Mechanical Vibrations

Contents

Introduction

Free Vibrations of Particles. Simple Harmonic Motion

Simple Pendulum (Approximate Solution)

Simple Pendulum (Exact Solution)

Sample Problem 19.1

Free Vibrations of Rigid Bodies

Sample Problem 19.2

Sample Problem 19.3

Principle of Conservation of Energy

Sample Problem 19.4

Forced Vibrations

Sample Problem 19.5

Damped Free Vibrations

Damped Forced Vibrations

Electrical Analogues

- *Mechanical vibration* is the motion of a particle or body which oscillates about a position of equilibrium. Most vibrations in machines and structures are undesirable due to increased stresses and energy losses.

- Time interval required for a system to complete a full cycle of the motion is the *period* of the vibration.

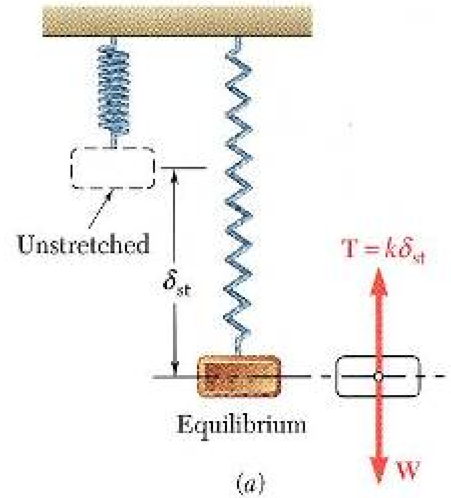
- Number of cycles per unit time defines the *frequency* of the vibrations.

- Maximum displacement of the system from the equilibrium position is the *amplitude* of the vibration.

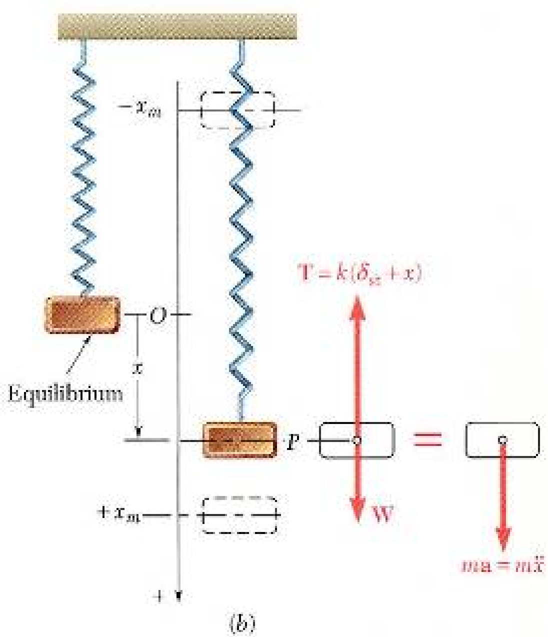
- When the motion is maintained by the restoring forces only, the vibration is described as *free vibration*. When a periodic force is applied to the system, the motion is described as *forced vibration*.

- When the frictional dissipation of energy is neglected, the motion is said to be *undamped*. Actually, all vibrations are *damped* to some degree.

Free Vibrations of Particles. Simple Harmonic Motion



(a)



(b)

- If a particle is displaced through a distance x_m from its equilibrium position and released with no velocity, the particle will undergo *simple harmonic motion*,

$$ma = F = W - k(\delta_{st} + x) = -kx$$

$$m\ddot{x} + kx = 0$$

- General solution is the sum of two *particular solutions*,

$$x = C_1 \sin\left(\sqrt{\frac{k}{m}}t\right) + C_2 \cos\left(\sqrt{\frac{k}{m}}t\right)$$

$$= C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t)$$

- x is a *periodic function* and ω_n is the *natural circular frequency* of the motion.

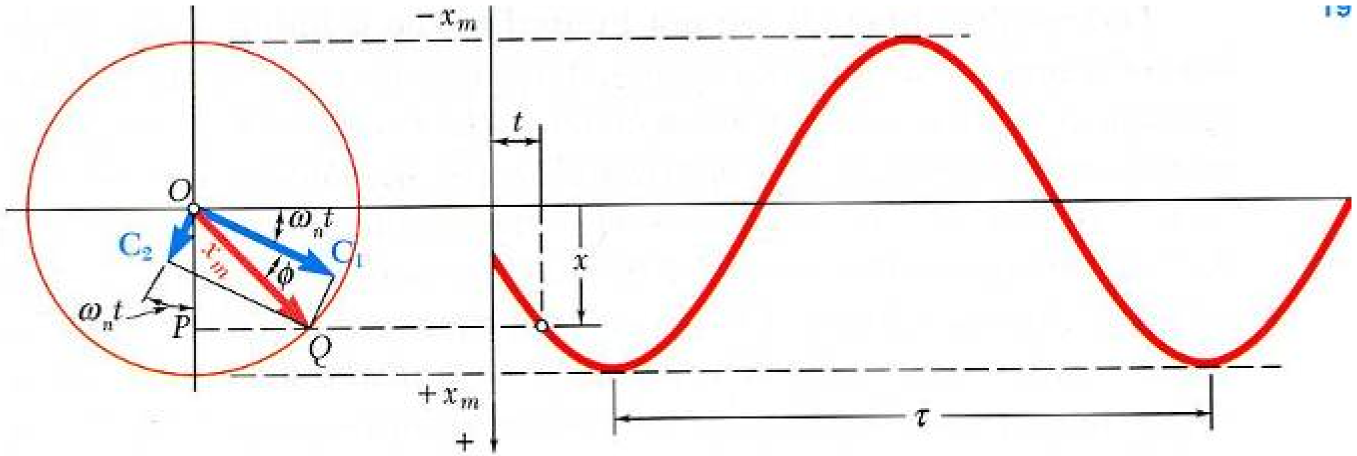
- C_1 and C_2 are determined by the initial conditions:

$$x = C_1 \sin(\omega_n t) + C_2 \cos(\omega_n t) \qquad C_2 = x_0$$

$$v = \dot{x} = C_1\omega_n \cos(\omega_n t) - C_2\omega_n \sin(\omega_n t) \qquad C_1 = v_0/\omega_n$$

$$C_1 = \frac{v_0}{\omega_n}$$

$$C_2 = x_0$$



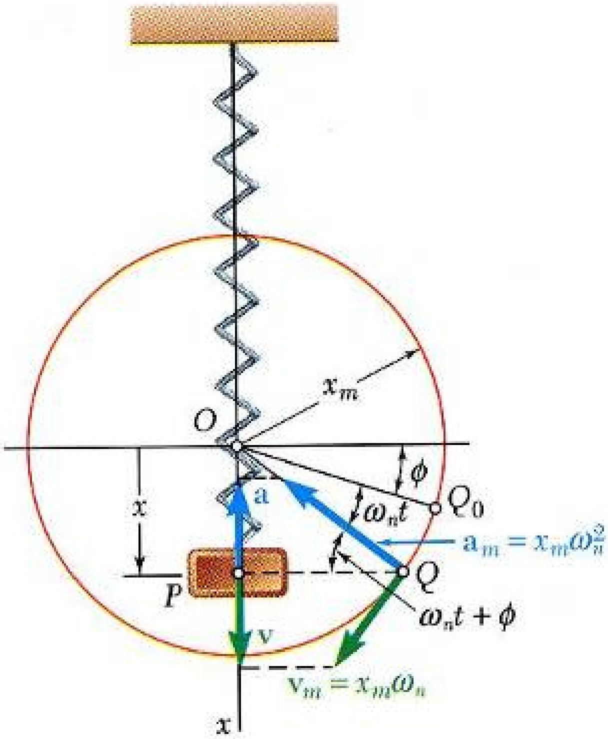
- Displacement is equivalent to the x component of the sum of two vectors $\vec{C}_1 + \vec{C}_2$ which rotate with constant angular velocity ω_n.

