations 等價關係

A relation ~ on *A* is an *equivalence relation* if it is reflexive, symmetric and transitive. 集合 A 上的『關係~ 』為等價關係（Equivalent relation），意指『關係~ 』滿足三種條件：

1. ∀x∈A，x~x（Reflexivity 反身性）

2. $\forall x, y \in A$，若 x~y，則 y~x（Symmetry 對稱性）

3. $\forall x, y, z \in A$，若 x~y 且 y~z，則 x~z（Transitivity 遞延性）

子集合（Subset）

若集合 B 是集合 A 的子集合（Subset），則可寫成 $B \subseteq A$ 或 $A \supseteq B$；意指集合 B 的每一元素都是集合 A 的元素。

若 $B \subseteq A$ 且 $A \neq B$，則集合 B 為集合 A 的真子集合（Proper subset）。

$A = B$ 若且為若 $B \subseteq A$ 且 $B \supseteq A$。

空集合（Empty set），以 $\varnothing$ 表示；意指不含任何元素的集合。

空集合是任一集合的「子集合」。

Unions and Intersections

兩個集合的聯集（Union），以 $A \cup B$ 表示；意指集合 A 與集合 B 內的元素共同形成的集合，寫成 $A \cup B = \{x : x \in A$ 或 $x \in B\}$。

兩個集合的交集（Intersection），以 $A \cap B$ 表示；意指同時出現在集合 A 與集合 B 內的元素所形成的集合，寫成 $A \cap B = \{x : x \in A$ 且 $x \in B\}$。

若兩個集合的交集為「空集合」，則稱兩個集合「互斥（Disjoint）」。

集合 A_1、A_2、…..、A_n 的聯集與交集定義為：

$$\bigcup_{i=1}^{n} A_i = \{x : x \in A_i, i = 1,2,\ldots,n\} \qquad \bigcap_{i=1}^{n} A_i = \{x : x \in A_i, i = 1,2,\ldots,n\}$$

函數（Function）

集合 A 與集合 B，由集合 A 映至集合 B 的函數（Function），寫成 $f : A \rightarrow B$，定義為「集合 A 的每一元素 x，在集合 B 中有唯一的元素與 x 對應」。

在集合 B 的唯一元素註記為 f(x)。

元素 f(x)為 x 的像（Image），x 為 f(x) 的前像（Preimage）。

若 f：A→B，則集合 A 為 f 的定義域（Domain），集合 B 為 f 的對應域（Codomain）。集合｛f(x)：x∈A｝稱為 f 的值域（Range）。

f 的值域是對應域 B 的子集合。

若 S⊆A（集合 S 是集合 A 的子集合），則集合 S 中所有元素的『像』集｛f(x)：x∈S｝，註記為 f(S)。

同理，若 T⊆B（集合 T 是集合 B 的子集合），則集合 T 中所有元素的『前像』集｛x∈A：f(x)∈T｝，註記為 $f^{-1}(T)$。

若二函數 f：A→B 及 g：A→B 相等（Equal），則 f = g，意指 f(x) = g(x)，∀x ∈ A 。

元素 1, 2, 3,及 4 所形成的集合，可寫成｛1, 2, 3, 4｝或｛x：x 是小於 5 的正整數｝。

EXAMPLE

Suppose that A = { x: x∈A, -10≦x≦10}. Let f: A→B be the function that assigns to each element s in A the element x^2+1 in R; that is, f is defined by $f(x) = x^2+1$.Then A is the domain of f, R is the codomain of f, and [1, 101] is the range of f.

設 A =｛x: x∈R, -10≦x≦10｝，令 f：A→R 定義為 A 中每一元素 x，在 R 中有唯一的元素 x^2+1 與 x 對應的函數。意即函數 f 定義為 $f(x) = x^2+1$。集合 A 是 f 的定義域，集合 R 是函數 f 的對應域，且 R=｛y: y∈R, 1≦y≦101｝是函數 f 的對應域。

由於 f(2) = 5，故 2 的像是 5，5 的前像為 2。-2 也是 5 的另一前像。

若 S = ｛x: x∈R,1≦x≦2｝（**定義域的子集合**）且 T = ｛y: y∈R, 82≦y≦101｝（**對應域的子集合**），則 f(S) = ｛y: y∈R, 2≦y≦5｝且 $f^{-1}(T)$ = ｛x: x∈R, -10≦x≦-9｝∪

｛x: x∈R, 9≦x≦10｝。

由 T = ｛y: y∈R, 82≦y≦101｝、$f^{-1}(T)$ = ｛x: x∈R, -10≦x≦-9｝⋃｛x: x∈R, 9≦x≦10｝來看，值域中的元素的前像不一定是唯一。

「值域的每一元素只有唯一前像」的函數，稱為「一對一（one-to-one）」函數。

若 f：A→B 為一對一函數，指「若 f(x) = f(y)，則 x = y」或「若 x≠y，則 f(x)≠f(y)」。

若 f：A→B，且 f(A) = B，則稱 f 為映成（Onto）函數。

若 f 是映成，則 f 的**值域等於對應域**。

EXAMPLE

令 f：｛x: x∈R, -1≦x≦1｝→｛y: y∈R, 0≦y≦1｝，f(x) = x^2，則此函數為映成（Onto）但非一對一（one-to-one），非一定一的理由為 f(-1) = f(1) = 1。**【一對一（one-to-one）的條件：值域的每一元素只有唯一前像。】**

若 S = ｛x: x∈R, 0≦x≦1｝(定義域的子集合)，則 f_S 為「映成且一對一」的函數。

集合 A、B、C，若 f：A→B 且 g：B→C 皆為函數，則 g‧f（Following f with g、**先 f 後 g**）：A→C 稱為 g 與 f 的合成（Composition）函數。

（g‧f）(x) = g (f(x))，∀x∈A。

For example, let A = B = C = R, f(x) = sin x, and g(x) = x^2+3. Then （g‧f）(x) = g (f(x)) = $\sin^2 x+3$, whereas （f‧g）(x) = f(g(x)) = sin(x^2+3). **Hence g‧f≠f‧g.（函數合成不具互換性。）**

Functional composition is **associative（函數合成有結合性）**, however; that is, if h: C→D is another function, then h‧(g‧f) = (h‧g)‧f.

函數 f：A→B 稱為可逆（Invertible），意即存在一函數 g：B→A 使得（f‧g）(y) =

y，$\forall y \in B$ 且（g‧f）(x) = x，$\forall x \in A$。若有這種函數 g 存在，則 g 為唯一，且稱 g 為 f 之反函數（Inverse）。記為 f^{-1}。

Function f is invertible if and only if f is both one-to-one and onto.

「函數 f 是可逆」若且為若條件為「f 為一對一且為映成」。

與可逆函數相關的事實：

1.若 f：A→B 為可逆（Invertible），則 f^{-1} 為可逆且$(f^{-1})^{-1} = f$。

2.若 f：A→B 及 g：B→C 皆為可逆，則 g‧f 為可逆且$(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$。

EXAMPLE

函數 f：R→R，f(x) = 3x+1，是一對一且映成，故 f 為可逆。f 之反函數是 f^{-1}：R→R，$f^{-1}(x) = (x-1)/3$。

體（Field）

Basically, a field is a set in which four operations (called addition, multiplication, substraction, and division) can be defined so that, with the exception of division by zero, the sum, product, difference, and quotient of any two elements in the set is an element of the set.

「Field」是一個集合，其元素為純量（Scalar），可分成實數體 R 與複數體 C。該集合內定義的加、減、乘、除等四種運算，除了使用零作為除數外，集合內任二元素加、減、乘、除的結果仍是集合內的元素。

DEFINTION

A field F is a set on which two operations + and ‧are defined so that, for each pair of elements, x, y in F, there are unique elemensts x+y and x‧y in F for which the following conditions hold for all elements a, b,c in F.

Field F 為一集合，＋（加法 Addition）與‧（乘法 Multiplication）為該集合上所定義的兩種運算。F 內任一對元素 x 與 y，透過運算可在 F 內獲得唯一的 x＋y 與 x‧y。因此，對 F 內任何 a、b、c、d 而言，下列條件成立：

【F1】a＋b = b＋a 且 a‧b = b‧a（加法與乘法的交換律）

【F2】(a＋b)＋c = a＋（b＋c）且(a‧b)‧c = a‧(b‧c)（加法與乘法的結合律）

【F3】存在 0 及 $1 \in F$ ，使得 0+a = a 且 1‧a = a

【F4】對任一元素 $a \in F$ 及任一非零元素 $b \in F$，存在 c 及 $d \in F$，使得 a＋c = 0 且

b・d = 1（加法與乘法的反元素）

【F5】a・(b＋c) = a・b＋a・c（乘法對加法分配律）

Theorem：消去律（Cancellation laws）

Let a and b be arbitrary elements of a field（a 與 b 是 Field 中任意元素）. Then each of the following statements are true:

1. a＋b = c＋b，則 a = c
2. a・b = c・b，則 a = c

Theorem

Let a and b be arbitrary elements of a field（a 與 b 是 Field 中任意元素）. Then each of the following statements are true:

1. a・0 = 0
2. (-a)・b = a・(-b) = -(a・b)
3. (-a)・(-b) = a・b

Chapter 1 Vector Spaces

Many familiar physical notion, such as forces, velocities, and accelerations, involve both a magnitude and a direction. Any such entity involving both magnitude and direction is called a “vector.”A vector is represented by an arrow whose length denotes the magnitude of the vector and whose direction represents the direction of the vector.

涉及大小與方向本質者皆可稱為「向量」，如作用力、速度與加速度等。常以一箭號表示向量，箭號的長度表示向量的大小，箭號的方向表示向量的方向。

Familiar situations suggest that when two like physical quantities act simultaneously at a point, the magnitude of their effect need not equal the sum of the magnitudes of the original quantities. Experiments show that if two like quantities act together, their effect is predictable. In this case, the vectors used to represent these quantities can be combined to form a resultant vector that represents the combined effects of the original quantities. This resultant vector is called the sum of the original vectors, and the rule for their combination is called the parallelgram law.

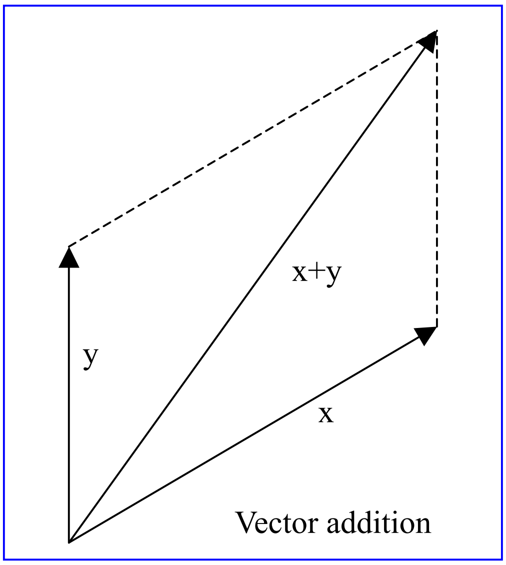
通常兩個相像的物理量同時作用在某一點時，所產生的效應大小，未必是原先兩個物理量的總和。物理實驗顯示兩個相像的物理量一起作用時，其總和效應是可以預期的。若把代表兩個物理量的向量組合在一起，所得到的新向量為兩向量和。兩個向量相加的規則，稱為平行四邊形定律。

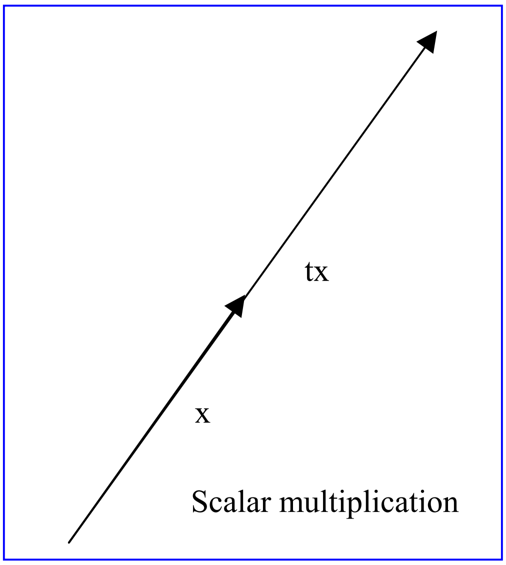
Parallelogram law for vector addition

The sum of two vectors x and y that act at the same point P is the vector beginning at P that is represented by the diagonal of parallelogram having x and y as adjacent sides.

Scalar multiplication

Multiplying a vector by a real number. If the vector x is represented by an arrow, the for any nonzero real number t, the vector tx is represented by an arrow in the same direction of $t > 0$ and in the opposite direction if $t < 0$. The length of the arrow tx is $|t|$ times the length of the arrow x.





The algebraic descriptions of vector addition and scalar multiplication for vectors in a plane yield the following properties:

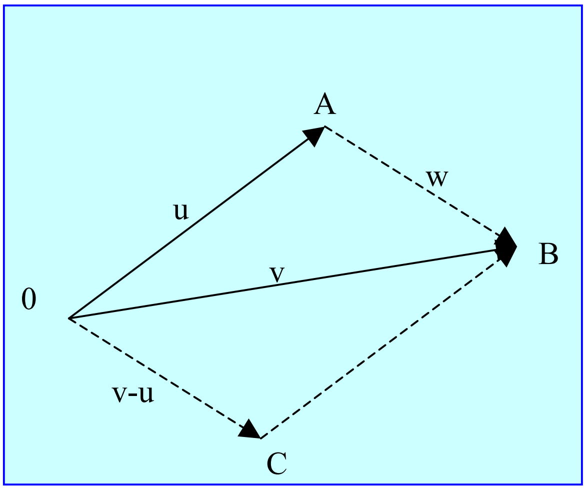
1. For all vectors x and y, x + y = y + x.
2. For all vectors x, y, and z, (x + y) + z = x + (y + z).
3. There exits a vector denoted 0 such that x + 0 = x for each vector x.
4. For each vector x, there is a vector y such that x + y = 0.
5. For each vector x, 1x = x.
6. For each pair real numbers a and b and each vector x, (ab) x = a (bx).
7. For each real number a and each pair of vectors x and y, a(x + y) = ax + by.
8. For each pair of real numbers a and b and each vector x, (a + b) x = ax + bx.

