Chapter 7 Dimensional Analysis
Modeling, and Similitude

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A typical fluid mechanics problem in which experimentation is required, consider the steady flow of an incompressible Newtonian fluid through a long, smooth-walled, horizontal, circular pipe.

An important characteristic of this system, which would be interest to an engineer designing a pipeline, is the pressure drop per unit length that develops along the pipe as a result of friction.
The first step in the planning of an experiment to study this problem would be to decide on the factors, or variables, that will have an effect on the pressure drop.

Pressure drop per unit length

$$\Delta p_{\ell} = f(D, \rho, \mu, V)$$

Pressure drop per unit length depends on FOUR variables: sphere size (D); speed (V); fluid density (\(\rho\)); fluid viscosity (\(\mu\)).
To perform the experiments in a meaningful and systematic manner, it would be necessary to change one of the variable, such as the velocity, which holding all other constant, and measure the corresponding pressure drop.

Difficulty to determine the functional relationship between the pressure drop and the various facts that influence it.
Series of Tests

出現一拖拉庫的資料與圖表

把實驗當作傳家寶，一代傳一代，代代當作使命來執行

就算已經完成實驗………

就算從中找出影響壓力降的因子與壓力降的關係
Fortunately, there is a much simpler approach to the problem that will eliminate the difficulties described above.

Collecting these variables into two nondimensional combinations of the variables (called dimensionless product or dimensionless groups)

\[
\frac{D \Delta p_l}{\rho V^2} = \phi \left( \frac{\rho VD}{\mu} \right)
\]

- Only one dependent and one independent variable
- Easy to set up experiments to determine dependency
- Easy to present results (one graph)

Dependent variable

應變數y與獨立變數x: y=f(x)
\[ \frac{D \Delta p_{\ell}}{\rho V^2} = \frac{L(F/L^3)}{(FL^{-4}T^2)(FT^{-1})^2} = F^0 L^0 T^0 \]

\[ \frac{\rho V D}{\mu} = \frac{(FL^{-4}T^2)(LT^{-1})(L)}{(FL^{-2}T)} = F^0 L^0 T^0 \]

dimensionless product or dimensionless groups
A fundamental question we must answer is how many dimensionless products are required to replace the original list of variables?

The answer to this question is supplied by the basic theorem of dimensional analysis that states:

If an equation involving $k$ variables is dimensionally homogeneous, it can be reduced to a relationship among $k-r$ independent dimensionless products, where $r$ is the minimum number of reference dimensions required to describe the variables.
Given a physical problem in which the dependent variable is a function of \( k - 1 \) independent variables.

\[
 u_1 = f(u_2, u_3, \ldots, u_k) 
\]

Mathematically, we can express the functional relationship in the equivalent form

\[
 g(u_1, u_2, u_3, \ldots, u_k) = 0 
\]

*Where \( g \) is an unspecified function, different from \( f \).*
The Buckingham Pi theorem states that: Given a relation among \( k \) variables of the form

\[
g(u_1, u_2, u_3, \ldots, u_k) = 0
\]

The \( k \) variables may be grouped into \( k-r \) independent dimensionless products, or \( \Pi \) terms, expressible in functional form by

\[
\Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})
\]

or

\[
\bar{\phi}(\Pi_1, \Pi_2, \Pi_3, \ldots, \Pi_{k-r}) = 0
\]

把\( k \)個變數併成\( k-r \)個無因次群組或乘積
Buckingham Pi Theorem

The number $r$ is usually, but not always, equal to the minimum number of independent dimensions required to specify the dimensions of all the parameters. Usually the reference dimensions required to describe the variables will be the basic dimensions $M, L, and T$ or $F, L, and T$.

The theorem does not predict the functional form of $\phi$ or $\varphi$. The functional relation among the independent dimensionless products $\Pi$ must be determined experimentally.

The $k-r$ dimensionless products $\Pi$ terms obtained from the procedure are independent.
**Buckingham Pi Theorem**

A \( \Pi \) term is not independent if it can be obtained from a product or quotient of the other dimensionless products of the problem. For example, if

\[
\Pi_5 = \frac{2\Pi_1}{\Pi_2 \Pi_3} \quad \text{or} \quad \Pi_6 = \frac{\Pi_1^{3/4}}{\Pi_3^2}
\]

then neither \( \Pi_5 \) nor \( \Pi_6 \) is independent of the other dimensionless products or dimensionless groups.

PI terms can be combined into new PI terms.
Several methods can be used to form the dimensionless products, or pi term, that arise in a dimensional analysis. The method we will describe in detail is called the METHOD of repeating variables. Regardless of the method to be used to determine the dimensionless products, one begins by listing all dimensional variables that are known (or believed) to affect the given flow phenomenon. Eight steps listed below outline a recommended procedure for determining the Π terms.

請follow下列步驟，就可以輕易找出PI terms……
Step 1 List all the variables.  
- List all the dimensional variables involved.
- Keep the number of variables to a minimum, so that we can minimize the amount of laboratory work.
- All variables must be independent. For example, if the cross-sectional area of a pipe is an important variable, either the area or the pipe diameter could be used, but not both, since they are obviously not independent.

\[ \gamma = \rho \times g, \text{ that is, } \gamma, \rho, \text{ and } g \text{ are not independent.} \]
Step 1 List all the variables. 

Let \( k \) be the number of variables.

Example: For pressure drop per unit length, \( k = 5 \). (All variables are \( \Delta p, D, \mu, \rho, \) and \( V \))

\[ \Delta p_\ell = f(D, \rho, \mu, V) \]
Step 2 Express each of the variables in terms of basic dimensions. Find the number of reference dimensions.

- Select a set of fundamental (primary) dimensions.
- For example: MLT, or FLT.
- Example: For pressure drop per unit length, we choose FLT.

\[
\begin{align*}
\Delta p_{\ell} & \equiv FL^{-3} \\
D & \equiv L \\
\rho & \equiv FL^{-4}T^2 \\
\mu & \equiv FL^{-2}T \\
V & \equiv LT^{-1}
\end{align*}
\]

\[r=3\]
Step 3 Determine the required number of pi terms.

- Let $k$ be the number of variables in the problem.
- Let $r$ be the number of reference dimensions (primary dimensions) required to describe these variables.
- The number of pi terms is $k - r$.

Example: For pressure drop per unit length $k=5$, $r = 3$, the number of pi terms is $k-r=5-3=2$.

有了$k$, 有了$r$, 當然就可以得出PI terms個數 $k-r$.
Step 4 Select a number of repeating variables, where the number required is equal to the number of reference dimensions.

- Select a set of $r$ dimensional variables that includes all the primary dimensions (repeating variables).

- These repeating variables will all be combined with each of the remaining parameters. No repeating variables should have dimensions that are powers of the dimensions of another repeating variable.

- Example: For pressure drop per unit length ($r = 3$) select $\rho$, $V$, $D$. 

$r$個重複變數必須包括problem中所涉及的所有primary dimensions
Step 5 Form a pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless.  

Set up dimensional equations, combining the variables selected in Step 4 with each of the other variables (nonrepeating variables) in turn, to form dimensionless groups or dimensionless product.

There will be $k - r$ equations.