$$x = x_m \sin(\omega_n t + \phi) \qquad x_m = \sqrt{(v_0/\omega_n)^2 + x_0^2} = \textit{amplitude}$$

$$\phi = \tan^{-1}(v_0/x_0\omega_n) = \textit{phase angle}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \textit{period}$$

$$f_n = \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} = \textit{natural frequency}$$



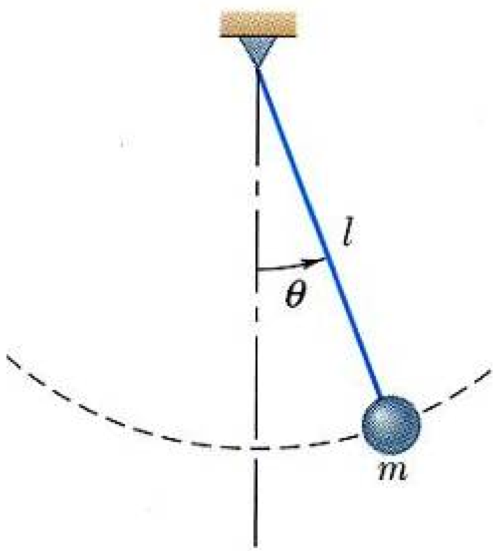
- Velocity-time and acceleration-time curves can be represented by sine curves of the same period as the displacement-time curve but different phase angles.

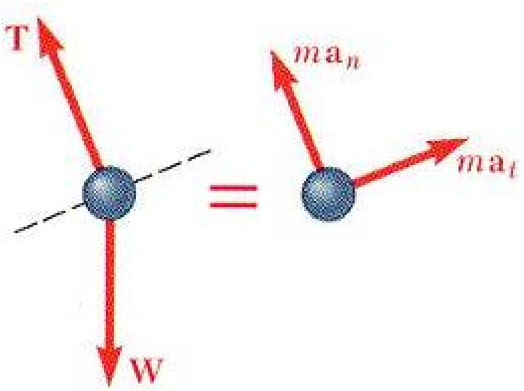
$$x = x_m \sin(\omega_n t + \phi)$$

$$\begin{aligned} v &= \dot{x} \\ &= x_m\omega_n \cos(\omega_n t + \phi) \\ &= x_m\omega_n \sin(\omega_n t + \phi + \pi/2) \end{aligned}$$

$$\begin{aligned} a &= \ddot{x} \\ &= -x_m\omega_n^2 \sin(\omega_n t + \phi) \\ &= x_m\omega_n^2 \sin(\omega_n t + \phi + \pi) \end{aligned}$$

Simple Pendulum (Approximate Solution)





- Results obtained for the spring-mass system can be applied whenever the resultant force on a particle is proportional to the displacement and directed towards the equilibrium position.
- Consider tangential components of acceleration and force for a simple pendulum,

$$\sum F_t = ma_t : \qquad -W\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

for small angles,

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

$$\theta = \theta_m \sin(\omega_n t + \phi)$$

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{l}{g}}$$

Simple Pendulum (Exact Solution)

An exact solution for $\ddot{\theta} + \frac{g}{l}\sin\theta = 0$

leads to $\tau_n = 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-\sin^2(\theta_m/2)\sin^2\phi}}$

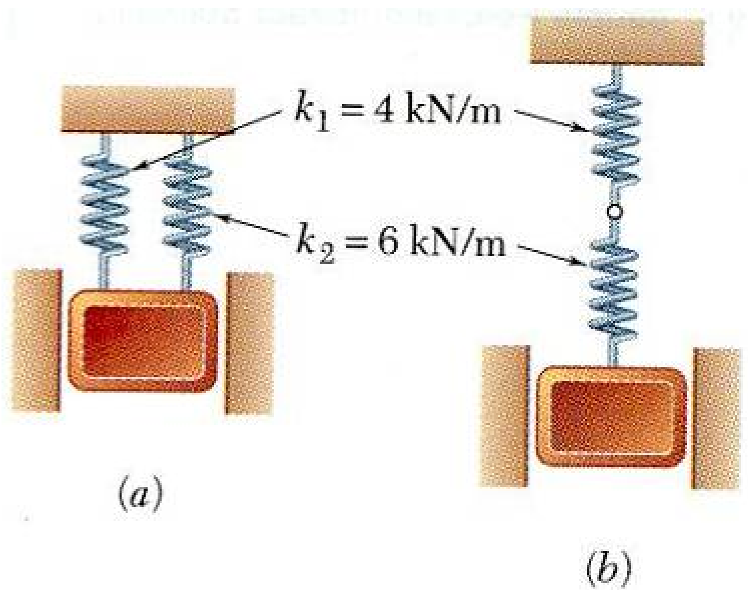
which requires numerical solution.

$$\tau_n = \frac{2K}{\pi}\left(2\pi\sqrt{\frac{l}{g}}\right)$$

Table 19.1. Correction Factor for the Period of a Simple Pendulum

θ_m	0°	10°	20°	30°	60°	90°	120°	150°	180°
K	1.571	1.574	1.583	1.598	1.686	1.854	2.157	2.768	∞
$2K/\pi$	1.000	1.002	1.008	1.017	1.073	1.180	1.373	1.762	∞

Sample Problem



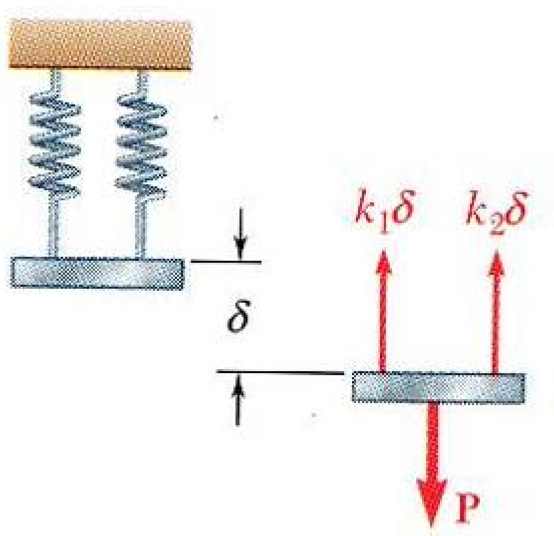
A 50-kg block moves between vertical guides as shown. The block is pulled 40mm down from its equilibrium position and released.

For each spring arrangement, determine *a*) the period of the vibration, *b*) the maximum velocity of the block, and *c*) the maximum acceleration of the block.

SOLUTION:

- For each spring arrangement, determine the spring constant for a single equivalent spring.
- Apply the approximate relations for the harmonic motion of a spring-mass system.