Arguments similar to the corresponding ones show that these eight properties, as well as the geometric interpretations of vector additions and scalar multiplication, are true also for vectors acting in space rather than in a plane. These results can be used to write equations of lines and planes in space.

上述八個性質及有關向量加法與純量乘積的幾何意義，對空間的向量亦成立。這些結果可用來撰寫空間的直線與平面方程式。

Consider first the equation of a line in space that passes through two distinct points A and B. Let O denote the origin of a coordinate system in space, and let u and v denote the vectors that begin at O and end at A and B, respectively. If w denotes the vector beginning at A and ending at B, and hence w = v – u. Since a scalar multiple of w is parallel to w but possibly of a different length than w, any point on the line joining A and B my be obtained as the endpoint of a vector that begins at A and has the form tw for some real number t. Thus an equation of the line through A and B is 通過 A 與 B 的線方程式：

$$\mathbf{x} = \mathbf{u} + t\mathbf{w} = \mathbf{u} + t\,(\mathbf{v} - \mathbf{u})$$



EXAMPLE

Let A and B be points having coordinates (-2, 0, 1) and (4, 5, 3), respectively. The end point C of the vector emanating from the origin and having the same direction as the vector beginning at A and terminating at B has coordinates (4, 5, 3) - (-2, 0, 1) = (6, 5, 2). Hence the equation of the line through A and B is

x = (-2, 0, 1) + t (6, 5, 2) 【連接 A 與 B 的線方程式】

EXAMPLE

Let A, B, and C be the points having coordinates (1, 0, 2), (-3, -2, 4), and (1, 8, -5), respectively. The endpoint of the vector emanating from the origin and having the same length and direction as the vector beginning at A and terminating at B is (-3, -2, 4) - (1, 0, 2) = (-4, -2, 2).

Similarly, the endpoint of a vector emanating from the origin havng the same length and direction as the vector beginning at A and terminating at C is (1, 8, -5) - (1, 0, 2) = (0, 8, -7).

Hence the equation of the plane containing the three given points is

x = (1, 0, 2) + s(-4, -2, 2) + t (0, 8, -7) 【連接 A、B、C 的面方程式】

Exercises 2(a), 3(a)

1-1 Vector Spaces

向量空間（Vector space）V 的元素為向量。

DEFINITION 1.1 Vector Space

A vector space V over field F consists of a set on which two operations are defined so that for each pair of elements x, y, in V there is a unique element x+y in V, and for each element a in F and each element x in V there is a unique ax in V, such that the following conditions hold.

(VS 1) For all x, y $\in$ **V**, x+y = y+x.

(VS 2) For all x, y, z $\in$ **V**, x+(y+z) = (x+y)+z.

(VS 3) There exists an element in V denoted by 0 such that x+0 = x for each x $\in$ **V.**

(VS 4) For each x $\in$ **V**, there exists y $\in$ **V**, such that x+y = 0.

(VS 5) For each x $\in$ **V**, 1x = x.

(VS 6) For each pair of elements a, b $\in$ **F**, for each x $\in$ **V**, (ab)x = a(bx)

(VS 7) For each element a $\in$ **F**, for each pair of elements x, y $\in$ **V**, a(x+y) = ax+ay.

(VS 8) For each pair of elements a, b in F, for each element x $\in$ **V**, (a+b)x = ax+bx.

佈於 Field F 的向量空間（Vector space）或線性空間（Linear space）V 係由定義有加法（Addition）與純量乘積（Scalar multiplication）運算的集合所組成。該加法（Addition）與純量乘積（Scalar multiplication）運算除滿足「V 內每一對元素 x, y，V 內存在唯一元素 x + y」與「V 內任一元素 x 與 F 內任一元素 a，V 內存在唯一元素 ax」，並使下列條件成立：

VS 1. V 內所有的 x、y ，x + y = y + x。

VS 2. V 內所有的 x、y 與 z ，x + (y + z) = (x + y) + z。

VS 3. V 內每一元素 x，有一元素 0，使得 x + 0 = x。

VS 4. V 內每一元素 x，V 內存在一元素 y，使得 x + y = 0。

VS 5. V 內每一元素 x，1 x = x。

VS 6. F 內每一對元素 a、b 與 V 內每一元素 x， (ab) x = a (bx)。

VS 7. F 內每一元素 a 與 V 內每一對元素 x、y，a (x + y) = ax + ay。

VS 8. F 內每一對元素 a、b 與 V 內每一元素 x，(a + b) x = ax + bx。

其中，x+y 與 ax 分別稱為 x 與 y 的和（sum）及 a 與 x 的積（product）。

The **elements** of the field F are called **scalars** and the **elements** of the vector space V are called **vectors**（F 的元素為純量，V 的元素為向量）.

n-tuples

DEFINITION 1.2 n-tuple

An object of the form ($a_1, a_2, \ldots\ldots, a_n$), where the entries $a_1, a_2, \ldots\ldots, a_n$ are elements of a field F, is called an n-tupe with entries from F. The elements $a_1, a_2, \ldots\ldots, a_n$ are called the entries or components of the n-tupe.

Two n-tuples ($a_1, a_2, \ldots\ldots, a_n$) and ($b_1, b_2, \ldots\ldots, b_n$) with entries from a field F are called equal if $a_i = b_i$, for i = 1, 2,…n.

有一型式為（$a_1, a_2, \ldots\ldots, a_n$）的實體，若 $a_1, a_2, \ldots\ldots, a_n$ 為 Field 的元素，則（$a_1, a_2, \ldots\ldots, a_n$）稱為選自 F 的 n-序組（n-tuple），$a_1, a_2, \ldots\ldots, a_n$ 為 n-tuple 的元素（entries）或分量（components）。

兩 n-tuples （$a_1, a_2, \ldots\ldots, a_n$）與（$b_1, b_2, \ldots\ldots, b_n$），若 $a_i = b_i$，$\forall i = 1,2,\ldots.n$ ，則兩 n-tuple（$a_1, a_2, \ldots\ldots, a_n$）與 n-tuple（$b_1, b_2, \ldots\ldots, b_n$）相等。

所有 entries 選自 Field F 的 n-tuple 所組成的集合，記為 F^n。
所有 entries 選自 Field F 的 mxn 矩陣所組成的集合，記為 $M_{m\times n}(F)$。
所有係數選自 Field F 的多項式所組成的集合，記為 P(F)。

EXAMPLE 1

The set of all n-tuples with entries from a field F is denoted by F^n. This set is a vector space over F with the operations of coordinatewise addition and scalar multiplication; that is, if u = （$a_1, a_2, \ldots\ldots, a_n$）$\in F^n$, v = （$b_1, b_2, \ldots\ldots, b_n$）$\in F^n$ and $c \in F$, then u+v =（$a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n$）and cu = （$ca_1, ca_2, \ldots\ldots, ca_n$）.

所有選自 Field F 的 n-tuples 所形成的集合，註記為 F^n。該集合為佈於 F 的向量空間，其上兩個運算分別為對應 entries 相加的加法與純量乘個別 entry 的純量乘積。

若 u＝ （$a_1, a_2, \ldots\ldots, a_n$）$\in F^n$，v＝ （$b_1, b_2, \ldots\ldots, b_n$）$\in F^n$，且 $c \in F$，

則 u+v＝ （$a_1+b_1, a_2+b_2, \ldots\ldots, a_n+b_n$）, cu＝ （$ca_1, ca_2, \ldots\ldots, ca_n$）。

Thus R^3 is a vector space over R. In this vector space, (3, -2, 0) ＋（-1, 1, 4） = (2, -1, 4) and -5 (1, -2, 0) = (-5, 10, 0) （R^3：Entries 為 real number 的 3-tuples 所組成的集合）

R^3 是一佈於 R 的向量空間，在 R^3 中 (3, -2, 0) ＋（-1, 1, 4） = (2, -1, 4) 、 -5 (1, -2, 0) = (-5, 10, 0)。

Similaryly, C^2 is a vector space over C. In this vector space (1+i, 2) ＋ (2-3i, 4i) = (3-2i, 2+4i) 且 i (1+i, 2) = (-1+i, 2i)。(C^2：Entries 為 complex number 的 2-tuples 所組成的集合)

C^2 是一佈於 C 的向量空間，在 C^2 中(1+i, 2) ＋ (2-3i, 4i) = (3-2i, 2+4i)、 i(1+i, 2) = (-1+i, 2i)。

Matrix

An m×n matrix with entries **from a field**（其元素是選自 Field F）is **rectangular array** of the form

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ : & : & : & : \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

where each entry a_{ij} ($1 \leqq i \leqq m$, $1 \leqq j \leqq n$) is an element of F. a_{ij} 是 F 的元素。

當 i＝j 時，a_{ij} 為矩陣的對角元素（Diagonal entries）。

The entries $a_{i1}, a_{i2}, a_{i3},\ldots, a_{in}$ compose the ith row of the matrix. 第 i 列（橫）。

The entries $a_{1j}, a_{2j},\ldots, a_{mj}$ compose the jth column of the matrix. 第 j 行（直）。

矩陣的行數與列數相等者，稱為方矩陣（Square matrix）。

當兩個 m×n 矩陣 A 與 B 相等（Equal），表示兩者對應的元素皆為相等，即 $A_{ij} = B_{ij}$（$1 \leqq i \leqq m$, $1 \leqq j \leqq n$）。

Zero matrix

The m×n matrix in which each entry equals to zero is called the **zero matrix** and denoted by $O_{m\times n}$。每一個元素皆為 0 的 mxn matrix，稱為零矩陣。註記為 $O_{m\times n}$。

$$O = \begin{pmatrix} 0 & 0 & . & . & 0 \\ 0 & 0 & & & 0 \\ : & : & & & : \\ : & : & & & : \\ 0 & 0 & . & . & 0 \end{pmatrix}$$

EXAMPLE 2

The set of all mxn matrices with entries from a field F is a vector space, which we denote by $M_{m\times n}(F)$, with the following operations of matrix addition and scalar multiplication:

For $A,B \in M_{m\times n}(F)$ and $c \in F$,

$(A+B)_{ij} = A_{ij}+B_{ij}$ and $(cA)_{ij} = cA_{ij}$ for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$.

Entries 皆選自 F 的 mxn 矩陣所組成的集合是一個向量空間，註記為 $M_{m\times n}(F)$。$M_{m\times n}(F)$具有下列矩陣加法與純量乘積運算：

For $A,B \in M_{m\times n}(F)$ and $c \in F$,

$(A+B)_{ij} = A_{ij}+B_{ij}$ （加法）and $(cA)_{ij} = cA_{ij}$（純量乘積） for $1 \leqq i \leqq m$, $1 \leqq j \leqq n$.

For instance, $\begin{pmatrix} 2 & 0 \\ 1 & -3 \end{pmatrix} + \begin{pmatrix} -5 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$ $\quad 3\begin{pmatrix} -5 & -2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} -15 & -6 \\ 9 & 12 \end{pmatrix}$

Polynomial

A polynomial with coefficients from a field F is a expression of the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

where n is a nonnegative integer and each a_k, called the coefficient of x^k, is in F. If f (x) = 0, that is, if $a_n = a_{n-1} = \ldots = a_0 = 0$, then f(x) is called the zero polynomial.

係數選自 Field F 的多項式，可表示為 $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$。其中，n 為非負整數，$a_k$ 稱為 x^k 的係數，為 Field F 的元素。若 f(x) = 0，且 $a_n = a_{n-1} = \ldots = a_0 = 0$，則 f(x)稱為零多項式（Zero polynomial）。

The degree of a polynomial is defined to be the largest exponent of x that appears in the representation

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$

with a nonzero coefficient.

多項式的 Degree 為係數非零的最大 x 指數。

NOTE: The polynomials of degress zero must be written in the form f(x) = c for some nonzero scalar c. 多項式的 degree 等於 0，可寫成 f(x) = c。

Two polynomials,

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0$

are called equal if m = n and $a_i = b_i$ for i = 0,1,….,n.

兩多項式相等的條件：(1) Degree 相等、(2) 對應係數相等。

EXAMPLE 3

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1} x^{m-1} + \ldots + b_1 x + b_0$ be polynomials with coefficients from a field F.

Suppose that m > n, and define $b_m = b_{m-1} = \ldots = b_{n+1} = 0$. Then g(x) can be written as $g(x) = b_n x^n + b_{n-1} x^{n-1} + \ldots + b_1 x + b_0$.

Define

$f(x)+g(x) = (a_n+b_n)x^n + (a_{n-1}+b_{n-1})x^{n-1} + \ldots + (a_1+b_1)x + (a_0+b_0)$ and for any $c \in F$,

$cf(x) = ca_n x^n + ca_{n-1} x^{n-1} + \ldots + ca_1 x + ca_0$.

With these operations of addition and scalar multiplication, the set of all polynomials with coefficients from F is a vector space, which we denote by P(F).

在加法與純量乘積運算定義下，所有係數選自 Field 的多項式所形成的集合為一向量空間，記為 P(F)。

Sequence

EXAMPLE 4

Let F be any field. **A sequence** in F is **a function σ** from the positive integers into F. **In this book, the sequence σ such that σ(n) = a_n for n = 1,2,… is denoted {a_n}.**

Let V consist of all sequences {a_n} in F that have only a finite number of nonzero terms.

If {a_n} and {b_n} are in V and t∈F, define

$\{a_n\}+\{b_n\} = \{a_n + b_n\}$ and $t\{a_n\} = \{ta_n\}$

With these operations V is a vector space.