Example: For pressure drop per unit length
nonrepeating variables

repeating variables

配对
Step 5 (Continued)^2

\[ \Pi_1 = \Delta p \ell D^a V^b \rho^c \]

\[ (FL^{-3})(L)^a (LT^{-1})^b (FL^{-4}T^2)^c \div F^0 L^0 T^0 \]

\begin{align*}
F &: 1 + c = 0 \\
L &: -3 + a + b - 4c = 0 \\
T &: -b + 2c = 0
\end{align*}

\[ \Rightarrow a = 1, b = -2, c = -1 \]

\[ \Pi_1 = \frac{\Delta p \ell D}{\rho V^2} \]
Step 6 Repeat Step 5 for each of the remaining nonrepeating variables.

\[ \Pi_2 = \mu D^a V^b \rho^c \]

\[ (FL^{-2}T)(L)^a (LT^{-1})^b (FL^{-4}T^2)^c \div F^0L^0T^0 \]

F : \( 1 + c = 0 \)

L : \(-2 + a + b - 4c = 0\)

T : \(1 - b + 2c = 0\)

\[ \Rightarrow a = -1, b = -1, c = -1 \]

\[ \Pi_2 = \frac{\mu}{DV\rho} \]
Step 7 Check all the resulting pi terms to make sure they are dimensionless.

Check to see that each group obtained is dimensionless.

Example: For pressure drop per unit length.

\[ \Pi_1 = \frac{\Delta p \ell D}{\rho V^2} \quad \hat{=} \quad F^0 L^0 T^0 \quad \hat{=} \quad M^0 L^0 T^0 \]

\[ \Pi_2 = \frac{\mu}{D V \rho} \quad \hat{=} \quad F^0 L^0 T^0 \quad \hat{=} \quad M^0 L^0 T^0 \]
Determination of Pi Terms

Step 8 Express the final form as a relationship among the pi terms, and think about what it means.

Express the result of the dimensional analysis.

\[ \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r}) \]

Example: For pressure drop per unit length.

\[ \frac{\Delta p}{\rho V^2} = \phi \left( \frac{\mu}{DV\rho} \right) \]

Dimensional analysis will not provide the form of the function. The function can only be obtained from a suitable set of experiments.

得靠實驗來找出兩者間的真正關係
The \( \Pi_2 \) terms can be rearranged. For example, \( \Pi_2 \), could be expressed as

\[
\Pi_2 = \frac{\rho VD}{\mu}
\]

\[
\frac{\Delta p \ell D}{\rho V^2} = \phi\left(\frac{\rho VD}{\mu}\right)
\]

The true relationship is found through experiments.
**Example 7.1 Method of Repeating Variables**

A thin rectangular plate having a width $w$ and a height $h$ is located so that it is normal to a moving stream of fluid. Assume that the drag, $D$, that the fluid exerts on the plate is a function of $w$ and $h$, the fluid viscosity, $\mu$, and $\rho$, respectively, and the velocity, $V$, of the fluid approaching the plate. Determine a suitable set of pi terms to study this problem experimentally.
Drag force on a PLATE

\[ D = f(w, h, \rho, \mu, V) \]

- **Step 1:** List all the dimensional variables involved.
  \[ D, w, h, \rho, \mu, V \quad k = 6 \text{ dimensional parameters.} \]

- **Step 2:** Select primary dimensions \( M, L, \) and \( T. \) Express each of the variables in terms of basic dimensions

\[
\begin{align*}
D &\equiv MLT^{-2} \\
w &\equiv L \\
h &\equiv L \\
\mu &\equiv ML^{-1}T^{-1} \\
\rho &\equiv ML^{-3} \\
V &\equiv LT^{-1}
\end{align*}
\]
Example 7.1 Solution\(^{2/5}\)

 kaz \Rightarrow \text{Step 3: Determine the required number of pi terms.}
\[
\text{k-r=6-3=3}
\]

\Rightarrow \text{Step 4: Select repeating variables w, V, \(\rho\).}

\Rightarrow \text{Step 5~6: Combining the repeating variables with each of the other variables in turn, to form dimensionless groups or dimensionless products.}
Example 7.1 Solution

\[ \Pi_1 = Dw^a V^b \rho^c = (MLT^{-2})(L)^a (LT^{-1})^b (ML^{-3})^c = M^0L^0T^0 \]

\( M : 1 + c = 0 \)
\( L : 1 + a + b - 3c = 0 \)
\( T : -2 - b = 0 \)

\( \Pi_1 = \frac{D}{w^2 V^2 \rho} \)

\( \Pi_2 = hw^a V^b \rho^c = L(L)^a (LT^{-1})^b (ML^{-3})^c = M^0L^0T^0 \)

\( M : c = 0 \)
\( L : 1 + a + b - 3c = 0 \)
\( T : b = 0 \)

\( \Pi_2 = \frac{h}{w} \)

\( >> a = -2, b = -2, c = -1 \)

\( >> a = -1, b = 0, c = 0 \)
Example 7.1 Solution\textsuperscript{4/5}

\[ \Pi_3 = \mu w^a V^b \rho^c = (ML^{-1}T^{-1})(L)^a (LT^{-1})^b (ML^{-3})^c = M^0 L^0 T^0 \]

M : 1 + c = 0
L : −1 + a + b − 3c = 0
T : −1 − b = 0
>> a = −1, b = −1, c = −1

\[ \Pi_3 = \frac{\mu}{wV\rho} \]
Example 7.1 Solution

Step 7: Check all the resulting pi terms to make sure they are dimensionless.

Step 8: Express the final form as a relationship among the pi terms.

The functional relationship is

\[ \Pi_1 = \bar{\phi}(\Pi_2, \Pi_3), \text{ or} \]

\[ \frac{D}{w^2 V^2 \rho} = \phi\left(\frac{h}{w}, \frac{\mu}{w V \rho}\right) \]
Selection of Variables

One of the most important, and difficult, steps in applying dimensional analysis to any given problem is the selection of the variables that are involved.

There is no simple procedure whereby the variable can be easily identified. Generally, one must rely on a good understanding of the phenomenon involved and the governing physical laws.

If extraneous variables are included, then too many pi terms appear in the final solution, and it may be difficult, time consuming, and expensive to eliminate these experimentally.

因次分析中最重要，也是最困難的步驟

沒有所謂簡單的程序可以決定那些變數要？那些不要？取決於對物理現象與 Physical laws 的了解
If important variables are omitted, then an incorrect result will be obtained; and again, this may prove to be costly and difficult to ascertain.

Most engineering problems involve certain simplifying assumptions that have an influence on the variables to be considered.

Usually we wish to keep the problems as simple as possible, perhaps even if some accuracy is sacrificed.
Selection of Variables

在簡化與正確間取得balance

- A suitable balance between simplicity and accuracy is an desirable goal.

- Variables can be classified into three general group:
  - Geometry: lengths and angles.
  - Material properties: relate the external effects and the responses.
  - External Effects: produce, or tend to produce, a change in the system. Such as force, pressure, velocity, or gravity.
When to determine the number of pi terms, it is important to know how many reference dimensions are required to describe the variables.

In fluid mechanics, the required number of reference dimensions is three, but in some problems only one or two are required.

In some problems, we occasionally find the number of reference dimensions needed to describe all variables is smaller than the number of basic dimensions. Illustrated in Example 7.2.
Example 7.2 Determination of \( \Pi \) Terms

An open, cylindrical tank having a diameter \( D \) is supported around its bottom circumference and is filled to a depth \( h \) with a liquid having a specific weight \( \gamma \). The vertical deflection, \( \delta \), of the center of the bottom is a function of \( D, h, d, \gamma, \) and \( E \), where \( d \) is the thickness of the bottom and \( E \) is the modulus of elasticity of the bottom material. Perform a dimensional analysis of this problem.

