Chap. 13
Kinetics of particles: force and acceleration

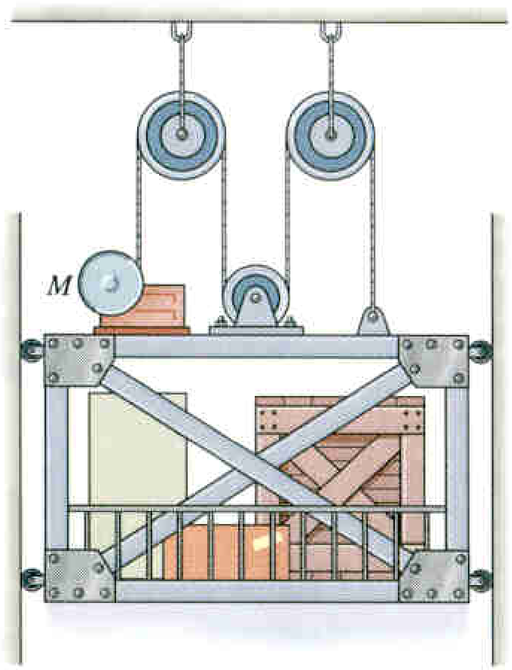
APPLICATIONS

The motion of an object depends on the forces acting on it.

A parachutist relies on the atmospheric drag resistance force to limit his velocity.

Knowing the drag force, how can we determine the acceleration or velocity of the parachutist at any point in time?

APPLICATIONS



A freight elevator is lifted using a motor attached to a cable and pulley system as shown.

How can we determine the tension force in the cable required to lift the elevator at a given acceleration?

Is the tension force in the cable greater than the weight of the elevator and its load?

一、牛頓運動定律

1^{st} Law：慣性

3^{rd} Law：作用力及反作用力

（1st Law、3rd Law：靜力）

2^{nd} Law：質點受一不平衡力(外力總和不爲0)作用時，會產生一加速度，其方向和力相同，大小則成正比。

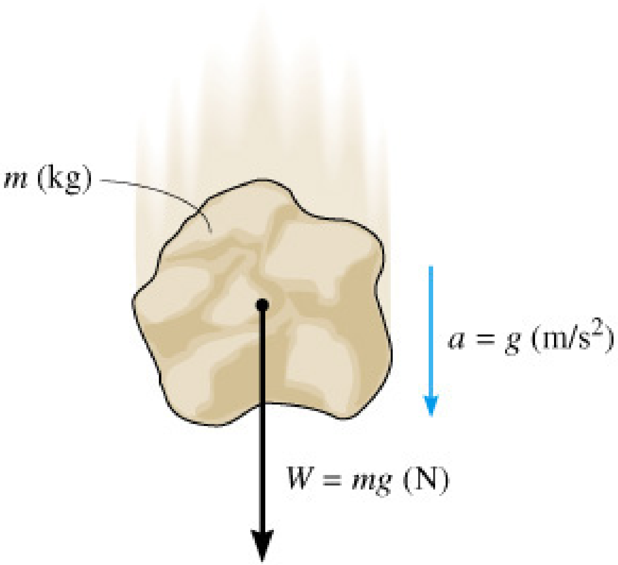
$$F = ma = m\frac{dv}{dt} = \frac{d(mv)}{dt}$$

$$ads = vdv \qquad a = \frac{vdv}{ds}$$

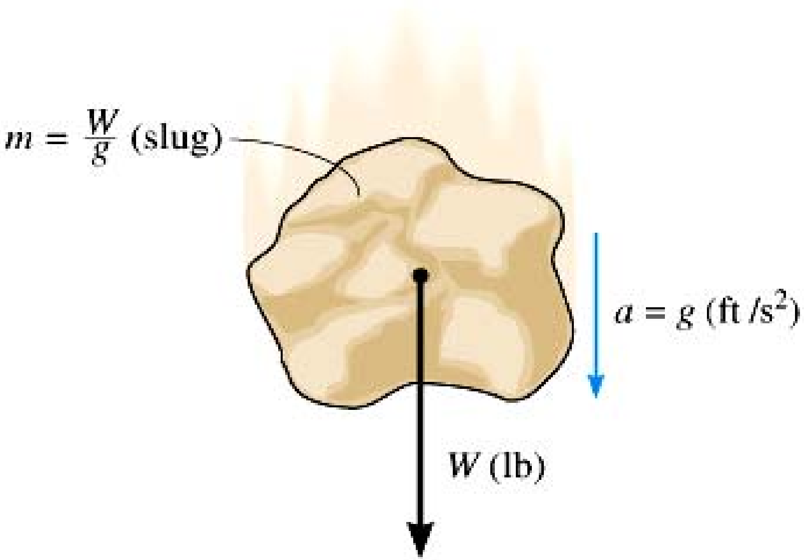
萬有引力定律

$$\mathbf{F} = \mathbf{G}\frac{\mathbf{m_1 m_2}}{\mathbf{r^2}}$$

$G=$萬有引力常數$66.73\times10^{-12}\quad m^3/(kg\cdot s^2)$



SI system
(a)



FPS system
(b)

二、運動方程式

直角座標系

直角座標系

$$\sum(F_x i + F_y j + F_z k) = m(a_x i + a_y j + a_z k)$$

$$\Sigma F_x = ma_x = m\ddot{x}, \quad \Sigma F_y = ma_y = m\ddot{y}, \quad \Sigma F_z = ma_z = m\ddot{z}$$

例：拋體運動中，物體僅受重力 -W**j**

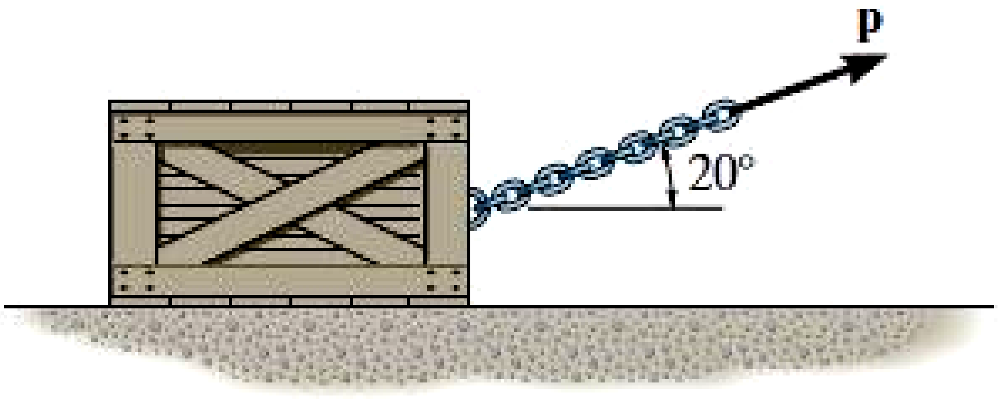
$$m\ddot{x} = 0, \quad m\ddot{y} = -w, \quad m\ddot{z} = 0$$

$$\text{or} \quad \ddot{x} = 0, \quad \ddot{y} = -\frac{w}{m} = -g, \quad \ddot{z} = 0$$

＊有兩個以上的物體時，須個別列出運動方程式。

p. 122, 13-10

The crate has a mass of 80 kg and is being towed by a chain which is always directed at 20° from the horizontal as shown. If the magnitude of **P** is increased until the crate begins to slide, determine the crate's initial acceleration if the coefficient of static friction is μ_S = 0.5 and the coefficient of kinetic friction is μ_K = 0.3.