Field 中的一序列可看成是由正整數映至 Field 的函數 σ。

將 σ 定義為 $\sigma(n) = a_n$ **for n = 1,2,…**，並註記為 {a_n}。

令 V 是所有序列{ a_n }的集合。

若{ a_n }與{b_n }為 V 中兩組序列、t∈F，且序列的加法與純量乘積定義為：

$\{a_n\}+\{b_n\} = \{a_n + b_n\}$ and $t\{a_n\} = \{ta_n\}$

則在此定義下，序列{ a_n }所形成的集合 V 為向量空間。

註：V 的元素為{a_n}。

EXAMPLE 5

Let S = {(a_1, a_2): a_1,a_2∈R}. For (a_1, a_2), (b_1, b_2) ∈S and c∈R, define $(a_1, a_2)+ (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$ and $c (a_1, a_2) = (ca_1, ca_2)$ Since (VS1), (VS2), and (VS8) fail to hold, S is not a vector space with these operations.

S 是序列(a_1, a_2)所形成的集合，若加法與純量乘積定義為：

$(a_1, a_2)+ (b_1, b_2) = (a_1 + b_1, a_2 - b_2)$

$c (a_1, a_2) = (ca_1, ca_2)$

S 是否為向量空間？

【答案】

在這種加法與純量乘積定義下，VS1、VS2 與 VS8 皆不成立，故 S 不是向量空間。

VS 1. V 內所有的 x、y ，x + y = y + x。

VS 2. V 內所有的 x、y 與 z ，x + (y + z) = (x + y) + z。

VS 8. F 內每一對元素 a、b 與 V 內每一元素 x，(a + b) x = ax + bx。

EXAMPLE 6

Let S = {(a_1,a_2): a_1,a_2∈R}. For (a_1,a_2), (b_1,b_2) ∈S and c∈R, define $(a_1, a_2)+ (b_1, b_2) = (a_1+b_1, 0)$ and $c (a_1, a_2) = (ca_1, 0)$. Since (VS3), (VS4), and (VS5) fail to hold, S is not a vector space with these operations.

S 是序列(a_1, a_2)所形成的集合，若加法與純量乘積定義為：

$(a_1, a_2)+ (b_1, b_2) = (a_1+b_1, 0)$

$c (a_1, a_2) = (ca_1, 0)$

S 是否為向量空間？

【答案】

在這種加法與純量乘積定義下，VS3、VS4 與 VS5 皆不成立，故 S 不是向量空間。

VS 3. V 內每一元素 x，有一元素 0，使得 x + 0 = x。

VS 4. V 內每一元素 x，V 內存在一元素 y，使得 x + y = 0。

VS 5. V 內每一元素 x，1 x = x。

Theorem 1-1 (Cancellation Law for Vector Addition)

If x, y and z are vectors in a vector space V such that x + z = y + z, then x = y.

【Proof】

V 是一向量空間，可援用向量空間所具備的特性：VS1~VS8。

「**(VS4) V 內每一元素 z，V 內存在一元素 v，使得 z+v = 0**」，因此由 VS2 與 VS3 可得知：x = x + 0 = x + (z + v) = (**x + z**) + v = (**y + z**) + v = y + (z + v) = y + 0 = y.

VS 2. V 內所有的 x、y 與 z ，x + (y + z) = (x + y) + z。

VS 3. V 內每一元素 x，有一元素 0，使得 x + 0 = x。

VS 4. V 內每一元素 x，V 內存在一元素 y，使得 x + y = 0。

Corollary 1

The vector O in (VS 3) is unique.

在 VS3 中所指的向量O是唯一的。

VS 3. V 內每一元素 x，有一元素 0，使得 x + 0 = x。

Corollary 2

The vector y in (VS 4) is unique.

在 VS4 中所指的向量 y 是唯一的。

VS 4. V 內每一元素 x，V 內存在一元素 y，使得 x + y = 0。

Theorem 1.2

In any vector space V, the following statements are true:

當 V 為向量空間，則下列敘述為真：

(a) ox = 0 for each x∈V.

(b) (-a)x = - (ax) = a (-x) for each a∈F and each x∈V.

(c) a0 = 0 for each a∈F.

【Proof】

(a) By VS8, VS3, and VS1, it follows that 0x + 0x = (0 + 0)x = 0x = 0x + 0 = 0 + 0x. Hence 0 x = 0 by Theorem 1.1.

(b) The vector –(ax) is the unique element of V such that ax+[-(ax)] = 0. Thus if ax + (-a)x = 0, Corollary 2 to Theorem 1.1 implies that (-a)x = -(ax). But by VS8, ax + (-a)x = 0x = 0 by (a),

Consequently (-a)x = -(ax).

By VS6, a(-x) = a[(-1)x] = [a(-1)]x = (-a)x.

VS 1. V 內所有的 x、y ，x + y = y + x。

VS 3. V 內每一元素 x，有一元素 0，使得 x + 0 = x。

VS 6. F 內每一對元素 a、b 與 V 內每一元素 x，（ab）x = a（bx）。

VS 8. F 內每一對元素 a、b 與 V 內每一元素 x，（a + b）x = ax + bx。

1-2 Subspaces 子空間

DEFINITION 1.3 Subspace

A subset W of a vector space V over a field is called a subspace of V if W is a vector space over F with the operations of addition and scalar multiplication defined on V.

W 為佈於 F 的向量空間 V 的子集合，W 為 V 的子空間（Subspace）的條件為：W 為向量空間且具有向量空間 V 所定義的加法與純量乘積運算。

In any vector space V, note that V and { 0 } are subspaces. { 0 } is called the zero subspace of V.

任意向量空間 V 中，V 與 { 0 } 是 V 的子空間，{ 0 } 稱為 V 的零子空間（Zero subspace）。

Fortunately it is not necessary to verify all of the vector space properties to prove that a sunset is a subspace. Because properties (VS 1), (VS 2), (VS 5), (VS 6), (VS 7), and (VS 8) hold for all vectors in the vector space, these properties automatically hold for the vectors in any subset. Thus a subset W of the vector space V is a subspace of V if and only if the following four properties hold.

要證明向量空間 V 的子集合 W 確實為一子空間，不必去驗證所有向量空間的性質（VS1~VS8），因為條件 VS1、VS2、VS5、VS6、VS7、VS8 適用於向量空間的元素，自然就自動適用於任意子集合的元素。因此，要驗證 V 的子集合 W 為一子空間，只要檢驗下列四個性質是否成立：【子空間 W 成立的「若且唯若」條件】

1. $x+y\in W$ whenever $x\in W$ and $y\in W$. 當 $x\in W$ 且 $y\in W$ 時，$x+y\in W$（W 在加法下是封閉的）。
2. $cx\in W$ whenever $c\in F$ and $x\in W$. 當 $c\in F$ 且 $x\in W$ 時，$cx\in W$（W 在純量乘積下是封閉的）。
3. W has a zero vector. W 擁有一零向量。
4. Each vector in W has an additive inverse in W. W 中每一向量，擁有一個加法反向量。

Theorem 1.3 W is a subspace of V

Let V be a vector space and W a subset of V. Then W is a subspace of V if and only if the following three conditions hold for the operations defined in V.

令 V 為向量空間，W 為 V 的子集合，則 W 為 V 的子空間「若且唯若」下列三個條件適用於 V 內所定義的運算：

(a) $0\in W$。

(b) $x+y\in W$ whenever $x\in W$ and $y\in W$. 當 $x\in W$ 且 $y\in W$ 時，則 $x+y\in W$。

(c) $cx\in W$ whenever $c\in F$ and $x\in W$. 當 $c\in F$ 且 $x\in W$ 時，$cx\in W$。

【Proof】

證明 W 是子空間→三條件成立…

If W is a subspace of V, then W is a vector space with the operations of addition and scalar multiplication defined on V.

Hence condition (b) and (c) holds, and there exists a vector $0' \in W$ such that $x+0' = x$ for each $x\in W$. But also $x+0 = x = x+0'$, and thus $0' = 0$ by Theorem 1.1. So codition (a) holds.

若 W 是 V 的子空間，則 W 為向量空間且具有向量空間 V 所定義的加法與純量乘積運算。因此條件（b）與（c）沒問題，條件（a）則引用 Theorem 1.1。

再證明三條件成立→W 是子空間…

Conversely, if conditions (a), (b), and (c) hold, the discusson preceeding this theorem shows that W is a subspace of V if additive inverse of each vector in W lie in W.

But if $x \in W$, then $(-1)x \in W$ by condition (c), and $-x = (-1)x$ by Theorem 1.2. Hence W is a subspace of V.

若條件（a）、（b）、（c）成立，依據前面的討論得知，只要 W 內每一元素的加法反元素仍屬於 W，則 W 為 V 的子空間。

1. 條件（c）：若 $x \in W$，則 $(-1)x \in W$。
2. 定理 1.2：$-x = (-1)x$。

▶ $-x \in W$（加法反元素仍屬於 W），故 W 是 V 的子空間。

Theorem 1.1 If x, y and z are vectors in a vector space V such that $x+z = y+z$, then $x = y$.

Theorem 1.2 In any vector space V, the following statements are true:

(a) $0x = 0$ for each $x \in V$.

(b) $(-a)x = -(ax) = a(-x)$ for each $a \in F$ and each $x \in V$.

(c) $a0 = 0$ for each $a \in F$.

Subspaces

❖ Let *V* be a vector space and *U* be a nonempty subset of *V*. *U* is said to be a **subspace** of *V* if **it is closed under addition and under scalar multiplication.**

❖ U is a subspace if it is closed under addition and under scalar multiplication. It then "inherits" the other vector space properties from V.

▶ For example, the subset U of R^3 consisting of vectors of the form (a, a, b) which have the first two components the same. (2, 2, 5) is in U while (1, 3, 9) is not in U.

Theorem

❖ If *U* be a subspace of a vector space *V*, then ***U* contains the zero vector of *V*.**

Proof

Let **u** be an arbitrary vector in *U* and **0** be the vector of *V*. Let 0 be the zero scalar.

By Theorem 4.5(a) we know that $0\mathbf{u} = \mathbf{0}$. Since *U* is closed under scalar multiplication, this means that **0** is in *U*.

Remark:

If a given subset does not contain the zero vector, it cannot be a subspace.

Example

❖ Let *U* be the subset of $\mathbf{R}^3$ consisting of all vectors of the **form $(a, 0, 0)$** (with zeros as second and third components). Show that *U* is a subspace of $\mathbf{R}^3$.

Let $(a, 0, 0)$ and $(b, 0, 0)$ be two elements of *U* and let *k* be a scalar. We get

$$(a, 0, 0) + (b, 0, 0) = (a + b, 0, 0) \in U$$
$$k(a, 0, 0) = (k\,a, 0, 0) \in U$$

Example

❖ Let W be the set of vectors of the form (a, a^2, b). Show that W is not a subspace of $\mathbf{R}^3$.

Let (a, a^2, b) and (c, c^2, d) be elements of W.

$$(a, a^2, b) + (c, c^2, d) = (a+c, a^2 + c^2, b + d)$$
$$\neq (a + c, (a + c)^2, b + d)$$

Thus $(a, a^2, b) + (c, c^2, d)$ is not an element of W. W is not closed under addition. W is not a subspaces.

Example 1/2

❖ Let P_n denote the set of real polynomial functions of degree $\le n$. Prove that P_n is a vector space if addition and scalar multiplication are defined on polynomials in a point-wise manner.

Let f and g be two elements of P_n defined by

$f = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

$g = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$

$(f+g)(x) = f(x) + g(x)$

$= [a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0] + [b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0]$

$= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0) \in P_n$

Example 2/2

$(cf)(x) = c[f(x)]$

$= c[a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0]$

$= ca_n x^n + ca_{n-1}x^{n-1} + \ldots + ca_1 x + ca_0$

(cf)(x) is a polynomial of degree $\le n$. Thus cf is an element of P_n. P_n is closed under scalar multiplication.

P_n is a subspace of V and therefore a vector space.

Transpose of a matrix 轉置矩陣

The transpose A^t of an m×n matrix A is the n×m matrix obtained from A by interchanging the rows with the columns; that is, $(A^t)_{ij} = A_{ji}$.

定義「轉置矩陣」：

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 5 & 1 \end{pmatrix}_{2\times 3} \Rightarrow A^t = \begin{pmatrix} 1 & 0 \\ -2 & 5 \\ 3 & 1 \end{pmatrix}_{3\times 2}$$

$$B = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \Rightarrow B^t = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \qquad B = B^t$$

若滿足 $A^t = A$，則 A 為一對稱矩陣（Symmetric matrix）

Clearly, a symmetric matrix must be square. 對稱矩陣必定為方矩陣。

The set W of all symmetric matrices in $M_{n\times n}(F)$ is subspace of $M_{n\times n}(F)$ since the conditions of Theorem 1.3 holds:

$M_{n\times n}(F)$中所有對稱矩陣所形成的集合 W 是 $M_{n\times n}(F)$的子空間？因為 Theorem 1.3 所列條件皆成立，故 W 是 $M_{n\times n}(F)$的子空間：

1. The zero matrix is equal to its transpose and hence belongs to W. 零矩陣的轉置矩陣為零矩陣，故零矩陣屬於 W。
2. If $A\in W$ and $B\in W$, then $A^t = A$ and $B^t = B$. Thus $(A+B)^t = A^t+B^t = A+B$, so that $A+B\in W$. 若 $A\in W$ 且 $B\in W$，則 $A^t = A$ 且 $B^t = B$，於是$(A+B)^t = A^t+B^t = A+B$，所以 $A+B\in W$。
3. If $A\in W$, then $A^t = A$. So for any $a\in F$, we have $(aA)^t = aA^t = aA$. Thus $aA\in W$. 若 $A\in W$，則 $A^t = A$。對任一 $a\in F$，$(aA)^t = aA^t = aA$，所以 $aA\in W$。

EXAMPLE 1

An $n \times n$ matrix M is called a diagonal matrix（稱為對角矩陣者） if $M_{ij} = 0$ whenever $i \neq j$; that is, if all its nondiagonal entries are zero. Clearly the zero matrix is a diagonal matrix because all of its entries are 0. Moreover, if A and B are diagonal $n \times n$ matrices, then whenever $i \neq j$,

$(A+B)_{ij} = A_{ij}+B_{ij} = 0+0 = 0$ and $(cA)_{ij} = cA_{ij} = c0 = 0$

For any scalar c. Hence A+B and cA are diagonal matrices for any scalar c. Therefore the set of diagonal matrices is a subspace of $M_{n \times n}(F)$ by Therorem 1.3.