$k_1 = 4\,\text{kN/m} \quad k_2 = 6\,\text{kN/m}$



$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10\,\text{kN/m} = 10^4\ \text{N/m}$$

SOLUTION:

- Springs in parallel:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4\,\text{N/m}}{20\,\text{kg}}} = 14.14\,\text{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\boxed{\tau_n = 0.444\,\text{s}}$$

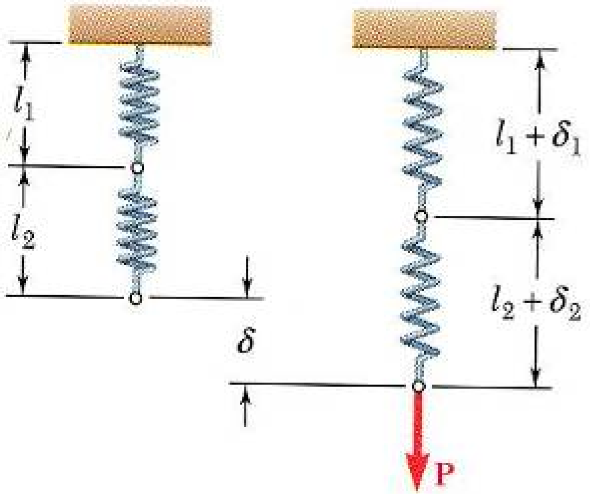
$$v_m = x_m\omega_n$$
$$= (0.040\,\text{m})(14.14\,\text{rad/s})$$

$$\boxed{v_m = 0.566\,\text{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\,\text{m})(14.14\,\text{rad/s})^2$$

$$\boxed{a_m = 8.00\,\text{m/s}^2}$$

$k_1 = 4\,\mathrm{kN/m} \quad k_2 = 6\,\mathrm{kN/m}$



$$P = k_1\delta + k_2\delta$$

$$k = \frac{P}{\delta} = k_1 + k_2$$

$$= 10\,\mathrm{kN/m} = 10^4\,\mathrm{N/m}$$

- Springs in series:
  - determine the spring constant for equivalent spring
  - apply the approximate relations for the harmonic motion of a spring-mass system

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2400\,\mathrm{N/m}}{20\,\mathrm{kg}}} = 6.93\,\mathrm{rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n}$$

$$\boxed{\tau_n = 0.907\,\mathrm{s}}$$

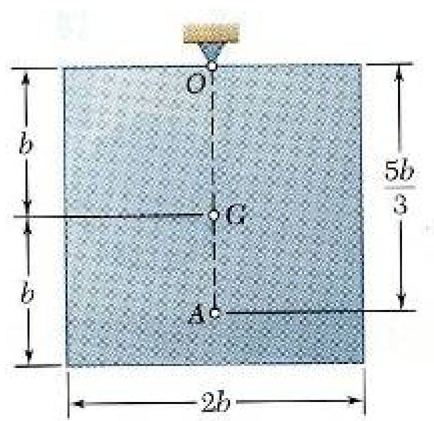
$$v_m = x_m \omega_n$$
$$= (0.040\,\mathrm{m})(6.93\,\mathrm{rad/s})$$

$$\boxed{v_m = 0.277\,\mathrm{m/s}}$$

$$a_m = x_m a_n^2$$
$$= (0.040\,\mathrm{m})(6.93\,\mathrm{rad/s})^2$$

$$\boxed{a_m = 1.920\,\mathrm{m/s^2}}$$

Free Vibrations of Rigid Bodies



- If an equation of motion takes the form

$$\ddot{x} + \omega_n^2 x = 0 \quad \text{or} \quad \ddot{\theta} + \omega_n^2 \theta = 0$$

  the corresponding motion may be considered as simple harmonic motion.

- Analysis objective is to determine ω_n.

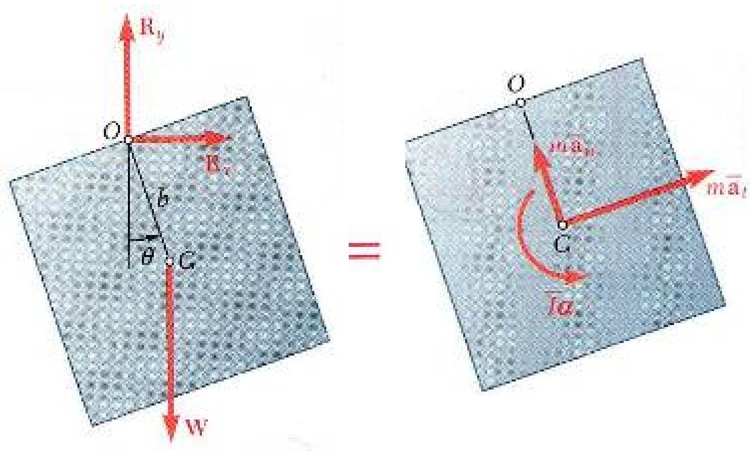
- Consider the oscillations of a square plate

$$+\circlearrowleft \; -W(b \sin\theta) = \left(mb\ddot{\theta}\right) + \bar{I}\ddot{\theta}$$

  but $\bar{I} = \frac{1}{12} m\left[(2b)^2 + (2b)^2\right] = \frac{2}{3} mb^2, \quad W = mg$

$$\ddot{\theta} + \frac{3}{5}\frac{g}{b}\sin\theta \cong \ddot{\theta} + \frac{3}{5}\frac{g}{b}\theta = 0$$

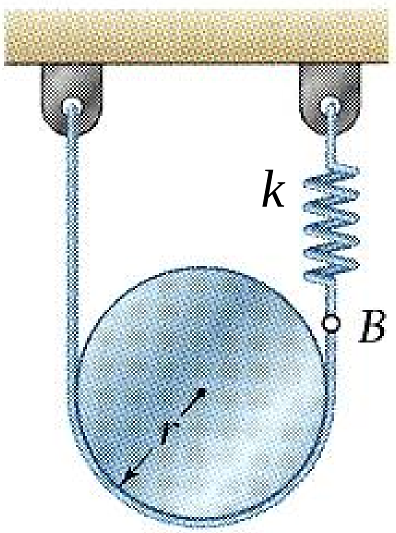
  then $\omega_n = \sqrt{\frac{3g}{5b}}, \quad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{5b}{3g}}$



- For an equivalent simple pendulum,

$$l = 5b/3$$

Sample Problem

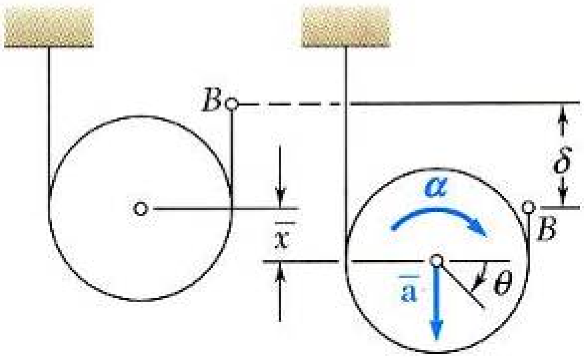


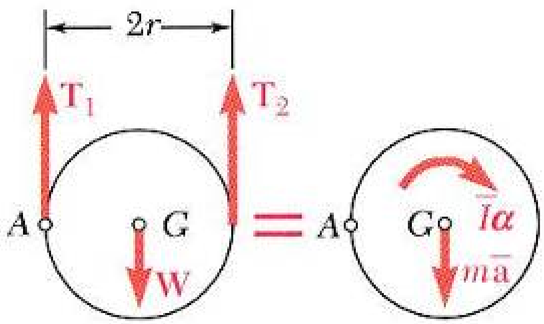
A cylinder of weight W is suspended as shown.

Determine the period and natural frequency of vibrations of the cylinder.

SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.
- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.





SOLUTION:

- From the kinematics of the system, relate the linear displacement and acceleration to the rotation of the cylinder.

$$\bar{x} = r\theta \qquad \delta = 2\bar{x} = 2r\theta$$

$$\vec{\alpha} = \ddot{\theta} \ \text{+}\!\circlearrowright \qquad \bar{a} = r\alpha = r\ddot{\theta} \qquad \vec{\bar{a}} = r\ddot{\theta} \ \text{+}\!\downarrow$$

- Based on a free-body-diagram equation for the equivalence of the external and effective forces, write the equation of motion.

$$+\circlearrowright \sum M_A = \sum (M_A)_{eff} : \qquad Wr - T_2(2r) = m\bar{a}r + \bar{I}\alpha$$

but $\quad T_2 = T_0 + k\delta = \frac{1}{2}W + k(2r\theta)$

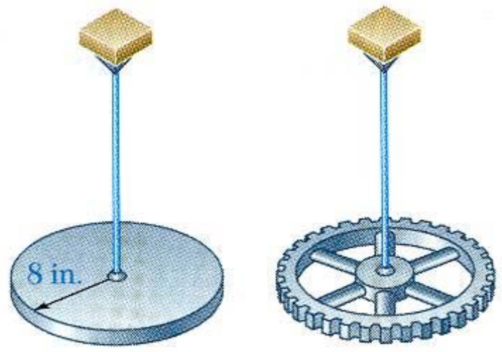
- Substitute the kinematic relations to arrive at an equation involving only the angular displacement and acceleration.