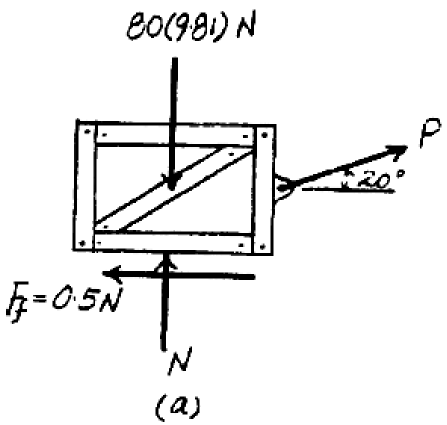
Equations of Equilibrium: If the crate is on the verge of slipping, $F_f = \mu_s N = 0.5N$. From FBD(a),

$$+\uparrow \Sigma F_y = 0; \qquad N + P\sin 20^\circ - 80(9.81) = 0 \qquad \textbf{(1)}$$

$$\overset{+}{\rightarrow} \Sigma F_x = 0; \qquad P\cos 20^\circ - 0.5N = 0 \qquad \textbf{(2)}$$

Solving Eqs.(1) and (2) yields

$$P = 353.29\text{ N} \qquad N = 663.97\text{ N}$$



Equations of Motion: The friction force developed between the crate and its contacting surface is $F_f = \mu_k N = 0.3N$ since the crate is moving. From FBD(b),

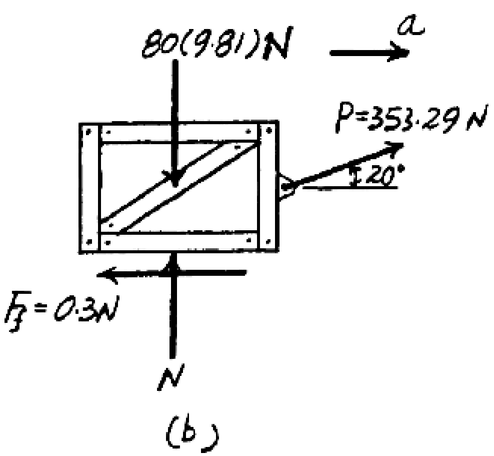
$$+\uparrow \Sigma F_y = ma_y; \qquad N - 80(9.81) + 353.29\sin 20^\circ = 80(0)$$

$$N = 663.97\text{ N}$$

Can be omitted

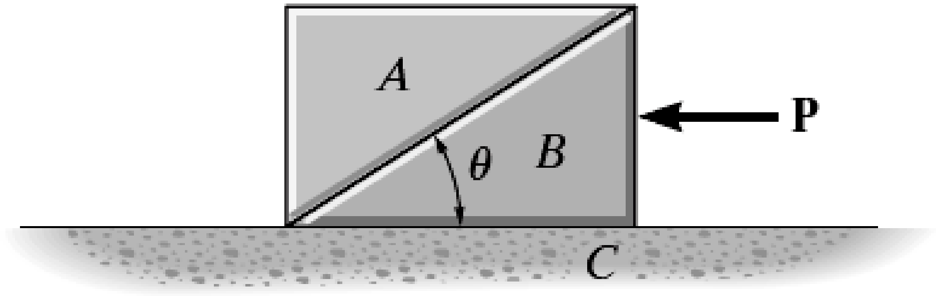
$$\overset{+}{\rightarrow} \Sigma F_x = ma_x; \qquad 353.29\cos 20^\circ - 0.3(663.97) = 80a$$

$$a = 1.66\text{ m/s}^2 \qquad \textbf{Ans.}$$



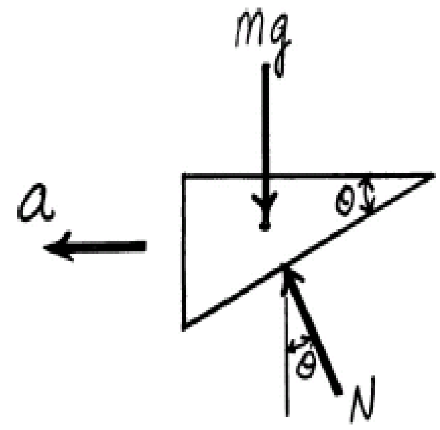
p. 128, 13-36

Blocks *A* and *B* each have a mass *m.* Determine the largest horizontal force **P** which can be applied to *B* so that *A* will not move relative to *B.* All surfaces are smooth.



$a_A = a_B = a$

Block A:

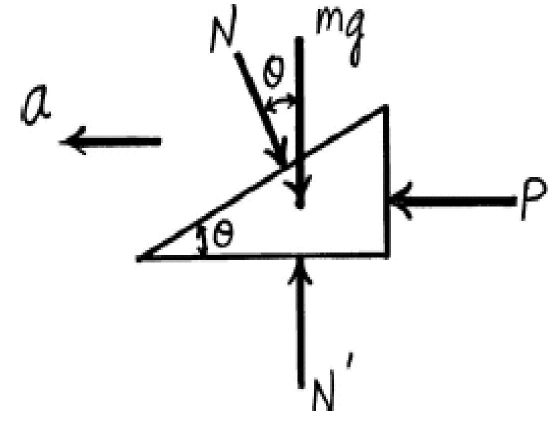


$+\uparrow \Sigma F_y = 0; \qquad N\cos\theta - mg = 0$

$\overset{+}{\leftarrow} \Sigma F_x = ma_x; \qquad N\sin\theta = ma$

$$a = g\tan\theta$$

Block B:

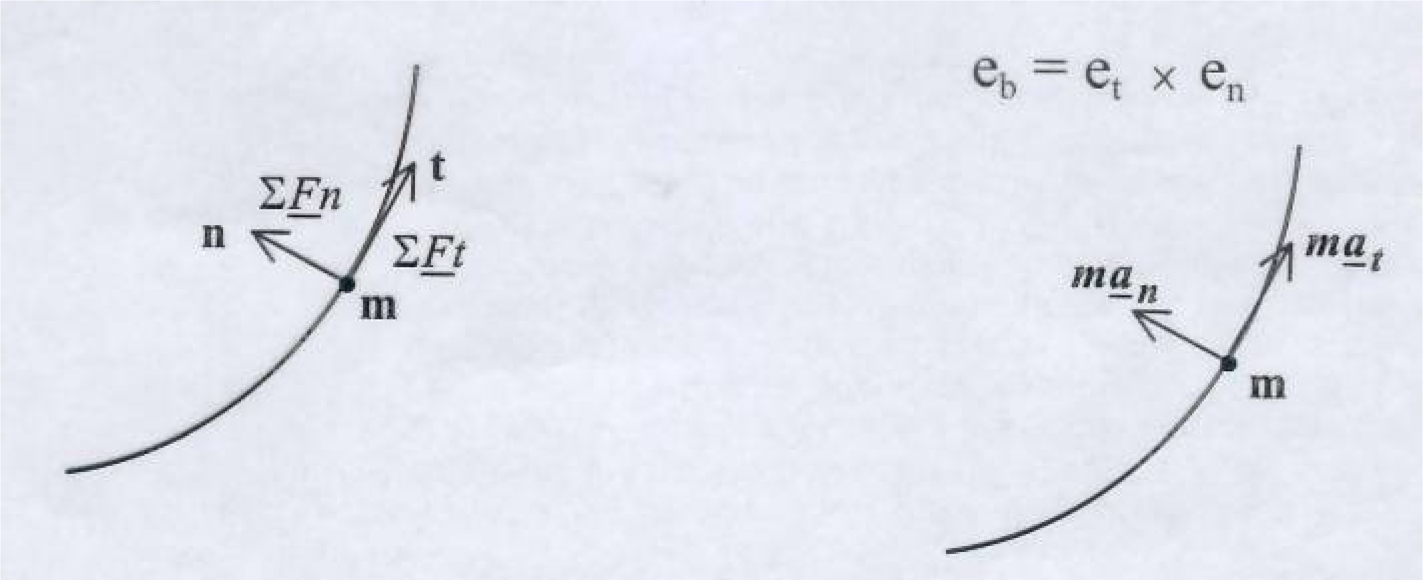


$\overset{+}{\leftarrow} \Sigma F_x = ma_x; \qquad P - N\sin\theta = ma$

$$P - mg\tan\theta = mg\tan\theta$$

$$P = 2mg\tan\theta$$

切線及法線向量



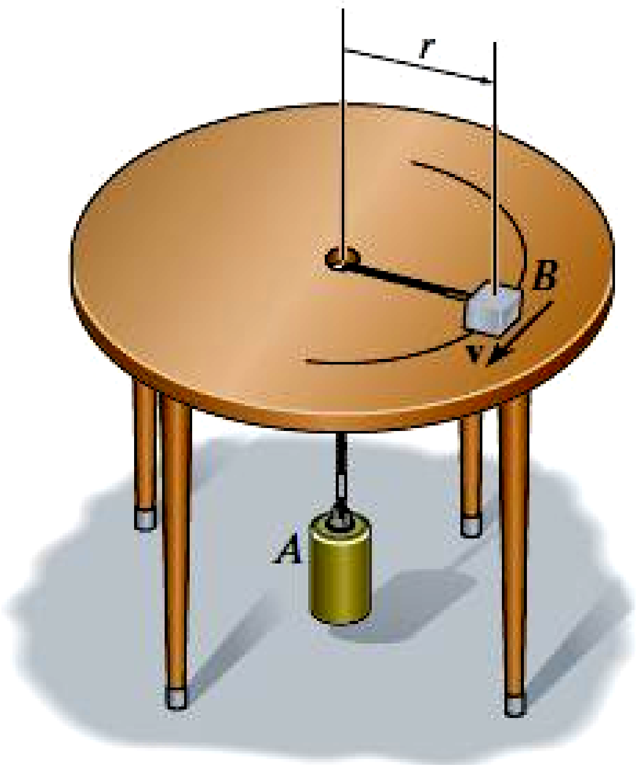
$$\Sigma F_t = ma_t, \qquad \Sigma F_n = ma_n$$

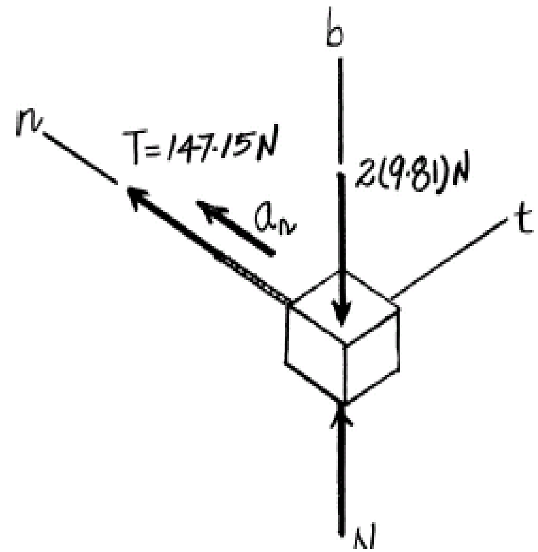
$$= m\frac{dv}{dt} \qquad = m\frac{v^2}{\rho}$$

（可解兩個未知數）

p. 138, 13-48

The 2-kg block *B* and 15-kg cylinder *A* areconnected to a light cord that passes through a hole in the center of the smooth table. If the block is given a speed of v = 10 m/s ,determine the radius r of the circular path along which it travels.





Free-Body Diagram: The free-body diagram of block B is shown in Fig. (a). The tension in the cord is equal to the weight of cylinder A, i.e., $T = 15(9.81)\text{N} = 147.15$ N. Here, $\mathbf{a}_n$ must be directed towards the center of the circular path (positive n axis).