若 A 與 B 均為對角矩陣，則

$(A+B)_{ij} = A_{ij}+B_{ij} = 0+0 = 0$ 且 $(cA)_{ij} = cA_{ij} = c0 = 0$ （$i \neq j$）

對任一純量而言，A+B 與 cA 均為對角矩陣。

依據定理 1.3，所有對角矩陣所成的集合為 $M_{n \times n}(F)$的子空間。

Theorem 1.3 令 V 為向量空間，W 為 V 的子集合，則 W 為 V 的子空間「若且唯若」下列三個條件適用於 V 內所定義的運算：

(a) $0 \in W$。

(b) $x+y \in W$ whenever $x \in W$ and $y \in W$. 當 $x \in W$ 且 $y \in W$ 時，則 $x+y \in W$。

(c) $cx \in W$ whenever $c \in F$ and $x \in W$. 當 $c \in F$ 且 $x \in W$ 時，$cx \in W$。

EXAMPLE 2

The trace of an $n \times n$ matrix M, denoted by tr(M), is the sum of the diagonal entries of M; that is $tr(M) = M_{11}+M_{22}+\ldots+M_{nn}$

矩陣 M 的跡，註記為 tr(M)，是 M 的對角元素的和。

EXAMPLE 3

The set of matrices in $M_{n \times n}(R)$ having nonnegative entries is not a subspace of $M_{n \times n}(R)$ because it is not closed under scalar multiplication (by negative scalars).

$M_{n \times n}(R)$內，元素為非負的矩陣所形成的集合，不是 $M_{n \times n}(R)$的子空間。理由為：乘上負的純量後，所得的純量乘積不屬於該集合，即該集合在純量乘積下「不具封閉性」。

Theorem 1.4

Any intersection of subspace of V is a subspace of V.

向量空間 V 的子空間的交集也是 V 的子空間。

【**Proof**】

Let C be a collection of subspaces of V, and let W denote the intersection of the subspaces in C. Since every subspace contains the zero vector, $0\in W$. Let $a\in F$ and $x, y\in W$. Then x and y contained in each subspace in C. Because each subspace in C is closed under addition and scalar multiplication, it follows that x+y and ax are contained in each subspace in C. Hence x+y and ax are also contained in W, so that W is a subspace of V by Theorem 1.3.

令 C 為 V 的子空間所形成的集合，並令 W 為所有子空間的交集。

因每一子空間均含有零向量，$0\in W$。

令 $a\in F$ 且 x、$y\in W$，則 x 及 y 皆為 C 內每一子空間的元素。

因為 C 內每一子空間在加法與純量乘積定義下是封閉的，於是 x+y 及 ax 均為 C 內每一子空間的元素，因此 x+y 及 ax 亦為 W 的元素。

由定理 1-3 得知，W 為 V 的子空間。

Theorem 1.3 令 V 為向量空間，W 為 V 的子集合，則 W 為 V 的子空間「若且唯若」下列三個條件適用於 V 內所定義的運算：

(a) $0\in W$。

(b) $x+y\in W$ whenever $x\in W$ and $y\in W$. 當 $x\in W$ 且 $y\in W$ 時，則 $x+y\in W$。

(c) $cx\in W$ whenever $c\in F$ and $x\in W$. 當 $c\in F$ 且 $x\in W$ 時，$cx\in W$。

1-3 Linear Combinations and System of Linear Equations

DEFINITION 1.4 Linear combination

Let V be a vector space and S a nonempty subset of V. A vector $v\in V$ is called a **linear combination** of vectors of S if there exist a finite number of vectors $u_1, u_2,\ldots, u_n$ in S and scalars $a_1, a_2,\ldots, a_n$ in F such that $v = a_1u_1+a_2u_2+\ldots+a_nu_n$. **In this case we also say that v is a linear combination of $u_1,u_2,\ldots,u_n$ and call $a_1,a_2,\ldots,a_n$ the coefficients of the linear combinations**.

令 V 為一向量空間、S 為 V 的非空子集合。V 內一個向量 v 要能為 S 內向量的線性組合，表示在 S 內具有有限個向量 $u_1, u_2,\ldots, u_n$ 且在 F 內具有純量 $a_1, a_2,\ldots, a_n$，使得 v 可以表達成：$v = a_1u_1+a_2u_2+\ldots+a_nu_n$。我們稱 v 是 $u_1, u_2,\ldots, u_n$ 的線性組合，$a_1, a_2,\ldots, a_n$ 稱為線性組合的係數。

In any vector space V, $ov = 0$ for each $v\in V$. Thus the zero vector is a linear combination of any nonempty subset of V.

任意向量空間 V，$0v = 0$。因此，零向量是任意向量空間 V 中任意非空子集合的線性組合。

EXAMPLE 1

How to solve a system of linear equations by showing how to determine if the vector (2, 6, 8) can be expressed as a linear combination of

$u_1 = (1, 2, 1)$, $u_2 = (-2, -4, -2)$, $u_3 = (0, 2, 3)$, $u_4 = (2, 0, -3)$, and $u_5 = (-3, 8, 16)$

將(2, 6, 8)表達成 $u_1, u_2, \ldots, u_5$ 的線性組合。

We must determine if there are scalars a_1,a_2,a_3,a_4, and a_5 such that

$(2,6,8) = a_1u_1+a_2u_2+a_3u_3+a_4u_4+ a_5u_5$

$= a_1(1, 2, 1)+a_2(-2, -4, -2)+a_3(0, 2, 3)+a_4(2, 0, -3)+ a_5(-3, 8, 16)$

$= (a_1-2a_2+2a_4-3a_5, 2a_1-4a_2+2a_3+8a_5, a_1-2a_2+3a_3-3a_4+16a_5)$

設法找出線性組合的係數 $a_1, a_2, \ldots, a_5$。

Hence (2, 6, 8) can be expressed as a linear combination of u_1, u_2, u_3, u_4, and u_5 if and only if there is a 5-tuple of scalars (a_1,a_2,a_3,a_4, a_5) satisfying the system of linear equations:

$$
\begin{aligned}
a_1-2a_2+\qquad 2a_4-3a_5 &= 2 \\
2a_1-4a_2+2a_3+\qquad 8a_5 &= 6 \qquad (1) \\
a_1-2a_2+3a_3-3a_4+16a_5 &= 8
\end{aligned}
$$

For any choice of scalars a_2 and a_5, a vector of the form

$(a_1,a_2,a_3,a_4,a_5) = (2a_2-a_5-4, a_2, -3a_5+7, 2a_5+3, a_5)$ is a solution to system of linear equations. In particular, the vector (-4,0, 7, 3, 0) obtained by setting $a_2 = 0$ and $a_5 = 0$ is a solution to (1). Therefore

$(2, 6, 8) = -4u_1 + 0u_2 + 7u_3 + 3u_4 + 0u_5$

so that (2, 6, 8) is a linear combination of u_1, u_2, u_3, u_4, and u_5.

EXAMPLE 2

We claim that $2x^3-2x^2+12x-6$ is a linear combination of x^3-2x^2-5x-3 and $3x^3-5x^2-4x-9$ in $P_3(R)$, but that $3x^3-2x^2+7x+8$ is not.

$2x^3-2x^2+12x-6$ 是 x^3-2x^2-5x-3 與 $3x^3-5x^2-4x-9$ 的線性組合？

$3x^3-2x^2+7x+8$ 非 x^3-2x^2-5x-3 與 $3x^3-5x^2-4x-9$ 的線性組合？

In the first case we wish to find scalar a and b such that $2x^3-2x^2+12x-6 = a(x^3-2x^2-5x-3)+b(3x^3-5x^2-4x-9)$ …. $a = -4$ $b = 2$.

Hence $2x^3-2x^2+12x-6 = -4(x^3-2x^2-5x-3)+2(3x^3-5x^2-4x-9)$

可找到 a 與 b。故 $2x^3-2x^2+12x-6$ 是 x^3-2x^2-5x-3 與 $3x^3-5x^2-4x-9$ 的線性組合。

In the second case, we wish to show that there are no scalars a and b for which $3x^3-2x^2+7x+8 = a(x^3-2x^2-5x-3)+b(3x^3-5x^2-4x-9)$.

找不到 a 與 b。故 $3x^3-2x^2+7x+8$ 非 x^3-2x^2-5x-3 與 $3x^3-5x^2-4x-9$ 的線性組合。

Linear Combinations of Vectors

- Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space *V*. We say that **v**, a vector in *V*, is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ if there exist scalars $c_1, c_2, \ldots, c_m$ such that **v** can be written

$$\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$$

Example

- Show that the vector (3, -4, -6) cannot be expressed as a linear combination of the vectors (1, 2, 3), (-1, -1, -2), and (1, 4, 5).

Consider the identity

$$c_1(1,2,3)+c_2(-1,-1,-2)+c_3(1,4,5)=(3,-4,-6)$$

This identity leads to the following system of linear equations.

$$\begin{aligned} c_1 - c_2 + c_3 &= 3 \\ 2c_1 - c_2 + 4c_3 &= -4 \\ 3c_1 - 2c_2 + 5c_3 &= -6 \end{aligned}$$

This system has no solution. Thus (3, -4, -6) is not a linear combination of the vectors (1, 2, 3), (-1, -1, -2), and (1, 4, 5).

Example 1/2

- Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space M_{22} of 2×2 matrices.

We examine the following identity.

$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}+c_2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}+c_3\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}=\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

We get

$$\begin{bmatrix} c_1+2c_2 & -3c_2+c_3 \\ 2c_1+2c_3 & c_1+2c_2 \end{bmatrix}=\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

Example 2/2

On equating corresponding elements, we get the following system of a linear equations.

$$\begin{aligned} c_1 + 2c_2 &= -1 \\ -3c_2 + c_3 &= 7 \\ 2c_1 + 2c_3 &= 8 \\ c_1 + 2c_2 &= -1 \end{aligned}$$

This system can be shown to have the unique solution $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. The given matrix is thus the following linear combination of the other three matrices.

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}=3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}-2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}+\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

If it had turned out that the above system of equations had no solution, then of course the given matrix would not have been a linear combination of the other matrices.

Span

DEFINITION 1.5 Span

Let S be a nonempty subset of a vector space V. The span of S, denoted span(S), is the set consisting of all linear combinations of the vectors in S. For convenience, we define span($\emptyset$) = {0}.

令 S 是向量空間 V 的非空子集合，則 S 的生成集（Span of S），註記為 Span(S)，為 S 中的向量經線性組合而成的所有向量所組成的集合。**Span(S) 是一個集合，該集合的元素是由 S 內的向量經線性組合而成**。若 S 為空集合 S = $\emptyset$，則 span($\emptyset$) = {0}。

Theorem 1.5

The span of any subset S of a vector space V is a subspace of V. Moreover, any subspace W of V that contains S must also contain the span of S.

向量空間 V 的任意子集合 S 的生成集（Span of S）為 V 的子空間。意即 span(S) 是 V 的子空間。

進而言之，包含 S 的向量空間 V（S 為 V 的子集合），其任意子空間 W 也必然包含 S 的生成集。意即 span(S) $\subseteq$W。

【Proof】

The result is immediate if S = $\emptyset$ because span($\emptyset$) = {0}, which is a subspace that is contained in any subspace of V.

If S$\neq\emptyset$, then S contains a vector z.

So 0z = 0 is in span(S).

Let x, y$\in$span(S). Then exist vectors $u_1, u_2,\ldots, u_m, v_1, v_2,\ldots, v_n$ in S and scalars $a_1, a_2,\ldots, a_m$、$b_1, b_2,\ldots, b_n$ such that

$x = a_1u_1+a_2u_2+\ldots+a_mu_m$ and $y = b_1v_1+b_2v_2+\ldots+b_nv_n$

Then $x+y = a_1u_1+a_2u_2+\ldots+a_mu_m + b_1v_1+b_2v_2+\ldots+b_nv_n$

and, for any scalar c,

$cx = (ca_1)u_1+(ca_2)u_2+\ldots+(ca_m)u_m$

are clearly linear combination of the vectors in S; so x+y and cx are in span(S).

Thus span(S) is a subspace of V.

Let W denote any subspace of V that contains S. If w$\in$span(S), then w has the form $w = c_1w_1+c_2w_2+\ldots+c_kw_k$ for some vectors $w_1,w_2,\ldots,w_k$ in S and some scalars $c_1, c_2,\ldots, c_k$. Since S$\subseteq$W, we have $w_1, w_2,\ldots,w_k\in W$. Therefore $w = c_1w_1+c_2w_2+\ldots+c_kw_k$ is in W. Because w, any arbitrary vector in span(S), belongs to W, it follows that span(S) $\subseteq$W.

若 S =$\emptyset$，因 span($\emptyset$) = {0}，{0}為一包含於 V 的任意子空間的子空間。

若 S$\neq\emptyset$，且 S 包含一向量 z。

因 0z = 0，故 0$\in$span(S)。（符合 Theorem 1.3 (a)）

若 x、y$\in$span(S)，則在 S 中存有向量 $u_1, u_2,\ldots, u_m$、$v_1, v_2,\ldots, v_n$ 及純量 $a_1, a_2,\ldots, a_m$、$b_1, b_2,\ldots, b_n$ 使得 $x = a_1u_1+a_2u_2+\ldots+a_mu_m$ 及 $y = b_1v_1+b_2v_2+\ldots+b_nv_n$

因 $x+y = a_1u_1+a_2u_2+\ldots+a_mu_m + b_1v_1+b_2v_2+\ldots+b_nv_n$

且對任一純量 c，$cx = (ca_1)u_1+(ca_2)u_2+\ldots+(ca_m)u_m$

均為 S 內的向量的線性組合。

故 x+y 與 cx 均為 span(S)的元素。（符合 Theorem 1.3 (b) 與 (c)）

可知，span(S) 是 V 的子空間。（依據 Theorem 1.3）

其次，令 W 為向量空間 V 的任意子空間，S 為 V 的子集合，若 w$\in$span(S)，則

w 可表達成 $w = c_1w_1+c_2w_2+....+c_kw_k$。其中，向量 $w_1, w_2,...., w_k \in S$ 且 $c_1, c_2,...., c_k$ 為純量。因 $S \subseteq W$，$w_1, w_2,...., w_k$ 當然也屬於 W，也當然使得 $w = c_1w_1+c_2w_2+....+c_kw_k$ 為 W 的元素。因 w 為 span(S) 的任意向量，屬於 W，所以 span(S) $\subseteq W$。

Theorem 1.3 令 V 為向量空間，W 為 V 的子集合，則 W 為 V 的子空間「若且唯若」下列三個條件適用於 V 內所定義的運算：

(a) $0 \in W$。

(b) $x+y \in W$ whenever $x \in W$ and $y \in W$. 當 $x \in W$ 且 $y \in W$ 時，則 $x+y \in W$。

(c) $cx \in W$ whenever $c \in F$ and $x \in W$. 當 $c \in F$ 且 $x \in W$ 時，$cx \in W$。

In R^3, for instance, the span of the set {(1, 0, 0), (0, 1, 0)} consists of all vectors in R^3 that have the form a(1, 0, 0) + b(0, 1, 0) = (a, b, 0) for some scalars a and b. Thus the span of {(1,0,0),(0,1,0)} contains all the points in the x-y plane. In this case, the span of the set is a subspace of R^3.