$$Wr - \left(\tfrac{1}{2}W + 2kr\theta\right)(2r) = m\left(r\ddot{\theta}\right)r + \tfrac{1}{2}mr^2\theta$$

$$\ddot{\theta} + \frac{8}{3}\frac{k}{m}\theta = 0$$

$$\omega_n = \sqrt{\frac{8k}{3m}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3m}{8k}} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{8k}{3m}}$$

Sample Problem



$W = 20 \text{ lb}$

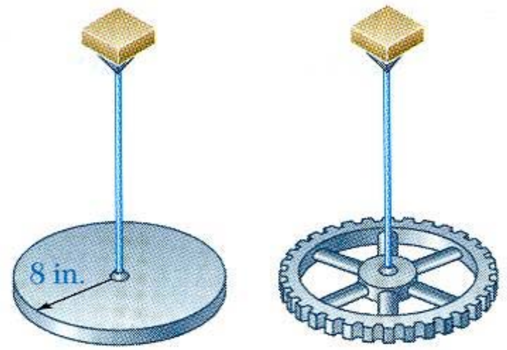
$\tau_n = 1.13 \text{s}$ $\tau_n = 1.93 \text{ s}$

The disk and gear undergo torsional vibration with the periods shown. Assume that the moment exerted by the wire is proportional to the twist angle.

Determine *a*) the wire torsional spring constant, *b*) the centroidal moment of inertia of the gear, and *c*) the maximum angular velocity of the gear if rotated through 90° and released.

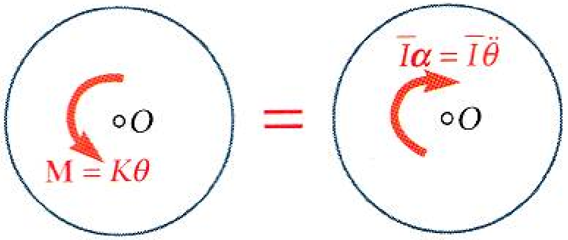
SOLUTION:

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.
- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.
- With natural frequency and spring constant known, calculate the moment of inertia for the gear.
- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.



$W = 20\text{ lb}$

$\tau_n = 1.13\text{s}$ $\quad\quad$ $\tau_n = 1.93\text{ s}$



SOLUTION:

- Using the free-body-diagram equation for the equivalence of the external and effective moments, write the equation of motion for the disk/gear and wire.

$$+\circlearrowleft \sum M_O = \sum (M_O)_{eff} : \qquad + K\theta = -\bar{I}\ddot{\theta}$$

$$\ddot{\theta} + \frac{K}{\bar{I}}\theta = 0$$

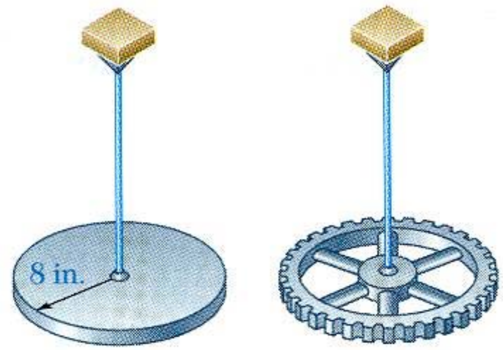
$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

- With the natural frequency and moment of inertia for the disk known, calculate the torsional spring constant.

$$\bar{I} = \tfrac{1}{2}mr^2 = \frac{1}{2}\left(\frac{20}{32.2}\right)\left(\frac{8}{12}\right)^2 = 0.138\,\text{lb}\cdot\text{ft}\cdot\text{s}^2$$

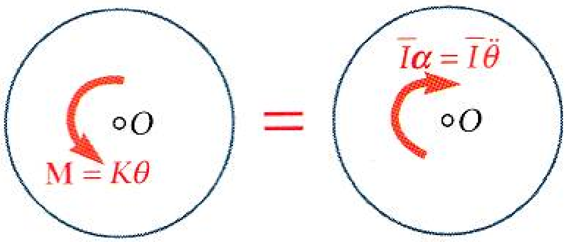
$$1.13 = 2\pi\sqrt{\frac{0.138}{K}}$$

$$\boxed{K = 4.27\ \text{lb}\cdot\text{ft}/\text{rad}}$$



$W = 20 \text{ lb}$

$\tau_n = 1.13 \text{s} \qquad \tau_n = 1.93 \text{ s}$



$$\omega_n = \sqrt{\frac{K}{\bar{I}}} \qquad \tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{\bar{I}}{K}}$$

$$K = 4.27 \text{ lb} \cdot \text{ft/rad}$$

- With natural frequency and spring constant known, calculate the moment of inertia for the gear.

$$1.93 = 2\pi\sqrt{\frac{\bar{I}}{4.27}}$$

$$\boxed{\bar{I} = 0.403 \, \text{lb} \cdot \text{ft} \cdot \text{s}^2}$$

- Apply the relations for simple harmonic motion to calculate the maximum gear velocity.

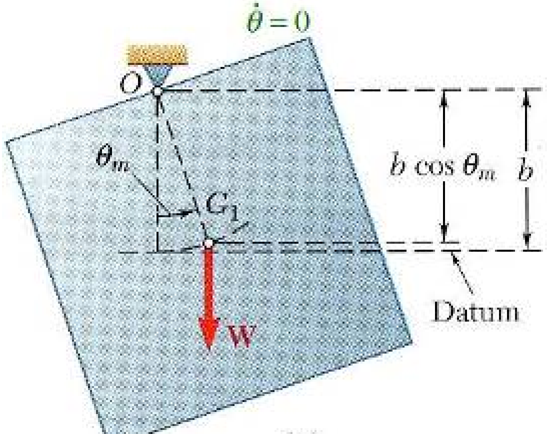
$$\theta = \theta_m \sin \omega_n t \qquad \omega = \theta_m \omega_n \sin \omega_n t \qquad \omega_m = \theta_m \omega_n$$

$$\theta_m = 90° = 1.571 \, \text{rad}$$

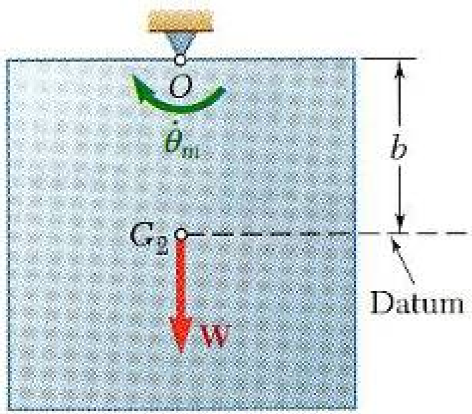
$$\omega_m = \theta_m \left(\frac{2\pi}{\tau_n}\right) = (1.571 \, \text{rad})\left(\frac{2\pi}{1.93 \, \text{s}}\right)$$

$$\boxed{\omega_m = 5.11 \, \text{rad/s}}$$

Principle of Conservation of Energy



(a)



(b)

- Resultant force on a mass in simple harmonic motion is conservative - total energy is conserved.

$$T + V = \text{constant} \qquad \tfrac{1}{2} m\dot{x}^2 + \tfrac{1}{2} kx^2 = \text{constant}$$

$$\dot{x}^2 + \omega_n^2 x^2 =$$

- Consider simple harmonic motion of the square plate,

$$T_1 = 0 \qquad V_1 = Wb(1-\cos\theta) = Wb\left[2\sin^2(\theta_m/2)\right]$$

$$\cong \tfrac{1}{2} Wb\theta_m^2$$

$$T_2 = \tfrac{1}{2} m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2 \qquad V_2 = 0$$