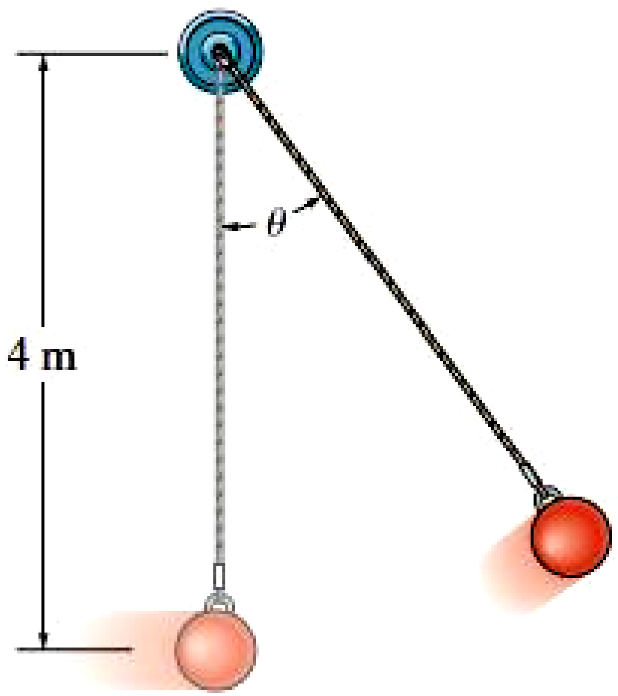
Equations of Motion: Realizing that $a_n = \dfrac{v^2}{\rho} = \dfrac{10^2}{r}$ and referring to Fig. (a),

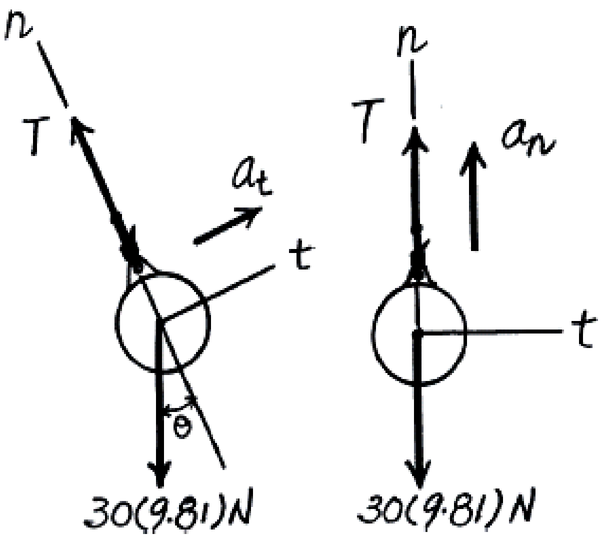
$$\Sigma F_n = ma_n; \qquad 147.15 = 2\left(\frac{10^2}{r}\right)$$

$$r = 1.36 \text{ m} \qquad \textbf{Ans.}$$

p. 140, 13-61

If the ball has a mass of 30kg and a speed at the instant it is at its lowest point, ,determine the tension in the cord at this instant .Also, determine the angle to which the balls wings and momentarily stops. Neglect the size of the ball.





$$+\uparrow \Sigma F_n = ma_n; \qquad T - 30(9.81) = 30\left(\frac{(4)^2}{4}\right)$$

$$\boxed{T = 414\text{ N}}$$

$$+\nearrow \Sigma F_t = ma_t; \qquad -30(9.81)\sin\theta = 30a_t$$

$$a_t = -9.81\sin\theta$$

$a_t\,ds = v\,dv$ Since $ds = 4\,d\theta$, then

$$-9.81\int_0^{\theta}\sin\theta(4\,d\theta) = \int_4^0 v\,dv$$

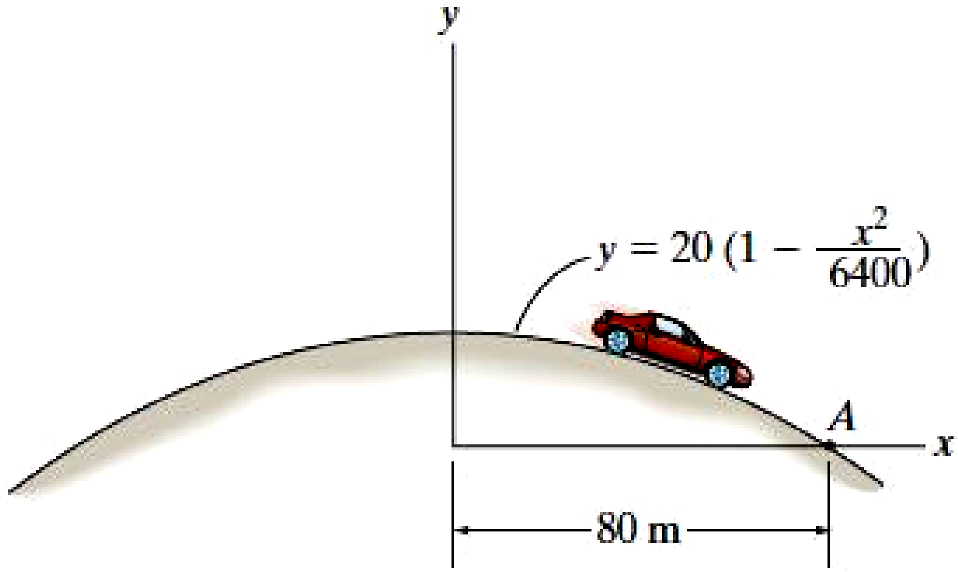
$$\left[9.81(4)\cos\theta\right]_0^{\theta} = -\frac{1}{2}(4)^2$$

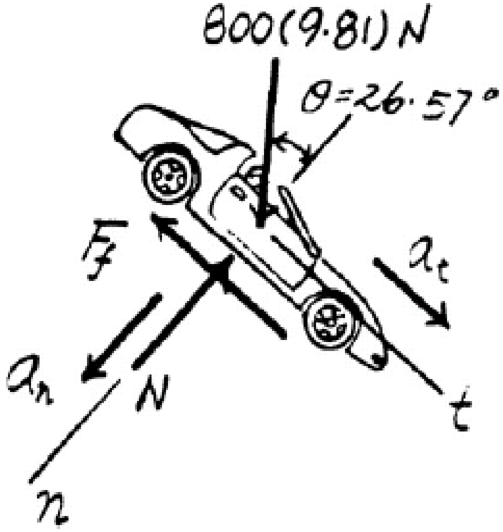
$$39.24(\cos\theta - 1) = -8$$

$$\boxed{\theta = 37.2^\circ}$$

p. 141, 13-72

The 0.8-Mg car travels over the hill having the shape of a parabola. If the driver maintains a constant speed of 9 m/s, determine both the resultant normal force and the resultant frictional force that all the wheels of the car exert on the road at the instant it reaches point *A*. Neglect the size of the car.