{(1, 0, 0), (0, 1, 0)}的生成集，為 (1, 0, 0) 與 (0, 1, 0) 經線性組合而成的向量的集合，線性組合而成的向量型態為 a(1,0,0)+b(0,1,0) = (a,b,0)。{(1,0,0),(0,1,0)}的生成集包含所有在 x-y 平面上的點，是 R^3 的子空間。

DEFINTION 1.6

A subset S of a vector space V generates (or spans) V if span(S) = V. In this case, we also say that the vectors of S generates (or span) V.

若 span(S) = V，則表示向量空間 V 的子集合 S 可以產生（生成）整個 V。在此情況下，亦可稱 S 的向量產生（生成）V。

EXAMPLE 3

The vectors (1, 1, 0), (1, 0, 1), and（0, 1, 1）generate R^3 since an arbitrary vector（a_1, a_2, a_3）in R^3 is a linear combination of the three given vectors; in fact, the scalars r, s, and t for which r(1, 1, 0)+s(1, 0, 1)+t（0, 1, 1） = （a_1,a_2,a_3）are $r = (a_1+a_2-a_3)/2$, $s = (a_1-a_2+a_3)/2$, and $t = (-a_1+a_2+a_3)/2$.

三個向量 (1, 1, 0)、(1, 0, 1)與（0, 1, 1）可以生成 R^3？

因為 R^3 中任意向量（a_1, a_2, a_3）可以是這三個向量的線性組合。

滿足 r(1, 1, 0)+s(1, 0, 1)+t（0, 1, 1） = （a_1,a_2,a_3）的純量 r、s、t 為：

$r = (a_1+a_2-a_3)/2$、$s = (a_1-a_2+a_3)/2$ 與 $t = (-a_1+a_2+a_3)/2$。

故**三個**向量(1, 1, 0)、(1, 0, 1)與（0, 1, 1）可以生成 R^3。

EXAMPLE 4

The polynomials x^2+3x-2, $2x^2+5x-3$, and $-x^2-4x+4$ generate $P_2(R)$ since each of the three given polynomials belongs to $P_2(R)$ and each polynomial ax^2+bx+c in $P_2(R)$ is a linear combination of these three polynomials.

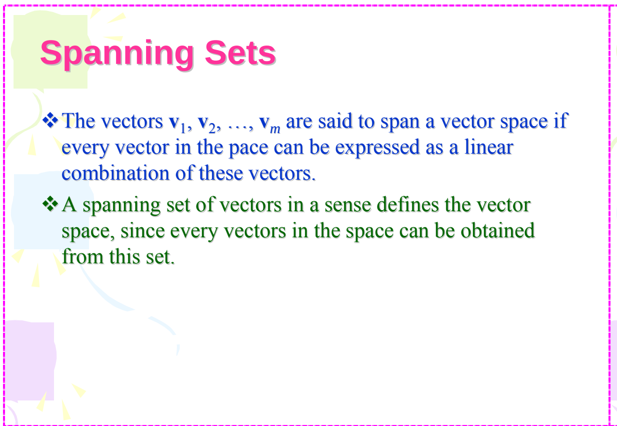
三個多項式 x^2+3x-2、$2x^2+5x-3$ 與 $-x^2-4x+4$ 可以生成 $P_2(R)$？

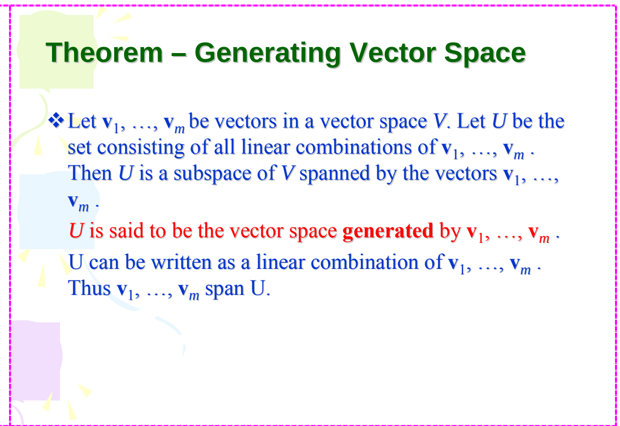
因為 $P_2(R)$中任意多項式 ax^2+bx+c 可以是這三個多項式的線性組合，故這**三個**多項式可以生成 $P_2(R)$。

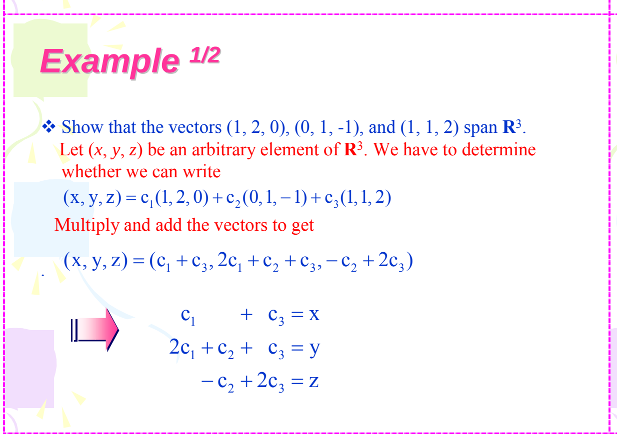
EXAMPLE 5

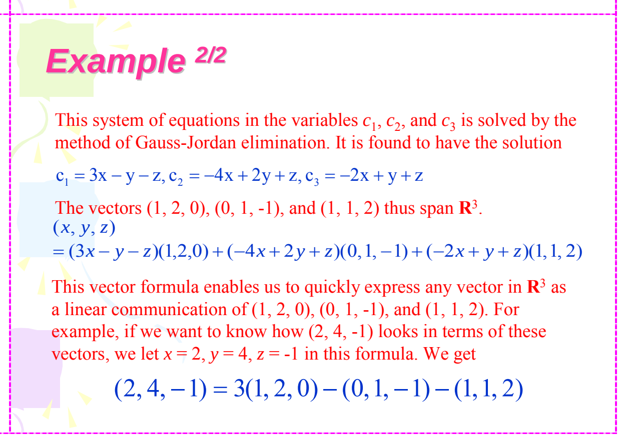
$\begin{bmatrix}1&1\\1&0\end{bmatrix}$、$\begin{bmatrix}1&1\\0&1\end{bmatrix}$、$\begin{bmatrix}1&0\\1&1\end{bmatrix}$、$\begin{bmatrix}0&1\\1&1\end{bmatrix}$生成 $M_{2\times2}(R)$？

因為 $M_{2\times2}(R)$中任意矩陣 $A = \begin{bmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{bmatrix}$可以是這四個矩陣的線性組合，故這四個矩陣可以生成 $M_{2\times2}(R)$。









Example

- Show that the following matrices span the vector space M_{22} of 2×2 matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be an arbitrary element of M_{22}.

We can express this matrix as follows.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

proving the result.

Example

- Let $\mathbf{v}_1$ and $\mathbf{v}_2$ span a subspace U of a vector space V. Let k_1 and k_2 be nonzero scalars. Show that $k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ also span U.

Let $\mathbf{v}$ be a vector in U. Since $\mathbf{v}_1$ and $\mathbf{v}_2$ span U, there exist scalars a and b such that $v = a\mathbf{v}_1 + b\mathbf{v}_2$

We can write

$$v = \frac{a}{k_1}(k_1 v_1) + \frac{b}{k_2}(k_2 v_2)$$

Thus the vectors $k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ span U.

1-4 Linear Dependence and Linear Independence

Suppose that V is a vector space over a infinite field and the W is a subspace of V. Unless W is the zero subspace, W is an infinite set. It is desirable to find a “small” finite subset S that generates W because we can describe each vector in W as a linear combination of the finite number of vectors in S. Indeed, the smaller that S is, the fewer computations that are required to represent vector in W.

設 V 是佈於 Infinite field 的向量空間且 W 是 V 的子空間。除非 W 是零子空間，否則 W 也是一無窮集合。我們期待可以找到一個「小的」有限子集合 S 來生成 W，屆時就可將 W 內的每一個向量描述為 S 內的有限個向量的線性組合。事實上，能用較小的集合 S，就可以用較少的計算來表示 W 的向量。

Consider, for example, the subspace W of R^3 generated by S = {u_1, u_2, u_3, u_4}, where u_1 = {2, -1, 4},u_2 = {1, -1, 3}, u_3 = {1, 1, -1}, u_4 = {1, -2, 1}. Let us attempt to find a proper subset of S that also generate W. The search for this subset is related to the question of whether or not some vectors in S. Now u_4 is a linear combination of the other vectors in S if and only if there are scalars a_1, a_2, and a_3 such that

$u_4 = a_1u_1+a_2u_2+ a_3u_3$ …. No such solution exists.

考慮由 S = {u_1, u_2, u_3, u_4}中找到一個 S 的真子集合（Proper subset），來生成 R^3 的子空間 W；在找尋此一真子集合時，如同探討 S 上是否有某一向量恰為 S 的其他向量的線性組合？欲檢驗 S 上是否有某一向量恰為 S 上其他向量的線性組合，即決定 u_1, u_2, u_3, u_4 中的某一向量是否可為其他向量的線性組合。過程中，我們可能需要解幾個不同的線性方程組，嘗試誰可？誰不可？例如，先從 u_4 切入，看看 u_4 能否恰為 u_1, u_2, u_3 的線性組合？經計算後若發現「不是」，則改由 u_3 切入，看看 u_3 是否恰為 u_1, u_2, u_4 的線性組合？若經計算後發現「是」，則將 u_3 確定；然後再試試其他的向量。為了節省時間，直接將零向量寫成 S 內的向量的線性組合：

$a_1u_1+a_2u_2+ a_3u_3+a_4u_4 = 0$

以 u_1 = {2, -1, 4},u_2 = {1, -1, 3}, u_3 = {1, 1, -1}, u_4 = {1, -2, 1}為例，得知 $a_1 = -2$、$a_2 = 3$、$a_3 = 1$、$a_4 = 0$；即並非所有係數均為零。由係數即可看出哪一個向量可以或不可以表達為其他向量的線性組合。例中，因 a_4 為零，其他係數不為零，所以 u_1、u_2 或 u_3 可被表達為其他三個向量的線性組合，u_4 則不可以。

Rather than asking whether some vectors in S is a linear combination of the other vectors in S, it is some efficient to ask whether the zero vector can be expressed as a linear combination of the vectors in S with coefficients that are not all zero.

因此要探討 S 內某一向量是否為 S 其他向量的線性組合，倒不如探討「零向量是否可被表達為 S 內向量的線性組合」來得迅速有效。

DEFINTION 1.7 Linear dependent

A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vector $u_1, u_2, \ldots, u_n$ in S and scalars $u_1, u_2, \ldots, u_n$ not all zero, such that $a_1u_1+a_2u_2+\ldots+a_nu_n = 0$. In this case we also that the vectors of S are linearly dependent.

就向量空間 V 內的子集合 S 而言，若 S 中存在有限個相異向量 $u_1, u_2, \ldots, u_n$ 與不全為零的純量 $a_1, a_2, \ldots, a_n$，使得 $a_1u_1+a_2u_2+\ldots+a_nu_n = 0$，則稱 S 中的向量為線性相依。

For any vectors $u_1, u_2, \ldots, u_n$, we have $a_1u_1+a_2u_2+\ldots+a_nu_n = 0$ if $a_1 = a_2 = \ldots = a_n = 0$. We call this the trivial representation of 0 as a linear combination of $u_1, u_2, \ldots, u_n$.

對任意向量 $u_1, u_2, \ldots, u_n$ 而言，若 $a_1 = a_2 = \ldots = a_n = 0$，則稱『$a_1u_1+a_2u_2+\ldots+a_nu_n = 0$』是「0 為 $u_1, u_2, \ldots, u_n$ 的線性組合的明顯表示式（trivial representation）」。

Thus, for a set to be linearly dependent, there must exist a nontrivial representation of 0 as a linear combination of vectors in the set. Consequently, **any subset of a vector space that contains the zero vector is linearly dependent**, because $0 = 1 \cdot 0$ is a nontrivial representation of 0 as a linear combination of vectors in the set.

若集合為線性相依，必存在「0 為集合內向量的線性組合的非明顯表示式（nontrivial representation）」，即 0 可被表達成該集合內向量的線性組合。又因為 $0 = 1 \cdot 0$（非明顯表示式），故向量空間內的任意子集合若僅包含零向量，則該子集合為線性相依。

EXAMPLE 1

Consider the set S = {(1, 3, -4, 2), (2, 2, -4, 0), (1, -3, 2, -4), (-1, 0, 1, 0)} in R^4.