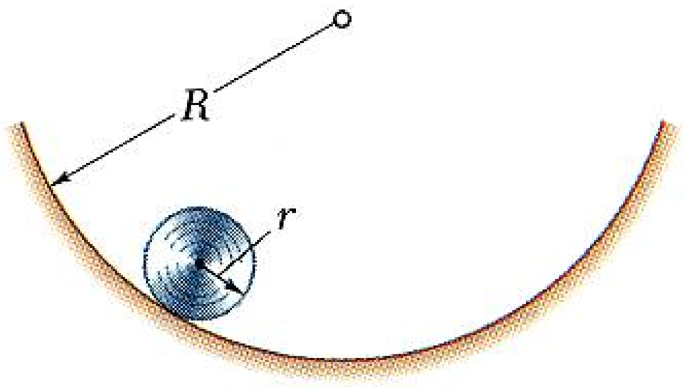
$$= \tfrac{1}{2} m\left(b\dot{\theta}_m\right)^2 + \tfrac{1}{2}\left(\tfrac{2}{3} mb^2\right)\omega_m^2$$

$$= \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\dot{\theta}_m^2$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + \tfrac{1}{2} Wb\theta_m^2 = \tfrac{1}{2}\left(\tfrac{5}{3} mb^2\right)\theta_m^2\omega_n^2 + 0 \qquad \omega_n = \sqrt{3g/5b}$$

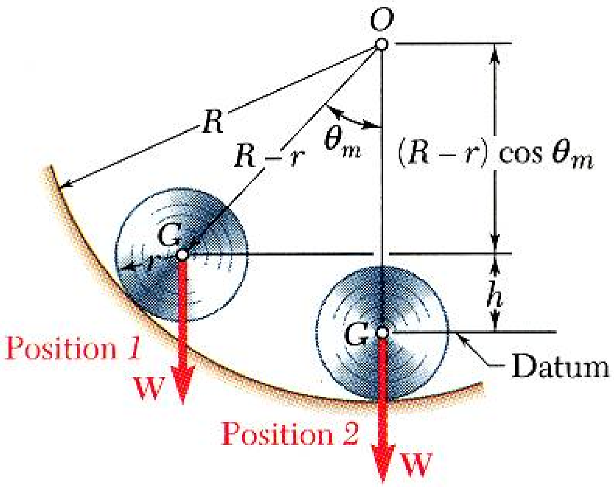
Sample Problem



Determine the period of small oscillations of a cylinder which rolls without slipping inside a curved surface.

SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.
- Solve the energy equation for the natural frequency of the oscillations.



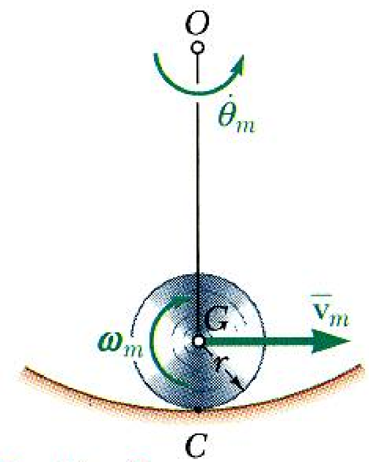
SOLUTION:

- Apply the principle of conservation of energy between the positions of maximum and minimum potential energy.

$$T_1 + V_1 = T_2 + V_2$$

$$T_1 = 0$$

$$V_1 = Wh = W(R-r)(1-\cos\theta) \cong W(R-r)\left(\theta_m^2/2\right)$$

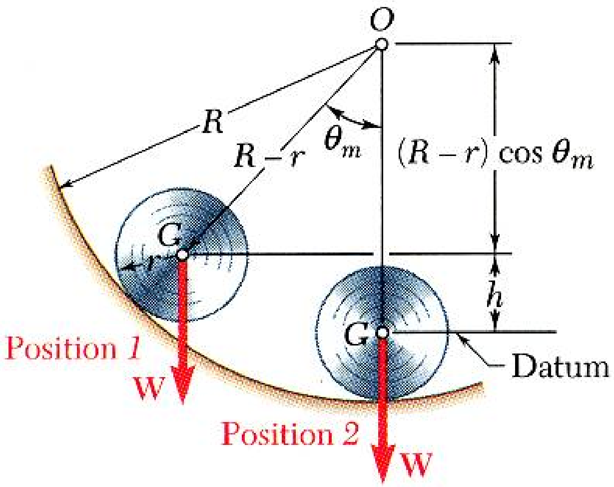


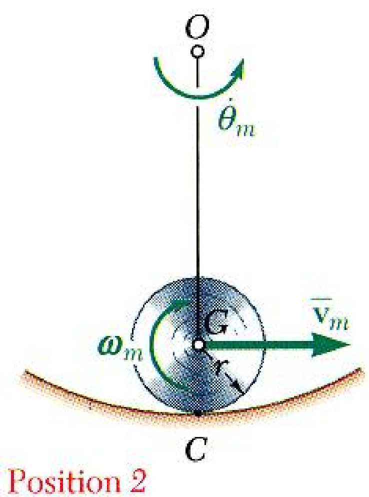
$$T_2 = \tfrac{1}{2}m\bar{v}_m^2 + \tfrac{1}{2}\bar{I}\omega_m^2$$

$$V_2 = 0$$

$$= \tfrac{1}{2}m(R-r)\dot{\theta}_m^2 + \tfrac{1}{2}\left(\tfrac{1}{2}mr^2\right)\left(\frac{R-r}{r}\right)^2\dot{\theta}_m^2$$

$$= \tfrac{3}{4}m(R-r)^2\dot{\theta}_m^2$$





- Solve the energy equation for the natural frequency of the oscillations.

$$T_1 = 0 \qquad V_1 \cong W(R-r)\left(\theta_m^2/2\right)$$

$$T_2 = \tfrac{3}{4} m (R-r)^2 \dot{\theta}_m^2 \qquad V_2 = 0$$

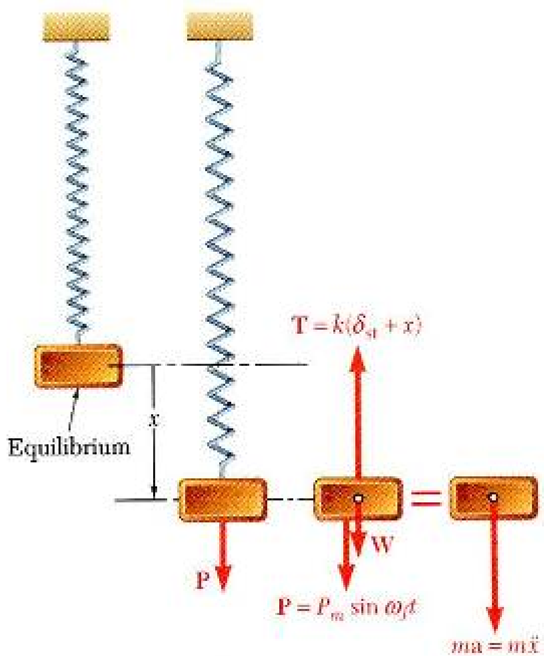
$$T_1 + V_1 = T_2 + V_2$$

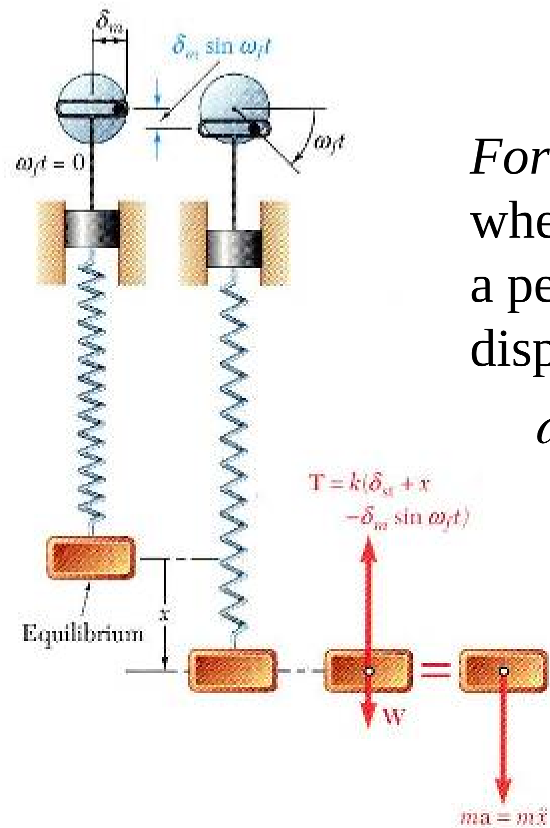
$$0 + W(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4} m (R-r)^2 \dot{\theta}_m^2 + 0$$