圓筒內裝liquid放在一個平臺上，平臺下凹深度\( \delta \)
Example 7.2 Solution

The vertical deflection

\[ \delta = f(D, h, d, \gamma, E) \]

平臺下凹深度\( \delta \)與那些變數有關？

使用不同的dimensional system

For F,L,T. Pi terms=6-2=4

For M,L,T Pi terms=6-3=3

\[ \delta \equiv L, \quad \delta \equiv L \]
\[ D \equiv L, \quad D \equiv L \]
\[ h \equiv L, \quad h \equiv L \]
\[ d \equiv L, \quad d \equiv L \]
\[ \gamma \equiv FL^{-3}, \quad \gamma \equiv ML^{-2}T^{-2} \]
\[ E \equiv FL^{-2}, \quad E \equiv ML^{-1}T^{-2} \]
Example 7.2 Solution\(^{2/3}\)

- For F,L,T system, Pi terms=6-2=4

  D and \(\gamma\) are selected as repeating variables

\[
\begin{align*}
\Pi_1 &= \delta D^{a_1} \gamma^{b_1} \\
\Pi_2 &= h D^{a_2} \gamma^{b_2} \\
\Pi_3 &= d D^{a_3} \gamma^{b_3} \\
\Pi_4 &= E D^{a_4} \gamma^{b_4}
\end{align*}
\]

\[
\begin{align*}
\Pi_1 &= \frac{\delta}{D} , \Pi_2 = \frac{h}{D} , \Pi_3 = \frac{d}{D} , \Pi_4 = \frac{E}{D\gamma} \Rightarrow \frac{\delta}{D} = \Phi \left( \frac{h}{D} , \frac{d}{D} , \frac{E}{D\gamma} \right)
\end{align*}
\]
Example 7.2 Solution

- For M, L, T system, Pi terms = 6 - 3 = 3 ?

A closer look at the dimensions of the variables listed reveal that only two reference dimensions, L and MT^2, are required.

直接目測法，看似3個，其實只有2個
Determination of Reference Dimension

EXAMPLE \[ \Delta h = f(D, \gamma, \sigma) \]

MLT SYSTEM

\[
\begin{array}{cccc}
\Delta h & D & \gamma & \sigma \\
L & L & \frac{M}{L^2T^2} & \frac{M}{T^2}
\end{array}
\]

Pi term = 4 - 3 = 1

FLT SYSTEM

\[
\begin{array}{cccc}
\Delta h & D & \gamma & \sigma \\
L & L & \frac{F}{L^3} & \frac{F}{L}
\end{array}
\]

Pi term = 4 - 2 = 2
Determination of Reference Dimension

Set Dimensional Matrix

\[
\begin{bmatrix}
\Delta h & D & \gamma & \sigma \\
M & 0 & 0 & 1 & 1 \\
L & 1 & 1 & -2 & 0 \\
T & 0 & 0 & -2 & -2 \\
\end{bmatrix}
\quad \text{MLT SYSTEM}
\]

\[
\begin{bmatrix}
\Delta h & D & \gamma & \sigma \\
F & 0 & 0 & 1 & 1 \\
L & 1 & 1 & -3 & -1 \\
T & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\quad \text{FLT SYSTEM}
\]

\[\Pi_1 = \frac{\Delta h}{D}\]

\[\Pi_2 = \frac{\sigma}{D^2 \gamma} \Rightarrow \frac{\Delta h}{D} = \Phi \left( \frac{\sigma}{D^2 \gamma} \right)\]

\[\text{Rank} = 2 \quad \text{Pi term} = 4 - 2 = 2\]
The Pi terms obtained depend on the somewhat arbitrary selection of repeating variables. For example, in the problem of studying the pressure drop in a pipe.

$$\Delta p_\ell = f(D, \rho, \mu, V)$$

Selecting $D, V, \rho$ as repeating variables:

$$\frac{\Delta p_\ell D}{\rho V^2} = \Phi_1 \left( \frac{\rho VD}{\mu} \right)$$

Selecting $D, V, \mu$ as repeating variables:

$$\frac{\Delta p_\ell D^2}{V \mu} = \Phi_2 \left( \frac{\rho VD}{\mu} \right)$$
Both are correct, and both would lead to the same final equation for the pressure drop. There is not a unique set of pi terms which arises from a dimensional analysis. The functions $\Phi_1$ and $\Phi_2$ are will be different because the dependent pi terms are different for the two relationships.

$$\frac{\Delta p \ell D}{\rho V^2} = \Phi_1 \left( \frac{\rho V D}{\mu} \right)$$

$$\frac{\Delta p \ell D^2}{V \mu} = \Phi_2 \left( \frac{\rho V D}{\mu} \right)$$
EXAMPLE \( \prod_1 = \Phi(\prod_2, \prod_3) \)

Form a new pi term \( \prod'_2 = \prod_2^a \prod_3^b \)

\[ \prod_1 = \Phi_1(\prod'_2, \prod_3) = \Phi_2(\prod_2, \prod'_2) \]

All are correct

前面已經談過，PI terms可以組合成新的PI term
Uniqueness of Pi Terms

\[
\frac{\Delta p \ell D}{\rho V^2} = \Phi_1\left(\frac{\rho V D}{\mu}\right)
\]

Selecting D, V, and \(\rho\) as repeating variables:

\[
\frac{\Delta p \ell D}{\rho V^2} \times \frac{\rho V D}{\mu} = \frac{\Delta p \ell D^2}{V \mu}
\]

Repeating variables

的最妥當選擇？

\[
\frac{\Delta p \ell D^2}{V \mu} = \Phi_2\left(\frac{\rho V D}{\mu}\right)
\]
### Common Dimensionless Groups

<table>
<thead>
<tr>
<th>Dimensionless Groups</th>
<th>Name</th>
<th>Interpretation (Index of Force Ratio Indicated)</th>
<th>Types of Applications</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\rho V \ell}{\mu} )</td>
<td>Reynolds number, Re</td>
<td>inertia force, viscous force</td>
<td>Generally of importance in all types of fluid dynamics problems</td>
</tr>
<tr>
<td>( \sqrt{\frac{V}{g \ell}} )</td>
<td>Froude number, Fr</td>
<td>inertia force, gravitational force</td>
<td>Flow with a free surface</td>
</tr>
<tr>
<td>( \frac{p}{\rho V^2} )</td>
<td>Euler number, Eu</td>
<td>pressure force, inertia force</td>
<td>Problems in which pressure, or pressure differences, are of interest</td>
</tr>
<tr>
<td>( \frac{p V^2}{E_v} )</td>
<td>Cauchy number, Ca</td>
<td>inertia force, compressibility force</td>
<td>Flows in which the compressibility of the fluid is important</td>
</tr>
<tr>
<td>( \frac{V}{c} )</td>
<td>Mach number, Ma</td>
<td>inertia force, compressibility force</td>
<td>Flows in which the compressibility of the fluid is important</td>
</tr>
<tr>
<td>( \frac{\omega \ell}{V} )</td>
<td>Strouhal number, St</td>
<td>inertia (local) force, inertia (convective) force</td>
<td>Unsteady flow with a characteristic frequency of oscillation</td>
</tr>
<tr>
<td>( \frac{p V^2 \ell}{\sigma} )</td>
<td>Weber number, We</td>
<td>inertia force, surface tension force</td>
<td>Problems in which surface tension is important</td>
</tr>
</tbody>
</table>

*The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.*
Froude Number \(1/2\)

\[ Fr = \frac{V}{\sqrt{gL}} \quad \text{and} \quad Fr^2 = \frac{V^2}{gL} = \frac{\rho V^2L^2}{\rho gL^3} \]

- In honor of William Froude (1810~1879), a British civil engineer, mathematician, and naval architect who pioneered the use of towing tanks for the study of ship design.
- Froude number is the ratio of the forces due to the acceleration of a fluid particles (inertial force) to the force due to gravity (gravity forces).
- Froude number is significant for flows with free surface effects.
- Froude number less than unity indicate subcritical flow and values greater than unity indicate supercritical flow.
Froude Number \(2/2\)

\[
\begin{align*}
\mathbf{a}_s &= \frac{dV_s}{dt} = V_s \frac{dV_s}{ds} = \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*} \\
F_l &= \frac{V^2}{\ell} V_s^* \frac{dV_s^*}{ds^*} \text{ m} \\
V_s^* &= \frac{V_s}{V} \quad s^* = \frac{s}{\ell} \\
Fr &= \frac{F_l}{F_G} = \frac{V^2}{g\ell} V_s^* \frac{dV_s^*}{ds^*} \equiv \frac{V^2}{g\ell} \equiv \frac{V}{\sqrt{g\ell}}
\end{align*}
\]
In honor of Osborne Reynolds (1842~1912), the British engineer who first demonstrated that this combination of variables could be used as a criterion to distinguish between laminar and turbulent flow.