We show that S is linearly dependent and then express one of the vectors in S as a linear combination of the other vectors in S. To show that S is linearly dependent, we must find scalars a_1,a_2,a_3,a_4 not all zero, such that $a_1(1, 3, -4, 2)+ a_2(2, 2, -4, 0)+ a_3(1, -3, 2, -$

4)+ $a_4(-1, 0, 1, 0) = 0$.

先證明 S 為線性相依，然後將 S 內一個向量表為其他向量的線性組合。

要證明 S 為線性相依，則必須找出一組不全為零的純量 a_1, a_2, a_3, a_4，使得 $a_1(1, 3, -4, 2)+ a_2(2, 2, -4, 0)+ a_3(1, -3, 2, -4)+ a_4(-1, 0, 1, 0) = 0$。

因 $a_1 = 4$、$a_2 = -3$、$a_3 = 2$、$a_4 = 0$，故 S 為 R^4 的線性相依子集合。

EXAMPLE 2

In $M_{2\times3}(R)$, the set $\left\{\begin{pmatrix}1 & -3 & 2\\ -4 & 0 & 5\end{pmatrix}, \begin{pmatrix}-3 & 7 & 4\\ 6 & -2 & -7\end{pmatrix}, \begin{pmatrix}-2 & 3 & 11\\ -1 & -3 & 2\end{pmatrix}\right\}$ is linearly dependent.

證明集合是線性相依。

DEFINTION 1.8 Linear independent

A subset S of a vector space that is not linear dependent is called linearly independent. As before, we also say that the vectors of S are linearly independent.

向量空間 V 內不為線性相依的子集合 S，稱為線性獨立。

The following facts about linearly independent sets are true in any vector space.
線性獨立的集合，具有下列事實：

1. The empty set is linearly independent, for linearly dependent sets must be nonempty.
   空集合為線性獨立，線性相依集合必為非空集合。
2. A set consisting of a single nonzero vector is linearly independent. For if {u} is linearly dependent, then au = 0 for some nonzero scalar a. Thus $u = a^{-1}(au) = a^{-1}0 = 0$.
   只含有一個非零向量的集合為線性獨立。若{ u }為線性相依，則 au = 0，a 為非零純量→於是 $u = a^{-1}(au) = a^{-1}0 = 0$（矛盾）。
3. A set is linearly independent if and only if the only representations of 0 as linear combinations of its vectors are trivial representations.
   一集合為線性獨立若且唯若「0 為集合內向量的線性組合表示式」為明顯表示式（Trivial representations）。

EXAMPLE 3

To prove that the set {(1, 0, 0, -1), (0, 1, 0, -1), (0, 0, 1, -1), (0, 0, 0, 1)} is linearly independent.

$a_1(1, 0, 0, -1)+ a_2(0, 1, 0, -1)+ a_3(0, 0, 1, -1)+ a_4(0, 0, 0, 1) = (0, 0, 0, 0)$

由係數是否全為零，來判斷是否為線性獨立？

EXAMPLE 4

For $k = 0, 1, \ldots, n$. Let $p_k(x) = x^k + x^{k-1} + \ldots + x^n$. The set $\{p_0(x), p_1(x), \ldots, p_n(x)\}$ is linearly independent in $P_n(F)$.

$a_0p_0(x)+a_1p_1(x)+\ldots+a_n\, p_n(x) = 0$

→ $a_0 = a_1 = \ldots = a_n = 0$ ？由係數是否全為零，來判斷是否為線性獨立？

Theorem 1.6

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_1 is linearly dependent, then S_2 is linearly dependent.

令 V 為一向量空間，且 $S_1 \subseteq S_2 \subseteq V$。若 S_1 為線性相依，則為 S_2 線性相依。

Corollary

Let V be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If S_2 is linearly independent, then S_1 is linearly independent.

令 V 為一向量空間，且 $S_1 \subseteq S_2 \subseteq V$。若 S_2 為線性獨立，則 S_1 為線性獨立。

前面提及找到「小的」有限集合 S 來生成 W，所找到的 S 是否為最小？

這個問題的答案就要看 S 內的部分向量是否可表達為其他向量的線性組合？或者說，要看看 S 是否為線性相依集合？

前面已經驗證 S 為線性相依（$-2u_1 + 3u_2 + u_3 - 0u_4 = 0$），其中，$u_3$ 可為 S 中其他向量的線性組合 $u_3 = 2u_1-3u_2+0u_4$。因此 S 內的向量的線性組合 $a_1u_1+a_2u_2+a_3u_3+a_4u_4 = 0$ 可以改寫成 u_1、u_2 與 u_4 的線性組合：

$$a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 = (a_1+2a_3)u_1 + (a_2-3a_3)u_2 + a_4u_4$$。

Thus the subset $S' = \{u_1, u_2, u_4\}$ of S has the same span as S.

故 S 的子集合 $S' = \{u_1, u_2, u_4\}$的生成集 與 S 的生成集相同。**用 $S' = \{u_1, u_2, u_4\}$與用 $S = \{u_1, u_2, u_3, u_4\}$來產生生成集（generates the span of S）的效果一樣（span(S)≡ span(S')）。**

More generally, suppose that S is any linearly dependent set containing two or more vectors. Then some vector $v \in V$ can be written as a linear combination of the other vectors in

S, and the subset obtained by removing v from S has the same span as S. If follows that **if no proper subset of S generates the span of S, then S must be linearly independent.**

若 S 為一線性相依且為含有二個或多個向量的集合，當 S 內的部分向量 v 可以表達為 S 內其他向量的線性組合時，則由 S 中移除 v 後所剩下來的子集合 S’，與 S 具有相同生成集（the same span as S）。相對地，若 S 內找不到子集合 S’（S 與 S’的生成集相同），則 S 必然是線性獨立的集合。

Theorem 1.7

Let S be a linearly independent subset of a vector space V, and let v be a vector in V that is not in S. Then S∪{v} **is linearly dependent if and only if v∈span(S).**

設 S 為向量空間 V 的一個線性獨立子集合，且令 v 為 V 中的一個、但不在 S 內的向量，則 **S∪{v}為線性相依『若且唯若』v∈span(S)**。

【Proof】

If S∪{v} is linearly dependent, then there are vectors $u_1, u_2, \ldots, u_n$ in S∪{v}such that $a_1u_1+a_2u_2+\ldots+a_nu_n = 0$ for some nonzero scalars $a_1, a_2, \ldots, a_n$. Because S is linearly independent, one of the u_i’s, say u_1, equals v. Thus $a_1v + a_2u_2 + \ldots + a_nu_n = 0$, and so

$$v = -a_1^{-1}(a_2u_2 + \ldots + a_nu_n)$$

Since v is linear combination of $u_2, \ldots, u_n$, which are in S, we have v∈span(S).

Conversely, let v∈span(S). Then there exist vectors $v_1, v_2, \ldots, v_m$ in S and scalars $b_1, b_2, \ldots, b_m$ such that $0 = b_1v_1 + b_2v_2 + \ldots + b_mv_m + (-1)v$.

Since $v \neq v_i$, for i = 1, 2,…,m, the coefficient of v in this linear combination is nonzero, and so the set $\{v_1, v_2, \ldots, v_m, v\}$ is linearly dependent. Therefore S∪{v} is linearly dependent by Theorem 1.6.

先證明 S∪{v}為線性相依→ v∈span(S)：

若 S∪{v}為線性相依，則在 S∪{v}中存在向量 $u_1, u_2, \ldots, u_n$ 及不全為零的純量 $a_1, a_2, \ldots, a_n$，使得 $a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0$。因 S 是線性獨立，故 $u_1, u_2, \ldots, u_n$ 中必然有一個不是 S 的元素，是 v，得抽離出來，若那個非 S 元素者為 u_1，則其餘 $u_2, \ldots, u_n \in$ S。

$a_1u_1 + a_2u_2 + \ldots + a_nu_n = 0 \rightarrow a_1v + a_2u_2 + \ldots + a_nu_n = 0$ 或 $v = -a_1^{-1}(a_2u_2 + \ldots + a_nu_n)$

既然 v 為 $u_2, \ldots, u_n$ 的線性組合，且 $u_2, \ldots, u_n$ 均為 S 的元素，所以 v∈span(S)。

其次證明 v∈span(S) → S∪{v}為線性相依：

反之，令 v∈span(S)，則存在向量 $v_1, v_2, \ldots, v_m \in$ S 及純量 $b_1, b_2, \ldots, b_m$，使得 $v = b_1v_1 + b_2v_2 + \ldots + b_mv_m \rightarrow 0 = b_1v_1 + b_2v_2 + \ldots + b_mv_m + (-1)v$。因 $v \neq v_i$，i = 1, 2,…,m（v 不在 S 內），故線性組合中的係數非全為零，所以集合$\{v_1, v_2, \ldots, v_m, v\}$為線性相依。由

定理 1.6 得知：S∪{v} 為線性相依。($\{v_1,v_2,\ldots,v_m,v\} \subseteq S \cup \{v\}$)
定理 1.6 令 V 為一向量空間，且 $S_1 \subseteq S_2 \subseteq V$。若 S_1 為線性相依，則為 S_2 線性相依。

1-5 Bases and Dimension

If S is a generating set for a subspace W and no proper subset of S is a generating set for W, then S must be linearly independent. A linearly independent generating set for W possesses a very useful property- every element in W can be expressed in one or only one way as a linear combination of the vectors in the set.

若 S 為子空間 W 的 Generating set（產生生成集 span(S)的集合）且 S 內沒有子集合可為 W 的 Generating set，則 S 必然為線性獨立。W 的一線性獨立 Generating set 具有一個非常有用的性質：**W 的每一元素可循唯一管道表達為 Generating set 的向量的線性組合。**

DEFINITION 1.9 Basis

A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.

向量空間 V 的基底 β 為生成 V 的**一線性獨立子集合**。若 β 為 V 的一組基底，則亦稱 β 的向量構成 V 的基底。

向量空間的基底是線性獨立集合，可用來生成向量空間。

EXAMPLE 1

Recalling that span(∅) = { 0 } and ∅ is linearly independent, we see that ∅ is a basis for the zero vector space.

已知 span(∅) = { 0 }且∅是線性獨立集合，故∅是**「零」向量空間{ 0 }的基底**。

EXAMPLE 2

In F^n, let $e_1 = (1, 0, 0,\ldots, 0)$, $e_2 = (0, 1, 0,\ldots, 0)$,…, $e_n = (0, 0, 0,\ldots, n)$; $\{e_1, e_2,\ldots, e_n\}$ is readily seen to be a basis for F^n and is called the standard basis for F^n.

在 F^n 中，$e_1 = (1, 0, 0,\ldots, 0)$、$e_2 = (0, 1, 0,\ldots, 0)$、….、$e_n = (0, 0, 0,\ldots, n)$。$\{e_1, e_2,\ldots, e_n\}$為 F^n 的一組基底，且為 F^n 的標準基底（Standard basis）。

EXAMPLE 3

In $M_{m\times n}(F)$, let E^{ij} denote the matrix whose only nonzero entry is a 1 in the ith row and jth column. Then $\{E^{ij}: 1\leqq m, 1\leqq j\leqq n\}$ is a basis for $M_{m\times n}(F)$. 。

在 $M_{m\times n}(F)$中，若 E^{ij} 為只有第 i 列第 j 行元素為 1 的矩陣，則 $\{E^{ij}: 1\leqq i\leqq m, 1\leqq j\leqq n\}$ 是 $M_{m\times n}(F)$的一組基底。

EXAMPLE 4

在 $P_n(F)$中，集合$\{1, x, x^2,\ldots, x^n\}$是 $P_n(F)$的一組基底，且為 $P_n(F)$的標準基底。

EXAMPLE 5

在 P(F)中，集合$\{1, x, x^2,\ldots\}$ 為一組基底。

Observe that Example 5 shows that a basis need not be finite. In fact, no basis for P(F) can be finite. Hence not every vector space has a finite basis.

Example 5 顯示基底不一定為有限。事實上，P(F)並沒有「有限」的基底。因此，並不是每一個向量空間皆有一個有限的基底。

Theorem 1.8 Basis

Let V be a vector space and $\beta = \{u_1, u_2,\ldots, u_n\}$ be a subset of V. Then β is a basis for V if and only if each $v\in V$ can be uniquely expressed as a linear combination of vectors of β, that is, can be expressed in the form $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ for unique scalars $a_1, a_2,\ldots, a_n$.

令 V 是一空間向量且 $\beta = \{u_1, u_2,\ldots, u_n\}$為 V 的子集合，則 β 為 V 基底的條件『若且惟若』「V 內的每一向量 v 皆能被唯一表達為 β 內的向量 $u_1, u_2,\ldots, u_n$ 的線性組合：$v = a_1u_1 + a_2u_2 +\ldots+ a_nu_n$，且線性組合的係數 $a_1, a_2, \ldots, a_n$ 是唯一的。

【Proof】

先證明 V 是一空間向量且 $\beta = \{u_1,u_2,\ldots,u_n\}$為 V 的子集合，若 β 為 V 的基底→ V 內的每一向量 v 皆能被唯一表達為 β 內的向量 $u_1, u_2,\ldots, u_n$ 的線性組合且線性組合的係數是唯一的。

Let β be a basis of V. If $v\in V$, then $v\in \text{span}(\beta)$ because span(β) = V【β 的生成集】. Thus v is a linear combination of the vectors of β.

Suppose that $v = a_1u_1+a_2u_2+\ldots+a_nu_n$ and $v = b_1u_1+b_2u_2+\ldots+b_nu_n$ are two such representation of v. Subtracting the second equation from the first gives $0 = (a_1-b_1)u_1+(a_2-b_2)u_2+\ldots+(a_n -b_n)u_n$. Since β is linear independent, it follows that $a_1-b_1 = a_2-b_2 = \ldots = a_n-b_n$

= 0. So v is uniquely repressible as a linear combination of the vectors of β.