Geometry: Here, $\frac{dy}{dx} = -0.00625x$ and $\frac{d^2y}{dx^2} = -0.00625$. The slope angle θ at point A is given by

$$\tan\theta = \left.\frac{dy}{dx}\right|_{x=80\text{ m}} = -0.00625(80) \qquad \theta = -26.57^\circ$$

and the radius of curvature at point A is

$$\rho = \frac{[1 + (dy/dx)^2]^{3/2}}{|d^2y/dx^2|} = \left.\frac{[1 + (-0.00625x)^2]^{3/2}}{|-0.00625|}\right|_{x=80\text{ m}} = 223.61\text{ m}$$

Equations of Motion: Here, $a_t = 0$. Applying Eq. 13–8 with $\theta = 26.57^\circ$ and $\rho = 223.61$ m, we have

$$\Sigma F_t = ma_t; \qquad 800(9.81)\sin 26.57^\circ - F_f = 800(0)$$

$$F_f = 3509.73\text{ N} = 3.51\text{ kN} \qquad \textbf{Ans.}$$

$$\Sigma F_n = ma_n; \qquad 800(9.81)\cos 26.57^\circ - N = 800\left(\frac{9^2}{223.61}\right)$$

$$N = 6729.67\text{ N} = 6.73\text{ kN} \qquad \textbf{Ans.}$$

三、徑向與橫向分量

$$\Sigma F_r = ma_r = m(\ddot{r} - r\dot{\theta}^2) \quad , \quad \Sigma F_\theta = ma_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

由 $\Sigma M_0 = r \times (\Sigma F_r + \Sigma F_\theta) = r \times \Sigma F_r + r \times \Sigma F_\theta = \dot{H}_0$

即 $r\Sigma F_\theta = \dfrac{d}{dt}(mr^2\dot{\theta}) \qquad H_0 = mr^2\dot{\theta}$

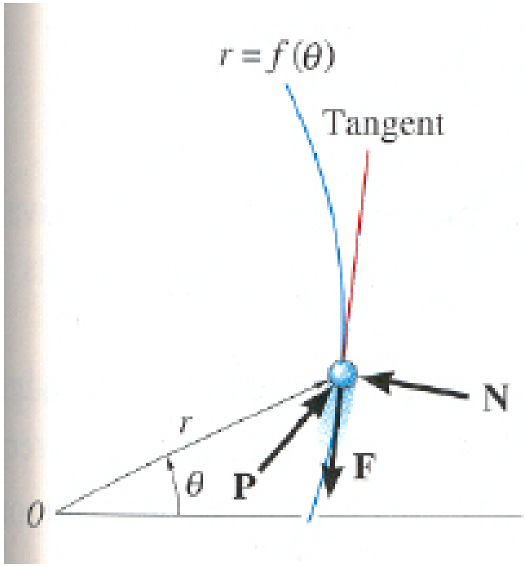
$$= m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$$

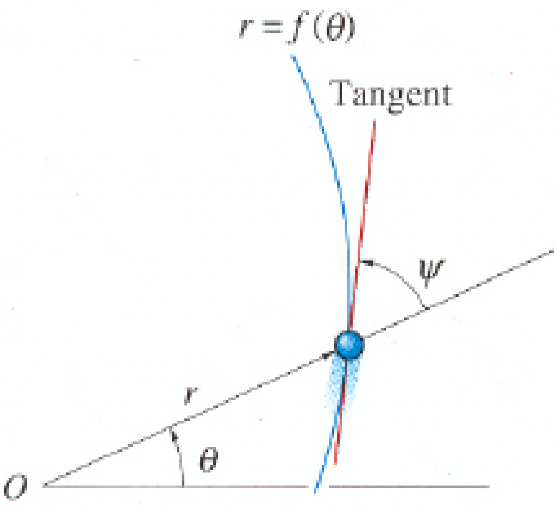
($\mathbf{r}$和ΣF_θ成直角，故$\left| r \times \Sigma F_\theta \right| = r\Sigma F_\theta \sin 90^0 = r\Sigma F_\theta$)

除以$r \Rightarrow \Sigma F_\theta = m(r\ddot{\theta} + 2\dot{r}\dot{\theta})$

TANGENTIAL AND NORMAL FORCES

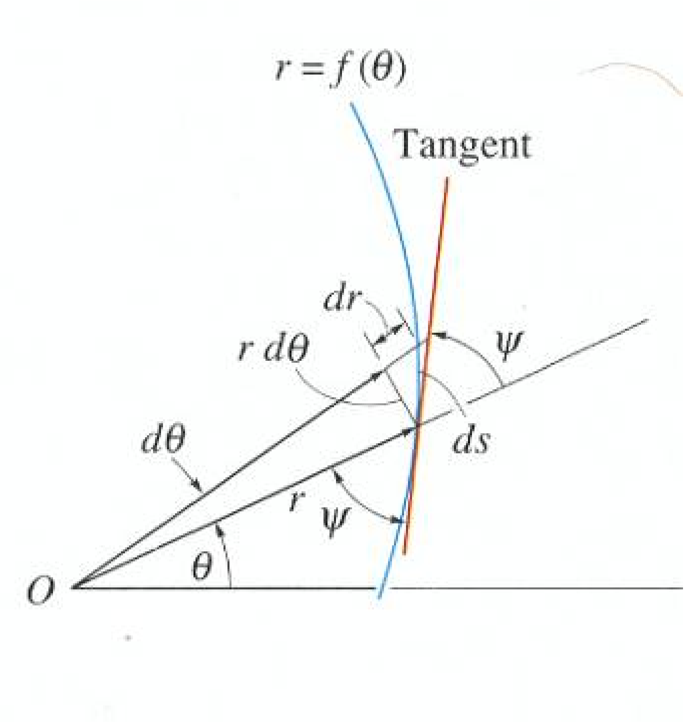
If a force ***P*** causes the particle to move along a path defined by $r = f(\theta)$, the normal force ***N*** exerted by the path on the particle is always perpendicular to the path's tangent. The frictional force ***F*** always acts along the tangent in the opposite direction of motion. The directions of ***N*** and ***F*** can be specified relative to the radial coordinate by using angle ψ.





DETERMINATION OF ANGLE ψ

The angle ψ, defined as the angle between the extended radial line and the tangent to the curve, can be required to solve some problems. It can be determined from the following relationship.



$$\tan \psi = \frac{r\, d\theta}{dr} = \frac{r}{dr/d\theta}$$

If ψ is positive, it is measured counterclockwise from the radial line to the tangent. If it is negative, it is measured clockwise.

四、中心力、角動量守恆

Def.