The Reynolds number is a measure of the ratio of the inertia forces to viscous forces.

If the Reynolds number is small (Re<<1), this is an indication that the viscous forces are dominant in the problem, and it may be possible to neglect the inertial effects; that is, the density of the fluid will no be an important variable.

\[ Re = \frac{\rho V \ell}{\mu} = \frac{V \ell}{\nu} \]
Flows with very small Reynolds numbers are commonly referred to as “creeping flows”.

For large Reynolds number flow, the viscous effects are small relative to inertial effects and for these cases it may be possible to neglect the effect of viscosity and consider the problem as one involving a “nonviscous” fluid.

Flows with “large” Reynolds number generally are turbulent. Flows in which the inertia forces are “small” compared with the viscous forces are characteristically laminar flows.
Euler number

\[ \text{Eu} = \frac{p}{\rho V^2} \equiv \frac{\Delta p}{\rho V^2} \]

- In honor of Leonhard Euler (1707~1783), a famous Swiss mathematician who pioneered work on the relationship between pressure and flow.
- Euler’s number is the ratio of pressure force to inertia forces. It is often called the \textit{pressure coefficient,} \( C_p. \)
For problems in which cavitation is of concern, the dimensionless group \((p_r - p_v)/\frac{1}{2}\rho V^2\) is commonly used, where \(p_v\) is the vapor pressure and \(p_r\) is some reference pressure.

The cavitation number is used in the study of cavitation phenomena.

The smaller the cavitation number, the more likely cavitation is to occur.

\[
Ca = \frac{p_r - p_v}{\frac{1}{2}\rho V^2}
\]

Ca越小，表示cavitation越易發生
The Cauchy number is named in honor of Augustin Louis de Cauchy (1789~1857), a French engineer, mathematician, and hydrodynamicist.

The Mach number is named in honor of Ernst Mach (1838~1916), an Austrian physicist and philosopher.

Either number may be used in problems in which fluid compressibility is important.
Both numbers can be interpreted as representing an index of the ratio of inertial force to compressibility force, where $V$ is the flow speed and $c$ is the local sonic speed.

Mach number is a key parameter that characterizes compressibility effects in a flow.

When the Mach number is relatively small (say, less than 0.3), the inertial forces induced by the fluid motion are not sufficiently large to cause a significant change in the fluid density, and in this case the compressibility of the fluid can be neglected.

For truly incompressible flow, $c=\infty$ so that $M=0$. 
Strouhal Number \(1/2\)

\[
St = \frac{\omega \ell}{V}
\]

- In honor of Vincenz Strouhal (1850~1922), who used this parameter in his study of “singing wires.” The most dramatic evidence of this phenomenon occurred in 1940 with the collapse of the Tacoma Narrow bridges. The shedding frequency of the vortices coincided with the natural frequency of the bridge, thereby setting up a resonant condition that eventually led to the collapse of the bridge.

- This parameter is important in unsteady, oscillating flow problems in which the frequency of the oscillation is \(\omega\).
Strouhal Number $^{2/2}$

- This parameter represents a measure of the ratio of inertial force due to the unsteadiness of the flow (local acceleration) to the inertial forces due to change in velocity from point to point in the flow field (convective acceleration). This type of unsteady flow may develop when a fluid flows past a solid body (such as a wire or cable) placed in the moving stream.

For example, in a certain Reynolds number range, a periodic flow will develop downstream from a cylinder placed in a moving stream due to a regular patterns of vortices that are shed from the body.
**Weber Number** $^1/_2$

$$\text{We} = \frac{\rho V^2 \ell}{\sigma}$$

- Named after Moritz Weber (1871~1951), a German professor of naval mechanics who was instrumental in formalizing the general use of common dimensionless groups as a basis for similitude studies.

- Weber number is important in problems in which there is an interface between two fluids. In this situation, the surface tension may play an important role in the phenomenon of interest.
Weber Number $^{2/2}$

- Weber number is the ratio of inertia forces to surface tension forces.
- Common examples of problems in which Weber number may be important include the flow of thin film of liquid, or the formation of droplets or bubbles.
- The flow of water in a river is not affected significantly by surface tension, since inertial and gravitational effects are dominant ($We>>1$).
Correlation of Experimental Data

- Dimensional analysis only provides the dimensionless groups describing the phenomenon, and not the specific relationship between the groups.
- To determine this relationship, suitable experimental data must be obtained.
- The degree of difficulty depends on the number of pi terms.
Problems with One Pi Term

- The functional relationship for one Pi term.

$$\Pi_1 = C$$

where $C$ is a constant. The value of the constant must still be determined by experiment.
Example 7.3 Flow with Only One Pi Term

Assume that the drag, $D$, acting on a spherical particle that falls very slowly through a viscous fluid is a function of the particle diameter, $d$, the particle velocity, $V$, and the fluid viscosity, $\mu$. Determine, with the aid of the dimensional analysis, how the drag depends on the particle velocity.
Example 7.3 Solution

- The drag

\[ D = f(d, V, \mu) \]

\[ D \propto C \mu Vd \]

For a given particle and fluids, the drag varies directly with the velocity.
Problems with two pi terms

\[ \Pi_1 = \Phi(\Pi_2) \]

the functional relationship among the variables can be determined by varying \( \Pi_2 \) and measuring the corresponding value of \( \Pi_1 \).

The empirical equation relating \( \Pi_2 \) and \( \Pi_1 \) by using a standard curve-fitting technique.

An empirical relationship is valid over the range of \( \Pi_2 \).

危险的 extrapolate beyond valid range.

利用 curve fitting 技术

有兩個 PI terms
Problems with three pi terms.

\[ \Pi_1 = \Phi(\Pi_2, \Pi_3) \]

To determine a suitable empirical equation relating the three pi terms.

To show data correlations on simple graphs.

Families curve of curves

三個PI terms：固定其中一個，找出其他兩個的關係，製作出Families curve of curves
Modeling and Similitude

To develop the procedures for designing models so that the model and prototype will behave in a similar fashion......

製造模型&比擬

探討設計模型的程序，使得設計出來的模型可用來預測雛型的行為，即模型可以用來比擬雛型。換言之，模型與雛型間必須達到所謂的「相似性（Similarity）」
Model vs. Prototype

- **Model?** A model is a representation of a physical system that may be used to predict the behavior of the system in some desired respect. Mathematical or computer models may also conform to this definition, our interest will be in physical model.

- **Prototype?** The physical system for which the prediction are to be made. Models that resemble the prototype but are generally of a different size, may involve different fluid, and often operate under different conditions.

- Usually a model is smaller than the prototype.

- Occasionally, if the prototype is very small, it may be advantageous to have a model that is larger than the prototype so that it can be more easily studied. For example, large models have been used to study the motion of red blood cells.
With the successful development of a valid model, it is possible to predict the behavior of the prototype under a certain set of conditions.

There is an inherent danger in the use of models in that predictions can be made that are in error and the error not detected until the prototype is found not to perform as predicted.

It is imperative that the model be properly designed and tested and that the results be interpreted correctly.