$$(mg)(R-r)\frac{\theta_m^2}{2} = \tfrac{3}{4} m (R-r)^2 (\theta_m \omega_n)_m^2$$

$$\omega_n^2 = \frac{2}{3}\frac{g}{R-r} \qquad \boxed{\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{3}{2}\frac{R-r}{g}}}$$

Forced Vibrations





Forced vibrations - Occur when a system is subjected to a periodic force or a periodic displacement of a support.

$$\omega_f = \textit{forced frequency}$$

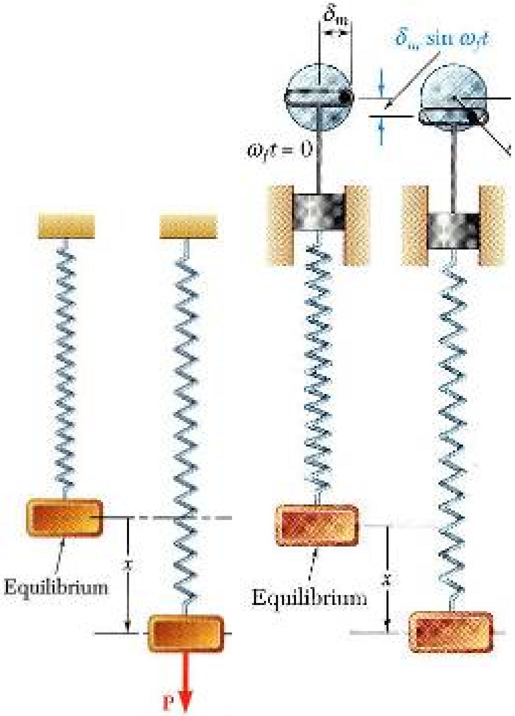
$$+\downarrow \sum F = ma:$$

$$P_m \sin \omega_f t + W - k(\delta_{st} + x) = m\ddot{x}$$

$$m\ddot{x} + kx = P_m \sin \omega_f t$$

$$W - k(\delta_{st} + x - \delta_m \sin \omega_f t) = m\ddot{x}$$

$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$



$$m\ddot{x} + kx = P_m \sin \omega_f t$$

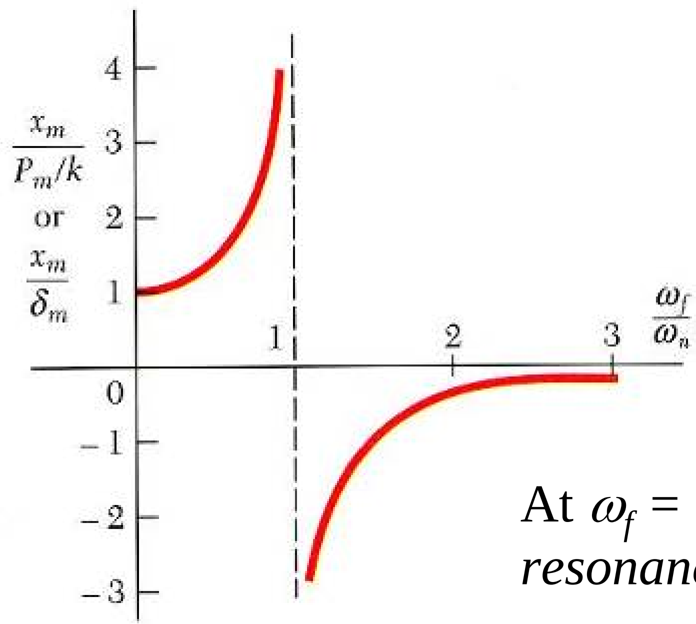
$$m\ddot{x} + kx = k\delta_m \sin \omega_f t$$

$$x = x_{complementary} + x_{particular}$$
$$= \left[C_1 \sin \omega_n t + C_2 \cos \omega_n t\right] + x_m \sin \omega_f t$$

Substituting particular solution into governing equation,

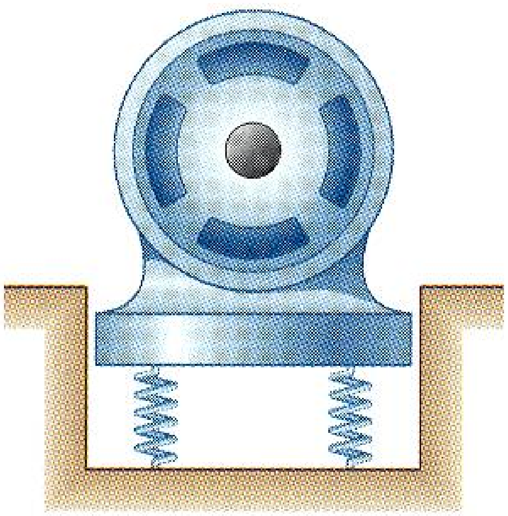
$$-m\omega_f^2 x_m \sin \omega_f t + kx_m \sin \omega_f t = P_m \sin \omega_f t$$

$$x_m = \frac{P_m}{k - m\omega_f^2} = \frac{P_m / k}{1 - \left(\omega_f / \omega_n\right)^2} = \frac{\delta_m}{1 - \left(\omega_f / \omega_n\right)^2}$$



At $\omega_f = \omega_n$, forcing input is in *resonance* with the system.

Sample Problem

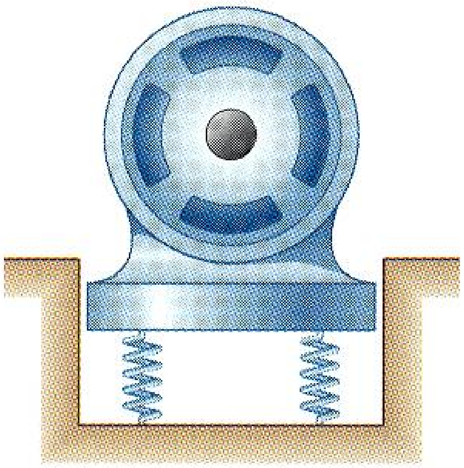


A motor weighing 350 lb is supported by four springs, each having a constant 750 lb/in. The unbalance of the motor is equivalent to a weight of 1 oz located 6 in. from the axis of rotation.

Determine *a*) speed in rpm at which resonance will occur, and *b*) amplitude of the vibration at 1200 rpm.

SOLUTION:

- The resonant frequency is equal to the natural frequency of the system.
- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.