作用於質點P的力F若通過定點C，則F稱為**中心力** (Central force)，C點稱為作用力中心 (Center of force)。

$\because$ F點通過O $\therefore \Sigma M_o = 0$ 即 $\dot{H}_O = 0$

$$\Rightarrow H_0 = constant = r \times mv$$

由以上知

$\therefore$ 受心作用的質點在一垂直於的固定平面上運動。

又 $r \perp H_0$

$$H_0 = const. = rmv\sin\phi = r_0 m v_0 \sin\phi_0$$ （行星運動的基本定理）

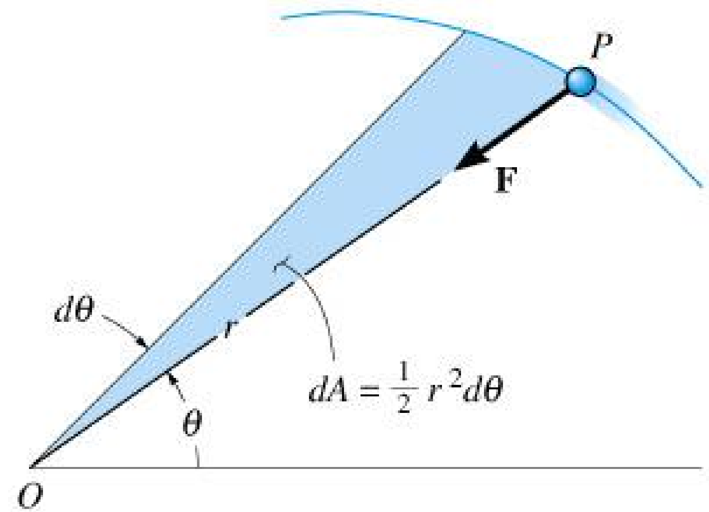
＊太陽對於行星的引力，太空船繞固定軌道迴轉皆為中心力。

$$H_0 = mr^2\dot{\theta} = const.$$

Def.　h為單位質量的角動量，則面積速度 (areal velocity)----面積的變化率

$$dA = \frac{1}{2} r^2 d\theta$$

$$p = \frac{dA}{dt}$$



areal velocity：

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{h}{2} = const.$$

五、受中心力運動(質點之軌跡)

$$\Sigma F = \Sigma F_r + \Sigma F_\theta$$

$$m(\ddot{r} - r\dot{\theta}^2) = -F \quad (\text{向心}) --- (1)$$

$$m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = 0 \quad \Rightarrow \frac{1}{r}\left[\frac{d}{dt}(r^2\frac{d\theta}{dt})\right] = 0$$

$$\Rightarrow r^2\dot{\theta} = h \quad or \quad r^2\frac{d\theta}{dt} = h \quad \Rightarrow \dot{\theta} = \frac{d\theta}{dt} = \frac{h}{r^2} --- (2)$$

$$\therefore \dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{h}{r^2}\frac{dr}{d\theta} = \frac{h}{d\theta}\frac{dr}{r^2}$$

$$= \frac{h}{d\theta}\left[-d\left(\frac{1}{r}\right)\right] = -h\frac{d}{d\theta}\left(\frac{1}{r}\right)$$

$$\ddot{r} = \frac{d\dot{r}}{dt} = \frac{h}{r^2}\frac{d\dot{r}}{d\theta} = \frac{h}{r^2}\frac{d}{d\theta}\left[-h\frac{d}{d\theta}\left(\frac{1}{r}\right)\right] \qquad dt = \frac{r^2 d\theta}{h}$$

(由(2))

$$= \frac{-h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) --- (3)$$

設　$u=\dfrac{1}{r}$將 (2)、(3) 代入 (1)

$$m\left[\frac{-h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)-r\frac{h^2}{r^4}\right]=-F$$

$$\Rightarrow m\left[h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3\right]=F$$

$$\Rightarrow \frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2}$$ (F爲吸引力時＋　阻力時–)

六、太空力學之應用

$$\because F = \frac{GMm}{r^2} = GMmu^2$$

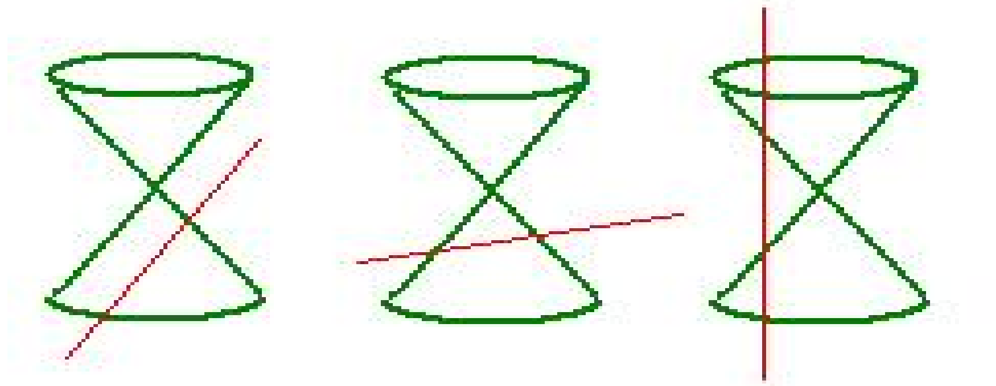
$$\therefore \frac{d^2u}{d\theta^2} + u = \frac{GM}{h^2}$$

$u = C_1 \cos\theta + C_2 \sin\theta$

general sol. $u = C\cos(\theta - \theta_0)$

particular sol. $u = \dfrac{GM}{h^2}$

若 $\theta_0 = 0 \Rightarrow u = \dfrac{GM}{h^2} + C\cos\theta$ ← Eq. of "conic section"



Def. (eccentricity) $\varepsilon = \dfrac{C}{GM/h^2} = \dfrac{Ch^2}{GM}$

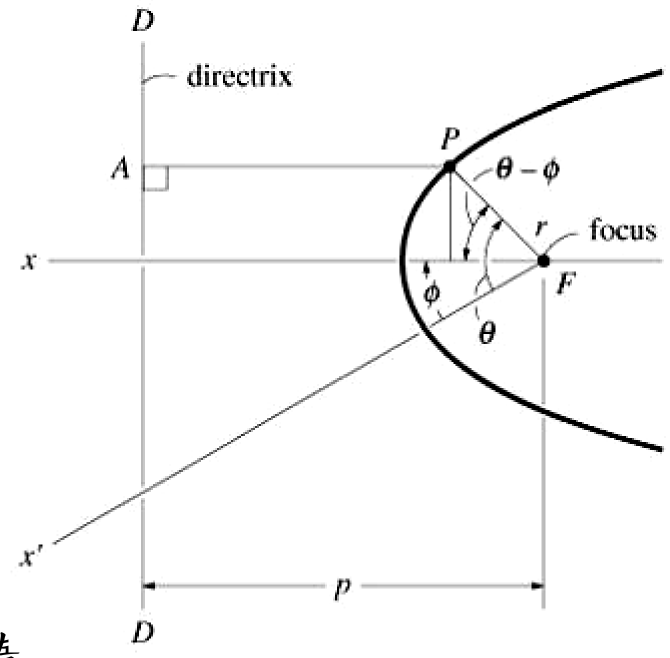
$$\therefore \frac{1}{r} = \frac{GM}{h^2}(1+\varepsilon\cos\theta)$$