所有在模型试验的结果不能保证移到prototype就会ok，因为有太多无法预测的变数造成风险的存在。即便如此，好好地设计模型，好好的进行模型试验，正确地解讀试验资料，仍然是有必要的。
Similarity of Model and Prototype

What conditions must be met to ensure the similarity of model and prototype?

Geometric Similarity
- Model and prototype have same shape.
- Linear dimensions on model and prototype correspond within constant scale factor.

Kinematic Similarity
- Velocities at corresponding points on model and prototype differ only by a constant scale factor.

Dynamic Similarity
- Forces on model and prototype differ only by a constant scale factor.

如何確保模型與雛型的相似性

在談「相似性」時，相似性包括「幾何上的相似性」、「運動上的相似性」與「動力上的相似性」

即 Model 與 prototype 間的... 與... 維持某一比例關係
The theory of models can be readily developed by using the principles of dimensional analysis. For given problem which can be described in terms of a set of pi terms as

\[ \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_n) \]

This equation applies to any system that is governed by the same variables.
A similar relationship can be written for a model of this prototype; that is,

$$\Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \ldots, \Pi_{nm})$$

where the form of the function will be the same as long as the same phenomenon is involved in both the prototype and the model.

The prototype and the model must have the same phenomenon.
Theory of Models 3/5

- Model design (the model is designed and operated) conditions, also called similarity requirements or modeling laws.

\[ \Pi_2 = \Pi_{2m} \quad \Pi_3 = \Pi_{3m} \ldots \Pi_n = \Pi_{nm} \]

- The form of \( \Phi \) is the same for model and prototype, it follows that

\[ \Pi_1 = \Pi_{1m} \]

This is the desired prediction equation and indicates that the measured of \( \Pi_{1m} \) obtained with the model will be equal to the corresponding \( \Pi_1 \) for the prototype as long as the other \( \Pi \) parameters are equal.
The prototype and the model must have the same phenomenon.

For prototype \[ \Pi_1 = \phi(\Pi_2, \Pi_3, \ldots, \Pi_n) \]

For model \[ \Pi_{1m} = \phi(\Pi_{2m}, \Pi_{3m}, \ldots, \Pi_{nm}) \]
The model is designed and operated under the following conditions (called design conditions, also called similarity requirements or modeling laws)

\[ \Pi_2 = \Pi_{2m} \quad \Pi_3 = \Pi_{3m} \ldots \Pi_n = \Pi_{nm} \]

才能用 Model 預測 Prototype

The measured of \( \Pi_{1m} \) obtained with the model will be equal to the corresponding \( \Pi_1 \) for the prototype.

\[ \Pi_1 = \Pi_{1m} \quad \text{Called prediction equation} \]
**Example: Considering the drag force on a sphere.**

\[ F = f(D, V, \rho, \mu) \quad \Rightarrow \quad \frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right) \]

- The prototype and the model must have the same phenomenon.

\[ \frac{F_m}{\rho_m V_m^2 D_m^2} = f_1\left(\frac{\rho_m V_m D_m}{\mu_m}\right) \quad \Rightarrow \quad \frac{F}{\rho V^2 D^2} = f_1\left(\frac{\rho V D}{\mu}\right)_{\text{prototype}} \]

- Design conditions.

\[ \left(\frac{\rho V D}{\mu}\right)_{\text{model}} = \left(\frac{\rho V D}{\mu}\right)_{\text{prototype}} \]

- Then …

\[ \left(\frac{F}{\rho V^2 D^2}\right)_{\text{model}} = \left(\frac{F}{\rho V^2 D^2}\right)_{\text{prototype}} \]
Example: Determining the drag force on a thin rectangular plate \((w \times h)\) in size

\[
D = f(w, h, \mu, \rho, V) \quad \Rightarrow \quad \frac{D}{w^2 \rho V^2} = \Phi \left( \frac{w}{h}, \frac{\rho Vw}{\mu} \right)
\]

\(\Rightarrow\) The prototype and the model must have the same phenomenon.

\[
\frac{D_m}{w_m^2 \rho_m V_m^2} = \Phi \left( \frac{w_m}{h_m}, \frac{\rho_m V_m w_m}{\mu_m} \right) \quad \Rightarrow \quad \frac{D}{w^2 \rho V^2} = \Phi \left( \frac{w}{h}, \frac{\rho Vw}{\mu} \right)_{\text{prototype}}
\]

\(\Rightarrow\) Design conditions.

\[
\frac{w_m}{h_m} = \frac{w}{h}, \quad \frac{\rho_m V_m w_m}{\mu_m} = \frac{\rho Vw}{\mu}
\]

\(\Rightarrow\) Then …

\[
\frac{D}{w^2 \rho V^2} = \frac{D_m}{w_m^2 \rho_m V_m^2} \quad \Rightarrow \quad D = \left( \frac{w}{w_m} \right)^2 \left( \frac{\rho}{\rho_m} \right) \left( \frac{V}{V_m} \right)^2 D_m
\]
A long structural component of a bridge has the cross section shown in Figure E7.5. It is known that when a steady wind blows past this type of bluff body, vortices may develop on the downwind side that are shed in a regular fashion at some definite frequency. Since these vortices can create harmful periodic forces acting on the structure, it is important to determine the shedding frequency. For the specific structure of interest, \( D = 0.1 \text{m} \), \( H = 0.3 \text{m} \), and a representative wind velocity 50km/hr. Standard air can be assumed. The shedding frequency is to be determined through the use of a small-scale model that is to be tested in a water tunnel. For the model \( D_m = 20 \text{mm} \) and the water temperature is 20°C.
Example 7.5 Prediction of Prototype Performance from Model Data 2/2

Determine the model dimension, \( H_m \), and the velocity at which the test should be performed. If the shedding frequency \( \omega \) for the model is found to be 49.9 Hz, what is the corresponding frequency for the prototype?

- For air at standard condition \( \mu = 1.79 \times 10^{-5} \text{ kg/m/s}, \rho = 1.23 \text{ kg/m}^3 \)
- For water at 20°C, \( \mu_{\text{water}} = 1 \times 10^{-3} \text{ kg/m/s}, \rho_{\text{water}} = 998 \text{ kg/m}^3 \)
Example 7.5 Solution

✓ Step 1: List all the dimensional variables involved. \( \omega \), \( D, H, V, \rho, \mu \). 6 dimensional variables.

✓ Step 2: Select primary dimensions \( F, L \) and \( T \). List the dimensions of all variables in terms of primary dimensions. 3 primary dimensions

\[
\begin{align*}
\omega & \doteq T^{-1} \\
D & \doteq L \\
H & \doteq L \\
V & \doteq LT^{-1} \\
\rho & \doteq FL^{-4}T^2 \\
\mu & \doteq ML^{-2}T
\end{align*}
\]
✓ Step 3: Determine the required number of pi terms.
   \[ k-r=6-3=3 \]

✓ Step 4: Select repeating variables \( D, V, \mu \).

✓ Step 5~6: Combining the repeating variables with each of the other variables in turn, to form dimensionless groups.

\[
\begin{align*}
\Pi_1 &= \omega D^{a_1} V^{b_1} \mu^{c_1} = \frac{\omega D}{V} \\
\Pi_2 &= HD^{a_2} V^{b_2} \mu^{c_2} = \frac{H}{D} \\
\Pi_3 &= \rho D^{a_3} V^{b_3} \mu^{c_3} = \frac{\rho VD}{\mu}
\end{align*}
\]
Example 7.5 Solution

✓ The functional relationship is

\[ \frac{\omega D}{V} = \phi \left( \frac{H}{D}, \frac{\rho V D}{\mu} \right) \]

Strouhal number

✓ The prototype and the model must have the same phenomenon.

\[ \frac{\omega_m D_m}{V_m} = \phi \left( \frac{H_m}{D_m}, \frac{\rho_m V_m D_m}{\mu_m} \right) \]
The Design conditions.

\[
\frac{H}{D} = \frac{H_m}{D_m} \quad \frac{\rho VD}{\mu} = \frac{\rho_m V_m D_m}{\mu_m}
\]

Then…. 

\[
H_m = \frac{H}{D} D_m = \ldots = 60 \text{mm} \quad V_m = \frac{\mu_m \rho}{\mu \rho_m} \frac{D}{D_m} V = \ldots = 13.9 \text{m/s}
\]

\[
\omega = \frac{V}{V_m} \frac{D_m}{D} \quad \omega_m = \ldots = 29.0 \text{Hz}
\]
The ratio of a model variable to the corresponding prototype variable is called the scale for that variable.