$W = 350$ lb

$k = 4(350$ lb/in)

SOLUTION:

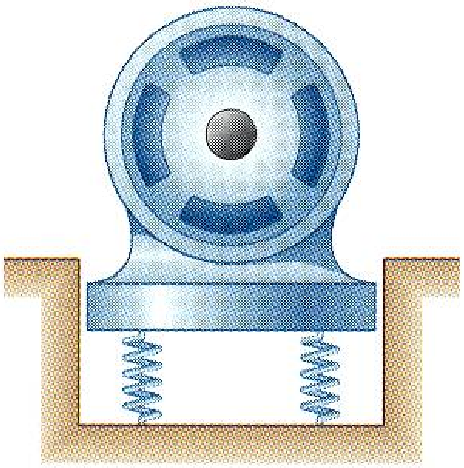
- The resonant frequency is equal to the natural frequency of the system.

$$m = \frac{350}{32.2} = 10.87\,\mathrm{lb \cdot s^2/ft}$$

$$k = 4(750) = 3000\,\mathrm{lb/in}$$
$$= 36{,}000\,\mathrm{lb/ft}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{36{,}000}{10.87}}$$
$$= 57.5\,\mathrm{rad/s} = 549\,\mathrm{rpm}$$

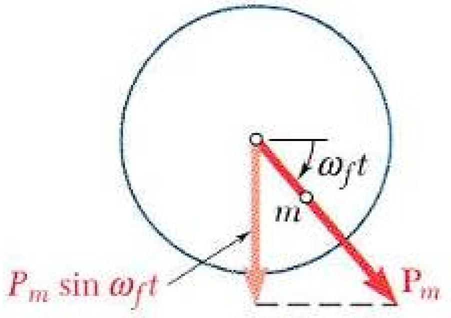
Resonance speed = 549 rpm



$W = 350$ lb

$k = 4(350$ lb/in)

$\omega_n = 57.5$ rad/s



- Evaluate the magnitude of the periodic force due to the motor unbalance. Determine the vibration amplitude from the frequency ratio at 1200 rpm.

$$\omega_f = \omega = 1200 \text{ rpm} = 125.7 \text{ rad/s}$$

$$m = (1\,\text{oz})\left(\frac{1\,\text{lb}}{16\,\text{oz}}\right)\left(\frac{1}{32.2\,\text{ft}/\text{s}^2}\right) = 0.001941\,\text{lb}\cdot\text{s}^2/\text{ft}$$

$$P_m = ma_n = mr\omega^2$$

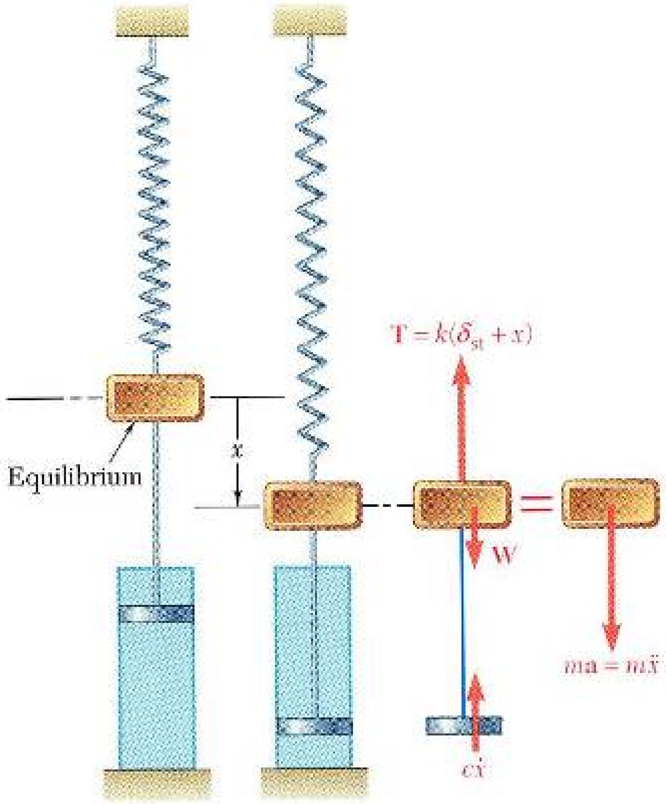
$$= (0.001941)\left(\tfrac{6}{12}\right)(125.7)^2 = 15.33\,\text{lb}$$

$$x_m = \frac{P_m/k}{1-(\omega_f/\omega_n)^2} = \frac{15.33/3000}{1-(125.7/57.5)^2}$$

$$= -0.001352\,\text{in}$$

$x_m = 0.001352$ in. (out of phase)

Damped Free Vibrations



- All vibrations are damped to some degree by forces due to *dry friction*, *fluid friction*, or *internal friction*.

- With *viscous damping* due to fluid friction,

$$+\downarrow \sum F = ma: \qquad W - k(\delta_{st} + x) - c\dot{x} = m\ddot{x}$$

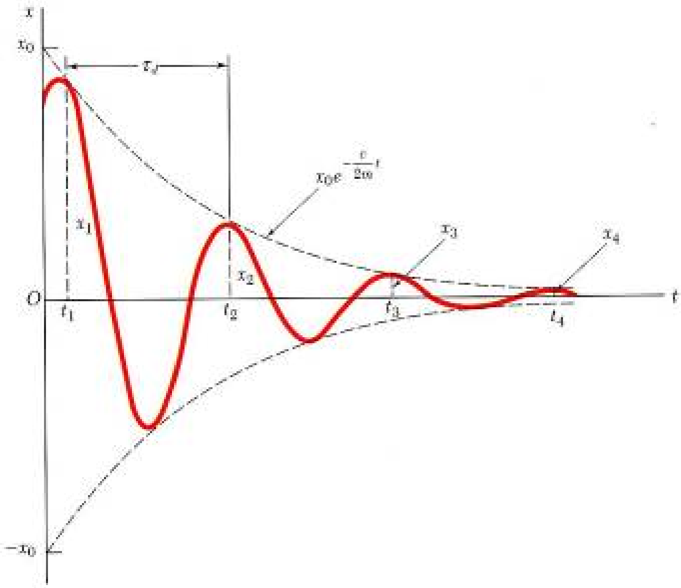
$$m\ddot{x} + c\dot{x} + kx = 0$$

- Substituting $x = e^{\lambda t}$ and dividing through by $e^{\lambda t}$ yields the *characteristic equation*,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- Define the critical damping coefficient such that

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0 \qquad c_c = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$



- Characteristic equation,

$$m\lambda^2 + c\lambda + k = 0 \qquad \lambda = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$c_c = 2m\omega_n = \text{critical damping coefficient}$$

- *Heavy damping*: $c > c_c$

$$x = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

  - negative roots
  - nonvibratory motion

- *Critical damping*: $c = c_c$

$$x = (C_1 + C_2 t) e^{-\omega_n t}$$

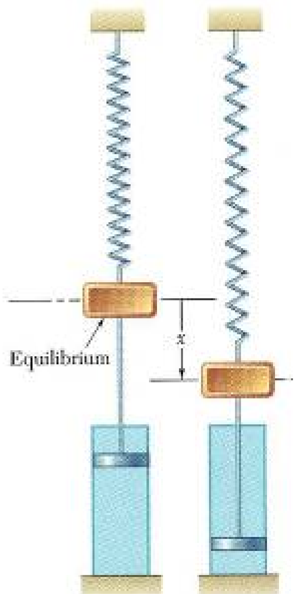
  - double roots
  - nonvibratory motion