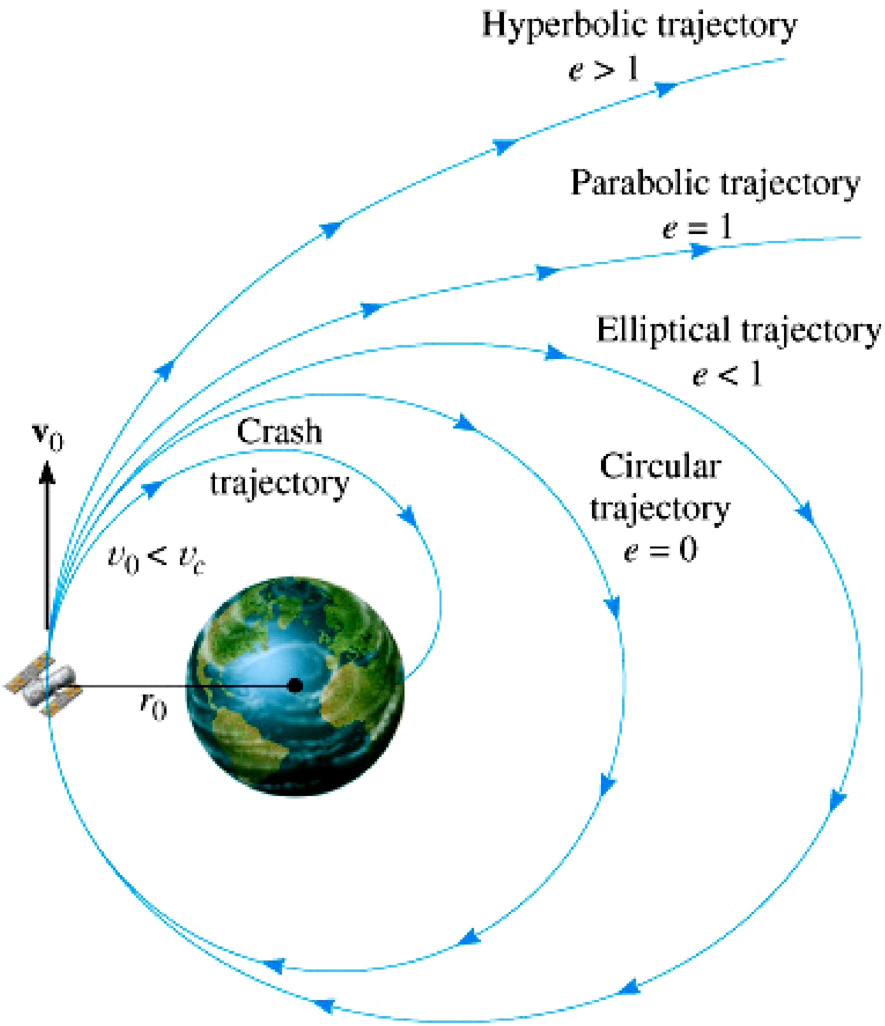
(1) $\varepsilon > 1 \quad or \quad C > \frac{GM}{h^2}$, $\cos\theta = -\frac{GM}{Ch^2}$

， **r =∞**， 雙曲線 (hyperbola)

(2) $\varepsilon = 1$， $\theta = 180$時， **r =∞**， 拋物線 (parabola)

(3) $\varepsilon < 1$ ，任何 θ ，**r 皆為有限值**，橢圓 (ellipse)





I.C.: $v_0 = r_0\theta_0$

由 $h = r_0^2\theta_0 = r_0\theta_0$

又 $GM = gR^2$

$$C = \frac{1}{r_0} - \frac{GM}{h^2} = \frac{1}{r_0} - \frac{GM}{r_0^2 v_0^2}$$

----- 若軌跡為拋物線， $C = \frac{GM}{h^2}$

$$\therefore v_0 = \sqrt{\frac{2GM}{r_0}}$$ ，為脫離速度 (escape velocity)

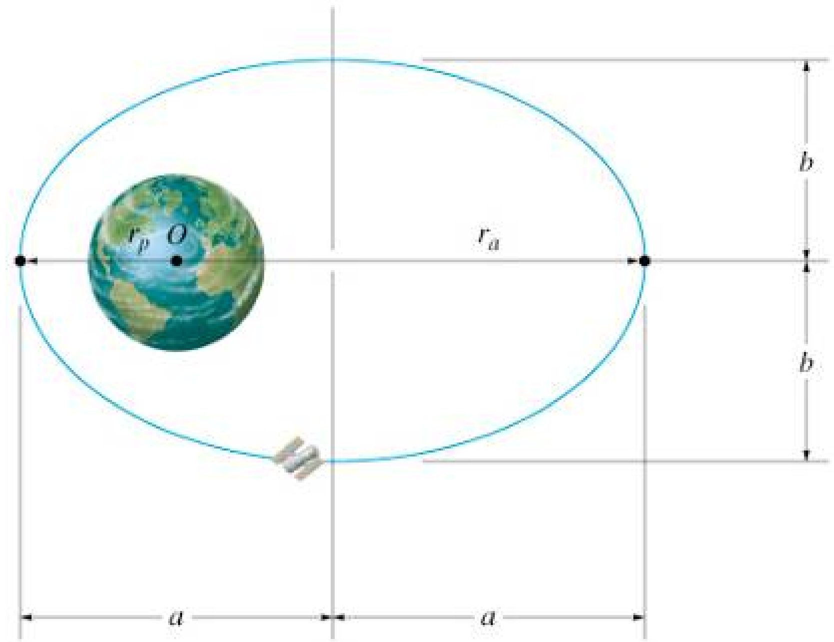
----- 圓形軌道時，C = 0

$$\Rightarrow v_{cir} = \sqrt{\frac{GM}{r_0}} or \sqrt{\frac{gR^2}{r_0}}$$

perigee 近地點，apogee 遠地點

----- 週期(時間) periodic time，繞軌道一周所需的時間

$$\mathbf{T} = \frac{軌道內面積}{面積速度} = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$$



$$r_0 + r_1 = 2a$$

$$a = \frac{r_0 + r_1}{2}$$

$$b = \sqrt{r_0 r_1}$$

七、喀卜勒行星運動定律

- 1. 行星以太陽為其橢圓軌道的焦點之一。
- 2. 以太陽為起點的半徑向量，相同的時間內所掃過的面積相同。
- *3.* $T^2 \sim a^3$ (need correct for text, should be major axis, not minor axis)