至於反向證明？

令 β 為 V 的基底。因 span(β) = V（**Defintion 1.9**），所以在 v∈V 下，v∈span(β)，即 v 可以表達成 β 內的向量的線性組合。

假設表達方式有二種：$v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$ 與 $v = b_1u_1 + b_2u_2 + \ldots + b_nu_n$，將兩個表達式相減，得知 $0 = (a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \ldots + (a_n - b_n)u_n$。因 β 為線性獨立，故 $a_1\text{-}b_1 = a_2\text{-}b_2 = \ldots = a_n\text{-}b_n = 0$，顯示 v 可以表達成 β 內的向量的線性組合方式是唯一的。

DEFINTION 1.6 A subset S of a vector space V generates (or spans) V if span(S) = V. In this case, we also say that the vectors of S generates (or span) V. 若 span(S) = V，則表示向量空間 V 的子集合 S 可以產生（生成）整個 V。在此情況下，亦可稱 S 的向量產生（生成）V。

DEFINTION 1.9 【Basis】A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.向量空間 V 的基底 β 為生成 V 的一線性獨立子集合。若 β 為 V 的一組基底，則亦稱 β 的向量構成 V 的基底。

從定理 1-8 可知，若向量 $u_1, u_2, \ldots, u_n$ 為空間向量 V 的基底，則 V 內每一個向量皆可被唯一表示為 $v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$。其中，$a_1, a_2, \ldots, a_n$ 是選定的適當純量。因此，v 決定一個唯一的 n-tuples 純量組 $\{a_1, a_2, \ldots, a_n\}$。反之，每一個 n-tuples 純量組決定一個唯一的向量 v∈V。

Theorem 1.9 Construction of Basis

If a vector space V is generated by a finite set S, then some subset of S is a basis for V. Hence V has a finite basis.

若向量空間 V 係由一有限集合 S 所生成，則 **S 的某些子集合會是 V 的一組基底**。意即，可以由 S 找出構成 V 基底的子集合。

【Proof】

Let S = ∅ or S = {0}, then V = {0} and ∅ is a subset of S that is a basis for V.

Otherwise S contains a nonzero vector u_1. By property 2 of linear independent set, $\{u_1\}$ is a linearly independent set.

Continuous, if possible, choosing vectors $u_2, \ldots, u_k$ in S such that $\{u_1, u_2, \ldots, u_k\}$ is linearly independent. Since S is a finite set, we must eventually reach a stage at which $\beta = \{u_1, u_2, \ldots, u_k\}$ is a linearly independent subset of S, but adjoining to β any vector in S not in β produces a linearly dependent set. We claim that β is a basis for V. Because β is linearly independent by construction, it suffices to show that β spans V. We need to show that S⊆span(β). Let v∈S. If v∈β, then v∈span(β). Otherwise, if v∉β, then the proceeding construction shows that β⋃{v} is linearly dependent. So v∈span(β) by Theorem 1.7. Thus S⊆span(β).

若 **S 為空集合**∅ **或 S = {0}**，則 V = {0}且∅是 S 的子集合，也是{0}的基底。

【Example 1 已知 span(∅) = { 0 }且∅是線性獨立集合，故∅是「零」向量空間{ 0 }的基底。**】**

否則，若 S 包含一個非零向量 u_1，則依線性獨立集合的特性 2：{u_1}為線性獨立。

其次，在 S 中找出 $u_2, \ldots, u_k$，連同{u_1}，使得 β = {$u_1, u_2, \ldots, u_k$}為 S 內的線性獨立子集合。若再加上一個在 S 但不在 β 的向量 v 到 β 裡頭，產生一個線性相依集合。

β是否為 V 的基底？【條件參考 Theorem 1.8 與 Definition 1.9】

由於β是線性獨立，β是否為 V 的基底，則必須看β是否可以生成 V？

即看看 S⊆span(β)是否成立？

令 v∈S。

若 v∈β，則 v∈span(β)。

若 v∉β，因 β⋃{v}是線性相依，故由 **Theorem 1.7** 得知 v∈span(β)。

→S⊆span(β)。

Theorem 1.5 向量空間 V 的任意子集合 S 的生成集（Span of S）為 V 的子空間。意即 span(S)是 V 的子空間。進而言之，包含 S 的向量空間 V（S 為 V 的子集合），其任意子空間 W 也必然包含 S 的生成集。意即 span(S) ⊆W。

Theorem 1.7 設 S 為向量空間 V 的一個線性獨立子集合，且令 v 為 V 中的一個、但不在 S 內的向量，則 **S∪{v}為線性相依『若且唯若』v∈span(S)**。

Theorem 1.8 令 V 是一空間向量且 β = {$u_1, u_2, \ldots, u_n$}為 V 的子集合，則 β 為 V 基底的條件『若且惟若』「V 內的每一向量 v 皆能被唯一表達為 β 內的向量 $u_1, u_2, \ldots, u_n$ 的線性組合：$v = a_1u_1 + a_2u_2 + \ldots + a_nu_n$，且線性組合的係數 $a_1, a_2, \ldots, a_n$ 是唯一的。

DEFINTION 1.9 【Basis】A basis β for a vector space V is a linearly independent subset of V that generates V. If β is a basis for V, we also say that the vectors of β form a basis for V.向量空間 V 的基底β為生成 V 的一線性獨立子集合。若β為 V 的一組基底，則亦稱β的向量構成 V 的基底。

◎線性獨立集合的特性：

1. 空集合為線性獨立。

2. 只含有一個非零向量的集合為線性獨立集合。
3. 一集合為線性獨立若且唯若「0 為集合內向量的線性組合表示式」為明顯表示式（Trivial representations）。

定理 1.9 是用來求基底 β 的方法。

EXAMPLE 6

Let S = {(2, -3, 5),(8, -12, 20),(1, 0, -2),(0, 2, -1),(7, 2, 0)}. It can be shown that S generates R^3. We can select a basis for R^3 that is a subset of S by the technique used in Theorem 1.9.

由 S 中找出可以構成 R^3 的基底？

To start, select any nonzero vector in S, say (2, -3, 5), to a vector in the basis. Which vectors can be included in the basis?

先由 **(2, -3, 5)** 出發，逐一確認可以加進來構成基底的向量。

(8, -12, 20) ? Since 4(2, -3, 5) = (8, -12, 20), the set {(2, -3, 5), (8, -12, 20)} is linearly dependent. Hence we do not include (8, -12, 20) in our basis

(1, 0, -2)? Since (1, 0, -2) is not a multiple of (2, -3, 5), the set {(1, 0, -2) ,(2, -3, 5)} is linearly independent. Thus we include (1, 0, -2) as part of our basis.

In a similar fashion, the final vector in S is included or excluded from our basis according to whether the set {(2, -3, 5),(1, 0, -2),(0, 2, -1),(7, 2, 0)}is linearly independent or linearly dependent.

→ We exclude (7, 2, 0) from our basis. We conclude that **{(2, -3, 5),(1, 0, -2),(0, 2, -1)}** is a subset of S that is a basis for R^3.

{(2, -3, 5),(1, 0, -2),(0, 2, -1)}為 S 的子集合，且為 R^3 的基底。

Theorem 1.10 (Replacement Theorem)

Let V be a vector space that is generated by a set G containing exactly n vectors, and let L be a linearly independent subset of V containing exactly m vectors. Then m≦n and there exists a subset H of G containing exactly n-m vectors such that L∪H generates V.

向量空間 V 係由集合 G 所生成，且 G 的向量數為 n。令 L 為 V 的線性獨立子集合，且含有向量數為 m，則 m≦n，且 G 含有向量數目為 n-m 的子集合 H，使得 L∪H 可以生成 V。

【Proof】

The proof is by mathematical induction on m.（採用數學上歸納法）

The induction begins with m = 0; for in this case L = $\emptyset$, and so taking H = G gives the desired result.

Now suppose that the theorem is true for some integer m≧0.

We prove that the theorem is true for m+1. Let L = $\{v_1,v_2,\ldots,v_{m+1}\}$ is a linearly independent subset of V containing m+1 vectors. By the corollary to Theorem 1.6. $\{v_1,v_2,\ldots,v_m\}$ is linearly independent, and so we may apply the induction hypothesis to conclude that m ≦ n and that there is a subset $\{u_1,u_2,\ldots,u_{n-m}\}$ of G such that $\{v_1,v_2,\ldots,v_m\}\cup\{u_1,u_2,\ldots,u_{n-m}\}$ generates V. Thus there exist scalars $a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_{n-m}$ such that $a_1v_1+a_2v_2+\ldots+a_mv_m+ b_1u_1+b_2u_2+\ldots+b_{n-m}u_{n-m} = v_{m+1}$.

Note that n-m>0, lest v_{m+1} be a linear combination of $v_1,v_2,\ldots,v_m$, which by Theorem 1.7 contradicts the assumption that L is linearly independent. Hence n>m; that is, n≧m+1. Moreover, some b_i, say b_1, is nonzero, for otherwise we obtain the same contradiction.

Solving for u_1 gives

$u_1 = (-b_1^{-1}a_1)v_1+(-b_1^{-1}a_2)v_2+\ldots+(-b_1^{-1}a_m)v_m+(-b_1^{-1})\, v_{m+1}+(-b_1^{-1}b_2)u_2+\ldots+(-b_1^{-1}b_{n-m})u_{n-m}$

Let H = $\{u_2,\ldots,u_{n-m}\}$. The $u_1\in$span(L∪H), and because $v_1,v_2,\ldots,v_m, u_2,\ldots,u_{n-m}$ are clearly in L∪H, it follows that

$\{ v_1,v_2,\ldots,v_m, u_1,u_2,\ldots,u_{n-m} \}\subseteq$ span(L∪H)

Because $\{v_1,v_2,\ldots,v_m,u_1,u_2,\ldots,u_{n-m}\}$ generates V, Defintion 1.6 implies that span(L∪H) = V. Since H is a subset of G that contains (n-m)-1 = n-(m+1) vectors, the theorem is true for m+1.

當 m = 0 時，L 是空集合，H 是 G 的子集合，含有 n 個向量，H = G，故 L∪H = L∪G = G，G 可以生成空間向量 V。

其次，假設本定理對於整數 m≧0 成立。

m+1 又如何？

令 L = $\{v_1, v_2,\ldots, v_{m+1}\}$為 V 中含有 m+1 個向量的線性獨立子集合。

因 $\{v_1, v_2, \ldots, v_m\}\subseteq\{v_1, v_2, \ldots, v_{m+1}\}$（Linearly independent）$\subseteq$V，依據定理 1.6 可知$\{v_1, v_2,\ldots, v_m\}$亦為線性獨立，且由歸納假設得知：m≦n 且 G 中存在一子集合 $\{u_1, u_2,\ldots, u_{n-m}\}$，使得$\{v_1, v_2,\ldots, v_m\}\cup\{u_1, u_2,\ldots, u_{n-m}\}$可以生成 V，即存在純量 $a_1, a_2, \ldots, a_m, b_1, b_2,\ldots, b_{n-m}$ 使得

$a_1v_1+a_2v_2+\ldots+a_mv_m+ b_1u_1+b_2u_2+\ldots+b_{n-m}u_{n-m} = v_{m+1}$。

n-m>0？

當然是！否則 v_{m+1} 就會為 $v_1, v_2,\ldots, v_m$ 的線性組合，由定理 1.7 得知$\{v_1, v_2,\ldots, v_m\}\cup\{v_{m+1}\}$就會成為線性相依，這與一開始假設 L = $\{v_1, v_2,\ldots, v_{m+1}\}$是線性獨立相互矛盾。

因此 $n>m$，即 $n \geqq m+1$。

此外，會有一些 b_i 不為零？當然不會，否則也會出現相同的矛盾。

令 $b_1 = b_i \neq 0$，由 $a_1v_1 + a_2v_2 + \ldots + a_mv_m + b_1u_1 + b_2u_2 + \ldots + b_{n-m}u_{n-m} = v_{m+1}$ 解出 u_1

$\rightarrow u_1 = (-b_1^{-1}a_1)v_1+(-b_1^{-1}a_2)v_2+\ldots+(-b_1^{-1}a_m)v_m+(-b_1^{-1})\, v_{m+1}+(-b_1^{-1}b_2)u_2+\ldots+(-b_1^{-1}b_{n-m})u_{n-m}$

令 $H = \{u_2, \ldots, u_{n-m}\}$，則 $u_1 \in span(L \cup H)$。（依據 Theorem 1.7）

又因 $v_1, v_2,\ldots, v_m, u_2, \ldots, u_{n-m}$ 均為 $span(L \cup H)$的向量，所以 $\{ v_1, v_2,\ldots, v_m, u_1, u_2,\ldots, u_{n-m} \} \subseteq span(L \cup H)$

因$\{v_1, v_2,\ldots, v_m, u_1, u_2,\ldots, u_{n-m}\}$ 可生成 V，依 Definition 1.6 得知 $span(L \cup H) = V$。又因 H 為 G 的子集合且包含 $(n-m)-1 = n-(m+1)$ 個向量，所以本定理對 $m+1$ 也是成立。

Theorem 1.6 令 V 為一向量空間，且 $S_1 \subseteq S_2 \subseteq V$。若 S_1 為線性相依，則為 S_2 線性相依。

Theorem 1.7 設 S 為向量空間 V 的一個線性獨立子集合，且令 v 為 V 中的一個、但不在 S 內的向量，則 **$S \cup \{v\}$為線性相依『若且唯若』$v \in span(S)$**。

DEFINTION 1.6 A subset S of a vector space V generates (or spans) V if span(S) = V. In this case, we also say that the vectors of S generates (or span) V. 若 span(S) = V，則表示向量空間 V 的子集合 S 可以產生（生成）整個 V。在此情況下，亦可稱 S 的向量產生（生成）V。

Corollary 1

Let V be a vector space having a finite basis. Then every basis for V containing the same number of vectors.

令 V 為一向量空間且具有有限基底，則 V 的每一個基底包含相同數量的向量。

【Proof】

Suppose that β is a finite basis for V that contains exactly n vectors, and let γ be any other basis for V. If γ contains more than n vectors, then we can select a subset S of γ containing exactly n+1 vectors. Since S is linearly independent and β generates V, the replacement theorem implies that $n+1 \leqq n$, a contradiction. Therefore γ is finite, and the number m of vectors in γ satisfies $m \leqq n$. Reversing the roles of β and γ and arguing as above, we obtain $n \leqq m$. Hence $m = n$.