**Length Scale**
\[
\frac{\ell_1}{\ell_2} = \frac{\ell_{1m}}{\ell_{2m}} \Rightarrow \lambda_\ell = \frac{\ell_{1m}}{\ell_1} = \frac{\ell_{2m}}{\ell_2}
\]

**Velocity Scale**
\[
\lambda_V = \frac{V_m}{V}
\]

**Density Scale**
\[
\lambda_\rho = \frac{\rho_m}{\rho}
\]

**Viscosity Scale**
\[
\lambda_\mu = \frac{\mu_m}{\mu}
\]
Validation of Models Design

- The purpose of model design is to predict the effects of certain proposed changes in a given prototype, and in this instance some actual prototype data may be available.

Validation of model design?

- The model can be designed, constructed, and tested, and the model prediction can be compared with these data. If the agreement is satisfactory, then the model can be changed in the desired manner, and the corresponding effect on the prototype can be predicted with increased confidence.
Distorted Models

- In many model studies, to achieve dynamic similarity requires duplication of several dimensionless groups.
- In some cases, complete dynamic similarity between model and prototype may not be attainable. If one or more of the similarity requirements are not met, for example, if \( \Pi_2 \neq \Pi_{2m} \), then it follows that the prediction equation \( \Pi_1 = \Pi_{1m} \) is not true; that is, \( \Pi_1 \neq \Pi_{1m} \).
- MODELS for which one or more of the similarity requirements are not satisfied are called DISTORTED MODELS.
Distorted Models EXAMPLE-1 1/3

- Determine the drag force on a surface ship, complete dynamic similarity requires that both Reynolds and Froude numbers be duplicated between model and prototype.

\[
\text{Fr}_m = \frac{V_m}{(g \ell_m)^{1/2}} = \text{Fr}_p = \frac{V_p}{(g \ell_p)^{1/2}} \quad \text{Froude numbers}
\]

\[
\text{Re}_m = \frac{V_m \ell_m}{\nu_m} = \text{Re}_p = \frac{V_p \ell_p}{\nu_p} \quad \text{Reynolds numbers}
\]

- To match Froude numbers between model and prototype

\[
\frac{V_m}{V_p} = \left( \frac{\ell_m}{\ell_p} \right)^{1/2}
\]
To match Reynolds numbers between model and prototype:

$$\frac{\nu_m}{\nu_p} = \frac{V_m}{V_p} \frac{\ell_m}{\ell_p} \Rightarrow \frac{\nu_m}{\nu_p} = \left( \frac{\ell_m}{\ell_p} \right)^{1/2} \quad \frac{\ell_m}{\ell_p} = \left( \frac{\ell_m}{\ell_p} \right)^{3/2}$$

If $\ell_m/\ell_p$ equals 1/100 (a typical length scale for ship model tests), then $\nu_m/\nu_p$ must be 1/1000.

>>> The kinematic viscosity ratio required to duplicate Reynolds numbers cannot be attained.
Distorted Models

- It is impossible in practice for this model/prototype scale of 1/100 to satisfy both the Froude number and Reynolds number criteria; **at best we will be able to satisfy only one of them.**

- If water is the only practical liquid for most model test of free-surface flows, a **full-scale test is required to obtain complete dynamic similarity.**
In the study of open channel or free-surface flows. Typically in these problems both the Reynolds number and Froude number are involved.

\[ \text{Fr}_m = \frac{V_m}{(g_m \ell_m)^{1/2}} = \text{Fr}_p = \frac{V_p}{(g_p \ell_p)^{1/2}} \]

\[ \text{Re}_m = \frac{\rho_m V_m \ell_m}{\mu_m} = \text{Re}_p = \frac{\rho_p V_p \ell_p}{\mu_p} \]

To match Froude numbers between model and prototype:

\[ \frac{V_m}{V_p} = \left( \frac{\ell_m}{\ell_p} \right)^{1/2} = \sqrt{\frac{\lambda}{\ell}} \]
Distorted Models

To match Reynolds numbers between model and prototype:

\[
\frac{V_m}{V_p} = \frac{\mu_m \rho_p \ell_p}{\mu_p \rho_m \ell_m}
\]

\[
\frac{V_m}{V_p} = \sqrt{\frac{\ell_m}{\ell_p}} = \frac{\mu_m / \mu_p}{\rho_p / \rho_m} \frac{\ell_p}{\ell_m}
\]

\[
\lambda^{3/2} = \frac{\mu_m / \mu_p}{\rho_p / \rho_m} = \frac{\nu_m}{\nu_p}
\]

If \(\ell_m / \ell_p\) equals 1/100 (a typical length scale for ship model tests), then \(\nu_m / \nu_p\) must be 1/1000.

>>> The kinematic viscosity ratio required to duplicate Reynolds numbers cannot be attained.
Typical Model Studies

- Flow through closed conduits.
- Flow around immersed bodies.
- Flow with a free surface.
Flow Through Closed Conduits

- This type of flow includes pipe flow and flow through valves, fittings, and metering devices.
- The conduits are often circular, they could have other shapes as well and may contain expansions or contractions.
- Since there are no fluid interfaces or free surface, the dominant forces are inertial and viscous forces so that the Reynolds number is an important similarity parameter.
For low Mach numbers (Ma<0.3), compressibility effects are usually negligible for both the flow of liquids or gases.

For flow in closed conduits at low Mach numbers, and dependent pi term, such as pressure drop, can be expressed as:

\[
\text{Dependent pi term} = \phi \left( \frac{l_i}{l'}, \frac{\varepsilon}{l}, \frac{\mu \rho Vl}{\mu} \right)
\]

Where \( l \) is some particular length of the system and \( l_i \) represents a series of length terms, \( \varepsilon / l \) is the relative roughness of the surface, and \( \rho Vl/\mu \) is the Reynolds number.
To meet the requirement of geometric similarity
\[ \frac{l_{im}}{l_m} = \frac{l_i}{l} \quad \frac{\varepsilon_m}{\varepsilon} = \frac{l_{im}}{l_i} = \frac{l_m}{l} = \frac{\varepsilon_m}{\varepsilon} = \lambda \ell \]

To meet the requirement of Reynolds number
\[ \frac{\rho_m V_m l_m}{\mu_m} = \frac{\rho V l}{\mu} \Rightarrow V_m = \frac{\mu_m}{\mu} \frac{\rho}{V} \frac{l}{l_m} \]

If the same fluid is used, then
\[ \frac{V_m}{V} = \frac{l}{l_m} \Rightarrow V_m = V / \lambda \ell \]
The fluid velocity in the model will be larger than that in the prototype for any length scale less than 1. Since length scales are typically much less than unity.