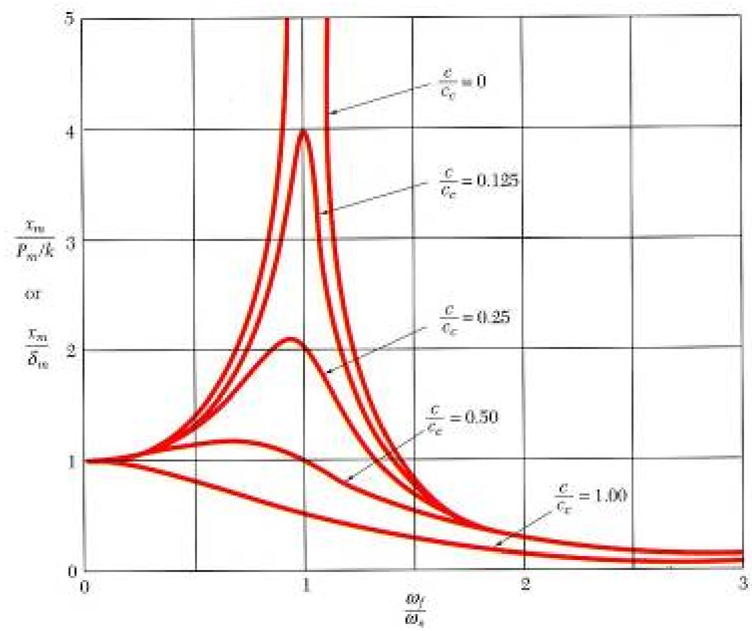
- *Light damping*: $c < c_c$

$$x = e^{-(c/2m)t}\left(C_1 \sin\omega_d t + C_2 \cos\omega_d t\right)$$

$$\omega_d = \omega_n \sqrt{1 - \left(\frac{c}{c_c}\right)^2} = \text{damped frequency}$$

Damped Forced Vibrations



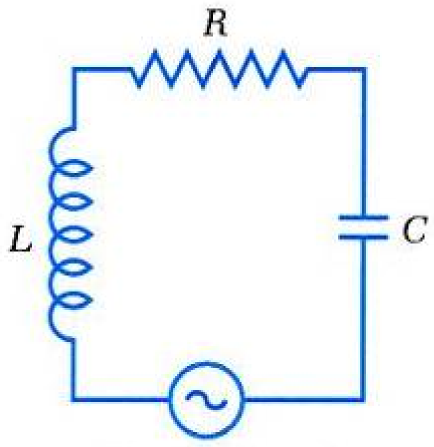


$$m\ddot{x} + c\dot{x} + kx = P_m \sin \omega_f t \qquad x = x_{complementary} + x_{particular}$$

$$\frac{x_m}{P_m/k} = \frac{x_m}{\delta} = \frac{1}{\sqrt{\left[1-\left(\omega_f/\omega_n\right)^2\right]^2 + \left[2\left(c/c_c\right)\left(\omega_f/\omega_n\right)\right]^2}} = \text{magnification factor}$$

$$\tan\phi = \frac{2\left(c/c_c\right)\left(\omega_f/\omega_n\right)}{1-\left(\omega_f/\omega_n\right)^2} = \text{phase difference between forcing and steady state response}$$

Electrical Analogues



$E = E_m \sin \omega_f t$

- Consider an electrical circuit consisting of an inductor, resistor and capacitor with a source of alternating voltage

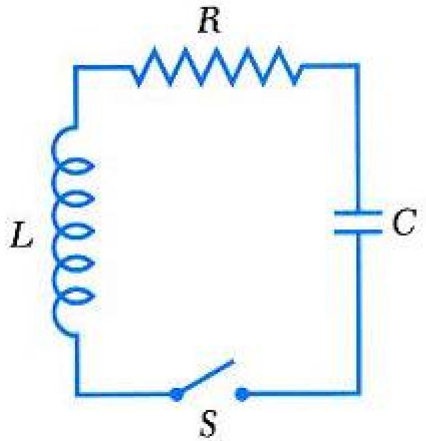
$$E_m \sin \omega_f t - L\frac{di}{dt} - Ri - \frac{q}{C} = 0$$

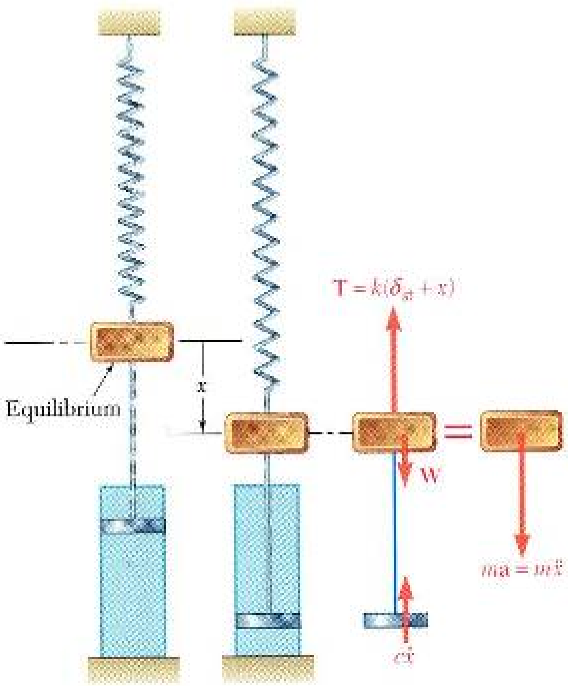
$$L\ddot{q} + R\dot{q} + \frac{1}{C}q = E_m \sin \omega_f t$$

- Oscillations of the electrical system are analogous to damped forced vibrations of a mechanical system.

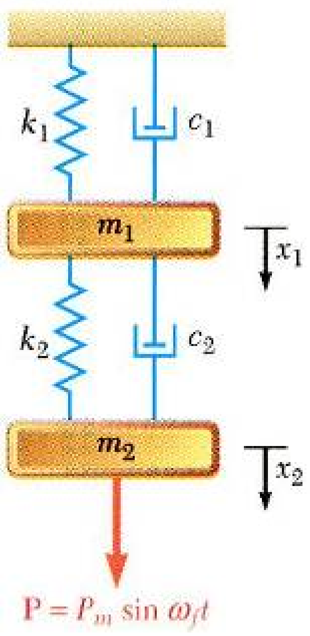
Table 19.2. Characteristics of a Mechanical System and of Its Electrical Analogue

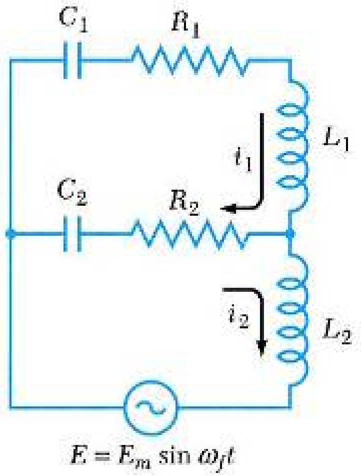
Mechanical System		Electrical Circuit	
m	Mass	L	Inductance
c	Coefficient of viscous damping	R	Resistance
k	Spring constant	$1/C$	Reciprocal of capacitance
x	Displacement	q	Charge
v	Velocity	i	Current
P	Applied force	E	Applied voltage





- The analogy between electrical and mechanical systems also applies to transient as well as steady-state oscillations.
- With a charge $q = q_0$ on the capacitor, closing the switch is analogous to releasing the mass of the mechanical system with no initial velocity at $x = x_0$.
- If the circuit includes a battery with constant voltage *E*, closing the switch is analogous to suddenly applying a force of constant magnitude *P* to the mass of the mechanical system.





- The electrical system analogy provides a means of experimentally determining the characteristics of a given mechanical system.
- For the mechanical system,

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + c_2\left(\dot{x}_1 - \dot{x}_2\right) + k_1 x_1 + k_2\left(x_1 - x_2\right) = 0$$

$$m_2\ddot{x}_2 + c_2\left(\dot{x}_2 - \dot{x}_1\right) + k_2\left(x_2 - x_1\right) = P_m \sin \omega_f t$$

- For the electrical system,

$$L_1\ddot{q}_1 + R_1\left(\dot{q}_1 - \dot{q}_2\right) + \frac{q_1}{C_1} + \frac{q_1 - q_2}{C_2} = 0$$

$$L_2\ddot{q}_2 + R_2\left(\dot{q}_2 - \dot{q}_1\right) + \frac{q_2 - q_1}{C_2} = E_m \sin \omega_f t$$

- The governing equations are equivalent. The characteristics of the vibrations of the mechanical system may be inferred from the oscillations of the electrical system.