假設 β 為 V 的有限基底且恰含有 n 個向量，令 γ 是 V 的任意其他基底，若 γ 含有向量數多於 n 個，則可由 γ 選出一個含有 n+1 個向量的子集合 S。因 S 是線性獨立且 β 可生成 V，由 Theorem 1.10 Replacement theorem 得知 $n+1 \leqq n$（矛盾），因此 γ 為有限且向量個數 m 滿足 $m \leqq n$。反之，將 β 與 γ 的角色互換，可得知 $n \leqq m$。因此

（m≦n 且 n≦m）m＝n。

DEFINTION 1.10 Finite-dimensional, Dimensions

A vector space is called finite-dimensional if it has a basis consisting of a finite number of vectors. The unique number of vectors in each basis for V is called the **dimension** of V and is denoted by dim(V). A vector space that is not finite-dimensional is called infinite-dimensional.

一向量空間稱為有限維度，意指該向量空間的基底含有「有限個數」的向量。V 的每一組基底的向量數目為唯一，該向量數目稱為向量空間 V 的維度。反之，若向量空間非有限維度，則為無限維度。

EXAMPLE 7

The vector space {0} has dimension zero.

向量空間{0}的維度為 0。

EXAMPLE 8

The vector space F^n has dimension n.

向量空間 F^n 的維度為 n。

EXAMPLE 9

The vector space $M_{m\times n}(F)$ has dimension mn

向量空間 $M_{m\times n}(F)$的維度為 m×n。

EXAMPLE 10

The vector space $P_n(F)$ has dimension n+1.

向量空間 $P_n(F)$的維度為 n+1。

The conclusion in the replacement theorem states that if V is a finite-dimensional vector space, then no linearly independent subset of V can contain more than dim(V).

由 Replacement theorem 的結論得知：若 V 是有限維度的向量空間，則 V 內任一線性獨立子集合的向量個數不會超過 dim(V)。

Corollary 2

Let V be a vector space with dimension n.令 V 是一向量空間，其維度為 n：

1. Any finite generating set for V contains at least n vectors, and a generating set for V that contains exactly n vectors is a basis for V.
   可生成 V 的任意 Generating set 至少含有 n 個向量，恰含有 n 個向量的 Generating set 為 V 的基底。
2. Any linearly independent subset of V that contains exactly n vectors is a basis for V.
   V 中恰含有 n 個向量的任意線性獨立子集合必為 V 的基底。
3. Every linearly independent subset of V can be extended to a basis for V.
   V 中每一個線性獨立子集合都可擴充成為 V 的基底。

EXAMPLE 11

$\{x^2+3x-2, 2x^2+5x-3, -x^2-4x+4\}$ is a basis for $P_2(R)$.

Since each of the three given polynomials belongs to $P_2(R)$ and each polynomial ax^2+bx+c in $P_2(R)$ is a linear combination of these three polynomials.

$\{x^2+3x-2, 2x^2+5x-3, -x^2-4x+4\}$為 $P_2(R)$的基底。

EXAMPLE 12

$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $M_{2\times 2}(R)$.

Since an arbitrary matrix A in $M_{2\times 2}(R)$ can be expressed as a linear combination of the four given matrices.

$\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ 為 $M_{2\times 2}(R)$的基底。

EXAMPLE 13

$\{(1, 0, 0, -1),(0, 1, 0, -1),(0, 0, 1, -1),(0, 0, 0, 1)\}$ is a basis for R^4.

Since $\{(1, 0, 0, -1),(0, 1, 0, -1),(0, 0, 1, -1),(0, 0, 0, 1)\}$ is linearly independent.

因$\{(1, 0, 0, -1),(0, 1, 0, -1),(0, 0, 1, -1),(0, 0, 0, 1)\}$為線性獨立，故為 R^4 的基底。

EXAMPLE 14

For k = 0, 1,…, n, let $p^k(x) = x^k + x^{k-1} + \ldots + x^n$. Then $\{p_0(x), p_1(x),\ldots, p_n(x)\}$ is a basis for Pn(F).

Bases

- A finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for a vector space V if the set spans V and is linearly independent.
- Intuitively, a basis is an efficient set for characterizing a vector space, in that any vector can be expressed as a linear combination of the basic vectors, and the basis vectors are independent of one another.

Standard Basis

- The set of n vectors $\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 1)\}$ is a basis for $\mathbf{R}^n$. This basis is called the **standard basis** for $\mathbf{R}^n$.

Proof:

To verify this result. We must show that this set spans R^n and that is linear independent.

(1) This set span $\mathbf{R}^n$:

Let $(x_1, x_2, \ldots, x_n)$ be an arbitrary element of R^n

$(x_1, x_2, \ldots, x_n) = x_1 (1, 0, \ldots, 0) + x_2 (0, 1, \ldots, 0) + \ldots + x_n (0, 0, \ldots, 1)$.

Thus the set span R^n.

(2) Linear independence:

$c_1 (1, 0, \ldots, 0) + c_2 (0, 1, \ldots, 0) + \ldots + c_n (0, 0, \ldots, 1) = (c_1, c_2, \ldots, c_n) = 0 \Rightarrow c_i = 0$.

Example 1/2

- Show that the set $\{(1, 0, -1), (1, 1, 1), (1, 2, 4)\}$ is a basis for **R**3.

Let us first show that the set spans $\mathbf{R}^3$.

Let (x_1, x_2, x_3) be an arbitrary element of $\mathbf{R}^3$. We try to find scalars a_1, a_2, a_3 such that

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(1, 2, 4)$$

This identity leads to the systems of equations

$$a_1 + a_2 + a_3 = x_1$$
$$a_2 + 2a_3 = x_2$$
$$-a_1 + a_2 + 4a_3 = x_3$$

This system of equations has the solution

$$a_1 = 2x_1 - 3x_2 + x_3, a_2 = -2x_1 + 5x_2 - 2x_3, a_3 = x_1 - 2x_2 + x_3$$

Thus the set spans the space.

Example 2/2

We now show that the set is linearly independent.

Consider the identity

$$b_1(1, 0, -1) + b_2(1, 1, 1) + b_3(1, 2, 4) = (0, 0, 0)$$

The identity leads to the system of equations

$$b_1 + b_2 + b_3 = 0$$
$$b_2 + 2b_3 = 0$$
$$-b_1 + b_2 + 4b_3 = 0$$

This system has the unique solution $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$. Thus the set is linearly independent.

We have shown that the set $\{(1, 0, -1), (1, 1, 1), (1, 2, 4)\}$ spans $\mathbf{R}^3$ and is linearly independent. It thus forms a basis for $\mathbf{R}^3$.

Dimension of Vector Space

- If a vector space V has a basis consisting of n vectors, then the **dimension** of V is said to be n. We write $\dim(V)$ for the dimension of V.
  - ⇨The set of n vectors $\{(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)\}$ forms a basis (the standard basis) for $\mathbf{R}^n$. Thus the dimension of $\mathbf{R}^n$ is n.
  - ⇨If a basis for a vector space is a finite set, then the vector space is finite dimensional.
  - ⇨If such a finite set does not exist, then the vector space is infinite dimensional.

Example

- Consider the set $\{\{1, 2, 3), (-2, 4, 1)\}$ of vectors in $\mathbf{R}^3$. These vectors generate a subspace V of $\mathbf{R}^3$ consisting of all vectors of the form

$$\mathbf{v} = c_1(1, 2, 3) + c_2(-2, 4, 1)$$

The vectors (1, 2, 3) and (-2, 4, 1) span this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector, the vectors are linearly independent.

Therefore $\{\{1, 2, 3), (-2, 4, 1)\}$ is a basis for V. Thus $\dim(V) = 2$. We know that V is, in fact, a plane through the origin.

1.5.1 An Overview of Dimension and Its Consequences

We summarize here the main results of this section in order to put them into better perspective.摘述本節的重要結果

1.A basis for a vector space V is a linearly independent subset of V that generates V.
一向量空間 V 的基底是是生成 V 的線性獨立子集合。

2.If V has a finite basis, then every basis for V contains the same number of vectors. The number is called the dimensions of V, and V is said to be finite-dimensional.
若 V 具有有限基底，則 V 的每一組基底含有相同向量數目。此數目稱為 V 的維度，且 V 可稱為具有有限維度的向量空間。

3.If the dimension of V is n, every basis for V contains exactly n vectors.

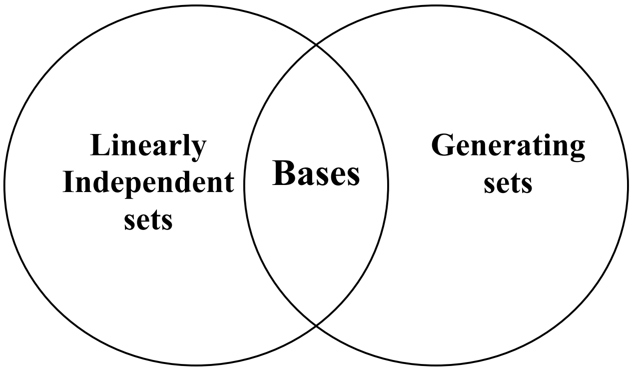
若 V 的維度為 n，則 V 的每一個基底恰含有 n 個向量。

4.Every linearly independent subset of V contains no more than n vectors and can be extended to a basis for V by including appropriately chosen vectors.

V 的每一線性獨立子集含有向量數不多於 n，但可藉由適當的向量選取擴張至成為 V 的一組基底。

5.Each generating set for V contains at least n vectors and can be reduced to a basis for V by excluding appropriately chosen vectors.

V 的 Generating set 含有至少 n 個向量，且可藉由刪去某些向量，將 Generating set 簡化成 V 的一組基底。



1.5.2 The Dimension of Subspaces

Theorem 1.11

Let W be a subspace of a finite-dimensional vector space V. Then W is finite-dimensional and dim(W)≦dim(V). Moreover, if dim(W) = dim(V), then V = W.

設 W 為有限維度的向量空間 V 內的子空間，則 W 為有限維度且 dim(W)≦dim(V)。若 dim(W) = dim(V)，則 W = V。

【Proof】

Let dim(V) = n. If W = {0}, then W is finite-dimensional and dim(W) = 0≦n.

Otherwise, W contains a nonzero vector x_1; so { x_1} is a linearly independent set.

Continue choosing vectors, x_1, x_2,…, x_k in W such that {x_1, x_2,…, x_k} is linearly independent. Since no linearly independent subset of V can contain more than n vectors, this process must stop at a stage where k≦n and {x_1, x_2,…, x_k} is linearly independent but adjoining any other vector from W produces a linearly dependent set. **Theorem 1.7** implies that {x_1, x_2,…, x_k} generates W, and hence it is a basis for W. Therefore dim(W) = k≦n.

If dim(W) = n, then a basis for W is linearly independent subset of V containing n vectors. But **Corollary 2 to Theorem 1.10** implies that this basis for W is also a basis for V; so W = V.

令 dim(V) = n。

若 W = {0}，則 W 為有限維度且 dim(W) = 0 ≦ n，否則 W 含有一非零向量 x_1。$\{x_1\}$為線性獨立集合。

繼續選取 W 的向量 $x_1, x_2,\ldots, x_k$ 使得 $\{x_1, x_2,\ldots, x_k\}$ 為線性獨立集合，並結合 W 內的任意其他向量組成一個線性相依集合。由 **Theorem 1.7** 得知$\{x_1, x_2,\ldots, x_k\}$生成 W，且是 W 的一組基底，因此 dim(W) = k≦n。

若 dim(W) = n，則 W 的一組基底也是 V 的線性獨立子集合，且含有 n 個向量，依據 Corollary to Theorem 1.10 得知此基底也是 V 的基底，故 W = V。

Theorem 1.7 設 S 為向量空間 V 的一個線性獨立子集合，且令 v 為 V 中的一個、但不在 S 內的向量，則 **S∪{v}為線性相依『若且唯若』v∈span(S)**。

EXAMPLE 1

Let $W = \{(a_1, a_2, a_3, a_4, a_5)\in F^5: a_1 + a_3 + a_5 = 0, a_2 = a_4\}$. It is easily shown that W is a subspace of F^5 having {(-1, 0, 1, 0, 0), (-1, 0, 0, 0, 1),(0, 1, 0, 1, 0)} as a basis. Thus dim(W) = 3.

W 是 F^5 的子空間且以{(-1, 0, 1, 0, 0), (-1, 0, 0, 0, 1),(0, 1, 0, 1, 0)}為基底，故 dim(W) = 3。

EXAMPLE 2

The set of diagonal nxn matrices is a subspace W of $M_{n\times n}(F)$. A basis for W is $\{E^{11}, E^{22}, \ldots, E^{nn}\}$, where E^{ij} is the matrix in which the only nonzero entry is a 1 in the i^{th} row and j^{th} column. Thus dim(W) = n.

$\{E^{11}, E^{22}, \ldots, E^{nn}\}$是 W 的基底。W 是 $M_{n\times n}(F)$的子空間。W 是 diagonal nxn matrices 的集合。

EXAMPLE 3

The set of symmetric nxn matrices is a subspace W of $M_{n\times n}(F)$. A basis for W is $\{A^{ij}: 1\leq i\leq j\leq n\}$, where A^{ij} is the nxn matrix having 1 in the i^{th} row and j^{th} column, and 0 elsewhere. It follows that dim(W) = n + (n - 1) +…+ 1 = n(n + 1)/2.

$\{A^{ij}: 1\leq i\leq j\leq n\}$是 W 的基底。

Corollary

If W is a subspace of a finite-dimensional vector space V, then any basis for W can be extended to a basis for V.

W 是 V 的子空間，則 W 的任一基底可以延伸為 V 的基底。

EXAMPLE 4

The set of all polynomials of the form $a_{18}x^{18} + a_{16}x^{16} + \ldots a_2x^2 + a_0$. where $a_{18}, a_{16}, \ldots, a_0 \in F$, is a subspace W of $P_{18}(F)$. A basis for W is $\{1, x^2, \ldots, x^{16}, x^{18}\}$, which is a subset of the standard basis of $P_{18}(F)$.