Reynolds number similarity may be difficult to achieve because of the large model velocities required.

\[ \frac{V_m}{V} = \frac{\ell}{\ell_m} \Rightarrow V_m = V/\lambda \ell \]

\[ V_m > V \]

\[ \lambda \ell < 1 \]
With these similarity requirements satisfied, it follows that the dependent pi term will be equal in model and prototype. For example,

Dependent pi term \( \Pi_1 = \frac{\Delta p}{\rho V^2} \)

The prototype pressure drop

\[ \Delta p = \frac{\rho}{\rho_m} \left( \frac{V}{V_m} \right)^2 \Delta p_m \]

In general \( \Delta p \neq \Delta p_m \)
Model test are to be performed to study the flow through a large valve having a 2-ft-diameter inlet and carrying water at a flowrate of 30 cfs. The working fluid in the model is water at the same temperature as that in the prototype. Complete geometric similarity exits between model and prototype, and the model inlet diameter is 30 in. Determine the required flowrate in the model.
Example 7.6 Solution

To ensure dynamic similarity, the model tests should be run so that

\[ \text{Re}_m = \text{Re} \]
\[ \frac{V_m D_m}{\nu_m} = \frac{VD}{\nu} \Rightarrow \frac{V_m}{V} = \frac{D}{D_m} \]

**Same fluid used in the model and prototype**

\[ \frac{Q_m}{Q} = \frac{V_m A_m}{V A} = ... = \frac{D_m}{D} \]

\[ Q_m = \frac{(3/12 \text{ft})}{(2 \text{ft})} (30 \text{ft}^3 / \text{s}) = 3.75 \text{cfs} \]
Flow Around Immersed Bodies

- This type of flow includes flow around aircraft, automobiles, golf balls, and building.
- For these problems, geometric and Reynolds number similarity is required.
- Since there are no fluid interfaces, surface tension is not important. Also, gravity will not affect the flow pattern, so the Froude number need not to be considered.
- For incompressible flow, the Mach number can be omitted.
A general formulation for these problems is

\[
\text{Dependent } \pi \text{ term } = \phi \left( \frac{l_i}{\ell}, \frac{\varepsilon}{\ell}, \frac{\rho V \ell}{\mu} \right)
\]

Where \( \ell \) is some characteristic length of the system and \( l_i \) represents other pertinent lengths, \( \varepsilon / \ell \) is the relative roughness of the surface, and \( \rho V \ell / \mu \) is the Reynolds number.

Model of the National Bank of Commerce, San Antonio, Texas, for measurement of peak, rms, and mean pressure distributions. The model is located in a long-test-section, meteorological wind tunnel.
Frequently, the dependent variable of interest for this type of problem is the drag, $D$, developed on the body.

The dependent pi terms would usually be expressed in the form of a drag coefficient

$$C_D = \frac{D}{\frac{1}{2} \rho V^2 l^2}$$

To meet the requirement of geometric similarity

$$\frac{l_{im}}{l_m} = \frac{l_i}{l} \quad \frac{\varepsilon_m}{l_m} = \frac{\varepsilon}{l} \Rightarrow \frac{l_{im}}{l_i} = \frac{l_m}{l} = \frac{\varepsilon_m}{\varepsilon} = \lambda_l$$
Flow Around Immersed Bodies

- To meet the requirement of Reynolds number similarity

\[ \frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu} \Rightarrow \frac{V_m}{V} = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{\ell}{\ell_m} = \frac{v_m}{v} \frac{\ell}{\ell_m} \]

\[ \frac{D}{\frac{1}{2} \rho V^2 \ell^2} = \frac{D_m}{\frac{1}{2} \rho_m V_m^2 \ell_m^2} \Rightarrow D = \frac{\rho}{\rho_m} \left( \frac{V}{V_m} \right)^2 \left( \frac{\ell}{\ell_m} \right)^2 D_m \]

The same fluid is used, then

\[ \frac{V_m}{V} = \frac{\ell}{\ell_m} \Rightarrow V_m = V / \lambda \ell \]
The fluid velocity in the model will be larger than that in the prototype for any length scale less than 1. Since length scales are typically much less than unity.

Reynolds number similarity may be difficult to achieve because of the large model velocities required.
How to reduce the fluid velocity in the model?

A different fluid is used in the model such that \( \frac{\nu_m}{\nu} < 1 \)

For example, the ratio of the kinematic viscosity of water to that of air is approximately 1/10, so that if the prototype fluid were air, test might be run on the model using water.

This would reduce the required model velocity, but it still may be difficult to achieve the necessary velocity in a suitable test facility, such as a water tunnel.
How to reduce the fluid velocity in the model?

Same fluid with different density.. \( \rho_m > \rho \)

An alternative way to reduce \( V_m \) is to increase the air pressure in the tunnel so that \( \rho_m > \rho \). The pressurized tunnels are obviously complicated and expensive.
Example 7.7 Model Design Conditions and Predicted Prototype Performance

- The drag on an airplane cruising at 240 mph in standard air is to be determined from tests on a 1:10 scale model placed in a pressurized wind tunnel. To minimize compressibility effects, the airspeed in the wind tunnel is also to be 240 mph. Determine the required air pressure in the tunnel (assuming the same air temperature for model and prototype), and the drag on the prototype corresponding to a measured force of 1 lb on the model.
Example 7.7 Solution

The Reynolds numbers in model and prototype are the same. Thus,

$$\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu}$$

$$V_m = V, \frac{\ell_m}{\ell} = 1/10$$

$$\frac{\rho_m}{\rho} = \frac{\mu_m}{\mu} \frac{V}{V_m} \frac{\ell}{\ell_m} = 10 \frac{\mu_m}{\mu}$$

The same fluid with \( \rho_m = \rho \) and \( \mu_m = \mu \) cannot be used if Reynolds number similarity is to be maintained.
Example 7.7 Solution

We assume that an increase in pressure does not significantly change the viscosity so that the required increase in density is given by the relationship

\[ \frac{\rho_m}{\rho} = 10 \]

For ideal gas

\[ \frac{p_m}{p} = \frac{\rho_m}{\rho} = 10 \]

\[ \frac{D}{\frac{1}{2} \rho V^2 \ell^2} = \frac{D_m}{\frac{1}{2} \rho_m V_m^2 \ell_m^2} \]

\[ D = \ldots \]
Flow Around Immersed Bodies at High Reynolds Number $^{1/3}$

- Unfortunately, in many situations the flow characteristics are not strongly influenced by the Reynolds number over the operating range of interest.
- Consider the variation in the drag coefficient with the Reynolds number for a smooth sphere of diameter $d$ placed in a uniform stream with approach velocity, $V$.
- At high Reynolds numbers, the drag is often essentially independent of the Reynolds number.
Flow Around Immersed Bodies at High Reynolds Number $^{2/3}$

The effect of Reynolds number on the drag coefficient, $C_D$, for a smooth sphere with $C_D = D/ \frac{1}{2} A \rho V^2$, where $A$ is the projected area of sphere, $\pi d^2/4$. 
Flow Around Immersed Bodies at High Reynolds Number $^{3/3}$

- For problems involving high velocities in which the Mach number is greater than about 0.3, the influence of compressibility, and therefore the Mach number (or Cauchy number), becomes significant.

- In this case complete similarity requires not only geometric and Reynolds number similarity but also Mach number similarity so that

$$\frac{V_m}{c_m} = \frac{V}{c} \Rightarrow \frac{c}{c_m} = \frac{\nu}{\nu_m} \frac{\ell_m}{\ell}$$
Flow with a Free Surface

- This type of flow includes flow in canals, rivers, spillways, and stilling basics, as well as flow around ships.

- For this class of problems, gravitational, inertial forces, and surface tension are important and, therefore, the Froude number and Weber number become important similarity parameters.

- Since there is a free surface with a liquid-air interface, forces due to surface tension may be significant, and the Weber number becomes another similarity parameter that needs to be considered along with the Reynolds number.
A general formulation for these problems is

$$\text{Dependent pi term} = \phi \left( \frac{\ell_i}{\ell}, \frac{\varepsilon}{\ell}, \frac{\rho V \ell}{\mu}, \frac{V}{\sqrt{g \ell}}, \frac{\rho V^2 \ell}{\sigma} \right)$$

Where $\ell$ is some characteristic length of the system and $\ell_i$ represents other pertinent lengths, $\varepsilon / \ell$ is the relative roughness of the surface, and $\rho V \ell / \mu$ is the Reynolds number.

To meet the requirement of Froude number similarity

$$\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g \ell}} \quad g_m = g \quad \frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda}$$
Flow with a Free Surface

To meet the requirement of Reynolds number and Froude number similarity

\[
\frac{\rho_m V_m \ell_m}{\mu_m} = \frac{\rho V \ell}{\mu} \Rightarrow \frac{V_m}{V} = \frac{\mu_m}{\mu} \frac{\rho}{\rho_m} \frac{\ell}{\ell_m} = \frac{\nu_m}{\nu} \frac{\ell}{\ell_m}
\]

\[
\frac{\nu_m}{\nu} = (\lambda_{\ell})^{3/2}
\]

The working fluid for the prototype is normally either freshwater or seawater and the length scale is small. It is virtually impossible to satisfy \( \frac{\nu_m}{\nu} = (\lambda_{\ell})^{3/2} \), so models involving free-surface flows are usually distorted.
The problem is further complicated if an attempt is made to model surface tension effects, since this requires the equality of Weber numbers, which leads to the condition

\[
\frac{\rho_m V_m^2 \ell_m}{\sigma_m} = \frac{\rho V^2 \ell}{\sigma} \Rightarrow \frac{\sigma_m / \rho_m}{\sigma / \rho} = \frac{V_m^2 \ell_m}{V^2 \ell} = (\lambda_\ell)^2
\]

For the kinematic surface tension \( \sigma / \rho \). It is evident that the same fluid cannot be used in model and prototype if we are to have similitude with respect to surface tension effects for \( \lambda_\ell \neq 1 \).
Fortunately, in many problems involving free-surface flows, both surface tension and viscous effect are small and consequently strict adherence to Weber and Reynolds number similarity is not required.

Certainly, surface tension is not important in large hydraulic structures and rivers.

Our only concern would be if in a model the depths were reduced to the point where surface tension becomes an important factor.

How to overcome this problem?
Different horizontal and vertical length scales, which introduce geometric distortion, are often used to eliminate surface tension effects in the model.

Although this approach eliminates surface tension effects in the model, it introduces geometric distortion that must be accounted for empirically, usually by increasing the model surface roughness.

Verification test in the model must be made. Model roughness can be adjusted to give satisfactory agreement between model and prototype.
For large hydraulic structures, such as dam spillways, the Reynolds numbers are large so that viscous forces are small in comparison to the force due to gravity and inertia. In this case Reynolds number similarity is not maintained and models are designed on the basis of Froude number similarity.
The model Reynolds numbers are large so that they are not required to equal to those of the prototype.

A scale hydraulic model (1:197) of the Guri Dam in Venezuela which is used to simulate the characteristics of the flow over and below the spillway and the erosion below the spillway.
A certain spillway for a dam is 20 m wide and is designed to carry 125 m³/s at flood stage. A 1:15 model is constructed to study the flow characteristics through the spillway. Determine the required model width and flowrate. What operating time for the model corresponds to 1 24-hr period in the prototype? The effects of surface tension and viscosity are to be neglected.
The width, \( w_m \), of the model of spillway is obtained from the length scale

\[
\frac{w_m}{w} = \lambda_{\ell} = \frac{1}{15} \quad \Rightarrow \quad w_m = \frac{20m}{15} = 1.33m
\]

With the neglect of surface tension and viscosity, the dynamic similarity will be achieved if the Froude numbers are equal between model and prototype

\[
\frac{V_m}{\sqrt{g_m \ell_m}} = \frac{V}{\sqrt{g \ell}} \quad \Rightarrow \quad g_m = g \quad \Rightarrow \quad \frac{V_m}{V} = \sqrt{\frac{\ell_m}{\ell}} = \sqrt{\lambda_{\ell}}
\]
Example 7.8 Solution

The flowrate

$$\frac{Q_m}{Q} = \frac{V_m A_m}{V A} = \sqrt{\frac{\ell_m}{\ell}} \left( \frac{\ell_m}{\ell} \right)^2$$

$$Q_m = (\lambda_\ell)^{5/2} Q = \ldots = 0.143 m^3 / s$$

$$\frac{V}{V_m} = \frac{\ell}{\ell_m} \frac{t_m}{t} \quad \frac{t_m}{t} = \frac{V}{V_m} \frac{\ell_m}{\ell} = \sqrt{\lambda_\ell}$$
Similitude Based on Governing Differential Equations

- For a steady incompressible two-dimensional flow of a Newtonian fluid with constant viscosity.

- The mass conservation equation is
  \[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]
  Has dimensions of \(1/\text{time}\).

- The Navier-Stokes equations are
  \[
  \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
  \]
  \[
  \rho \left( u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} - \rho g + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
  \]
  Has dimensions of force/volume.
Similitude Based on Governing Differential Equations

How to non-dimensionalize these equations?

\[
x^* = \frac{x}{\ell} \quad y^* = \frac{y}{\ell} \quad u^* = \frac{u}{V} \quad v^* = \frac{v}{V} \quad p^* = \frac{p}{p_0} \quad t^* = \frac{t}{\tau}
\]

\[
\frac{\partial u}{\partial x} = \frac{V}{\ell} \frac{\partial u^*}{\partial x^*} \quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} \right) = \frac{V}{\ell^2} \frac{\partial^2 u^*}{\partial x^*^2}
\]

The mass conservation equation

\[
\frac{\partial u^*}{\partial x^*} + \frac{\partial v^*}{\partial y^*} = 0
\]
Similitude Based on Governing Differential Equations

The Navier-Stokes equations

\[
\left[ \frac{\ell}{\tau V} \right] \frac{\partial u^*}{\partial t^*} + u^* \frac{\partial u^*}{\partial x^*} + v^* \frac{\partial u^*}{\partial y^*} = -\left[ \frac{p_0}{\rho V^2} \right] \frac{\partial p^*}{\partial x^*} + \left[ \frac{\mu}{\rho V\ell} \right] \left( \frac{\partial^2 u^*}{\partial x^*^2} + \frac{\partial^2 u^*}{\partial y^*^2} \right)
\]

\[
\left[ \frac{\ell}{\tau V} \right] \frac{\partial v^*}{\partial t^*} + u^* \frac{\partial v^*}{\partial x^*} + v^* \frac{\partial v^*}{\partial y^*} = -\left[ \frac{p_0}{\rho V^2} \right] \frac{\partial p^*}{\partial y^*} - \left[ \frac{g\ell}{V^2} \right] + \left[ \frac{\mu}{\rho V\ell} \right] \left( \frac{\partial^2 v^*}{\partial x^*^2} + \frac{\partial^2 v^*}{\partial y^*^2} \right)
\]

Reynolds number
Strouhal number
Euler number
Reciprocal of the square of the Froude number
Similitude Based on Governing Differential Equations

From these equations it follows that if two systems are governed by these equations, then the solutions (in terms of \( u^*, v^*, p^*, x^*, y^*, \) and \( t^* \)) will be the same if the four parameters are equal for the two systems.

The two systems will be dynamically similar. Of course, boundary and initial conditions expressed in dimensionless form must also be equal for the two systems, and this will require complete geometric similarity.
These are the same similarity requirements that would be determined by a dimensional analysis if the same variables were considered. These variables appear naturally in the equations.

All the common dimensionless groups that we previously developed by using dimensional analysis appear in the governing equations that describe fluid motion when these equations are expressed in term of dimensionless variables.

The use of governing equations to obtain similarity laws provides an alternative to conventional dimensional analysis.