- 證明：

$$\theta = 0°, \qquad \frac{1}{r_0} = \frac{GM}{h^2}(1+\varepsilon) ----(1)$$

$$\theta = 180°, \qquad \frac{1}{r_1} = \frac{GM}{h^2}(1-\varepsilon) ----(2)$$

$$(1)+(2), \qquad \frac{1}{r_0} + \frac{1}{r_1} = \frac{r_0 + r_1}{r_0 r_1} = \frac{2GM}{h^2}$$

又 $r_0 + r_1 = 2a, \quad r_0 r_1 = b^2$

$$\therefore \frac{2a}{b^2} = \frac{2GM}{h^2} \qquad or \qquad h^2 = \frac{GMb^2}{a}$$

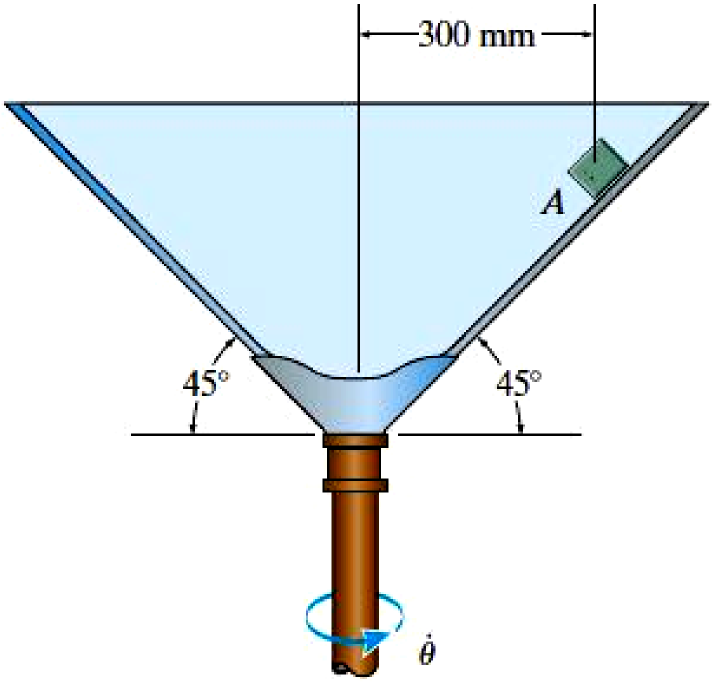
$$\because T = \frac{2\pi ab}{h}$$

$$\therefore T^2 = \frac{4\pi^2 a^2 b^2}{h^2} = \frac{4\pi^2 a^2 b^2}{GM\frac{b^2}{a}} = \frac{4\pi^2 a^3}{GM}$$

$$\Rightarrow T^2 \propto a^3$$

p. 151, 13-92

If the coefficient of static friction between the conical surface and the block of mass *m* is , determine the minimum constant angular velocity so that the block does not slide downwards.



Free-Body Diagram: The free-body diagram of the block is shown in Fig. (a). Since the block is required to be on the verge of sliding down the conical surface, $F_f = \mu_k N = 0.2N$ must be directed up the conical surface. Here, $\mathbf{a}_r$ is assumed to be directed towards the positive r axis.

Equations of Motion: By referring to Fig. (a),

$$+\uparrow \Sigma F_z = ma_z; \quad N\cos 45^\circ + 0.2N\sin 45^\circ - m(9.81) = m(0) \qquad N = 11.56m$$

$$\overset{+}{\rightarrow} \Sigma F_r = ma_r; \quad 0.2(11.56m)\cos 45^\circ - (11.56m)\sin 45^\circ = ma_r \quad a_r = -6.54\ \text{m/s}^2$$

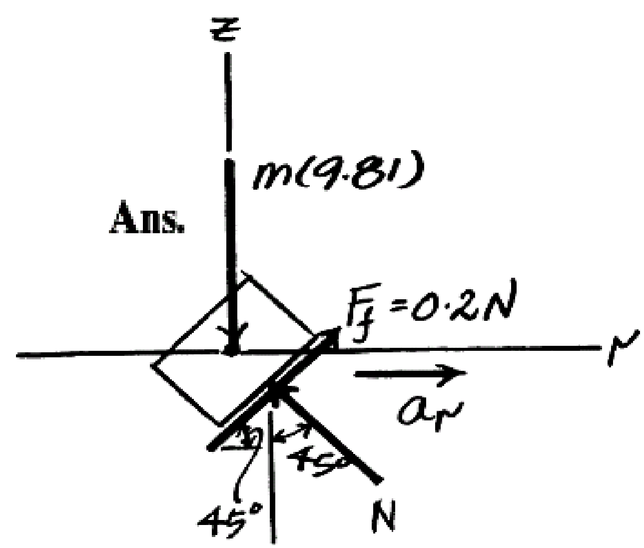
Kinematics: Since $r = 0.3$ m is constant, $\dot{r} = \ddot{r} = 0$.

$$a_r = \ddot{r} - r\dot{\theta}^2$$

$$-6.54 = 0 - 0.3\dot{\theta}^2$$

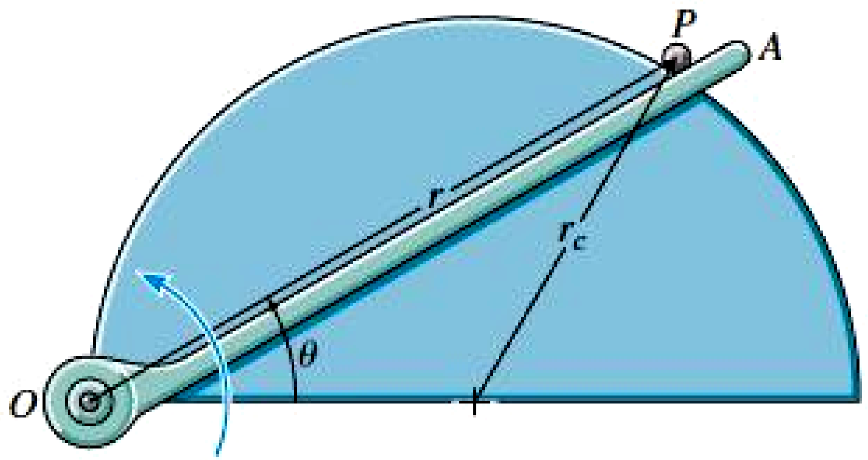
$$\dot{\theta} = 4.67\ \text{rad/s}$$

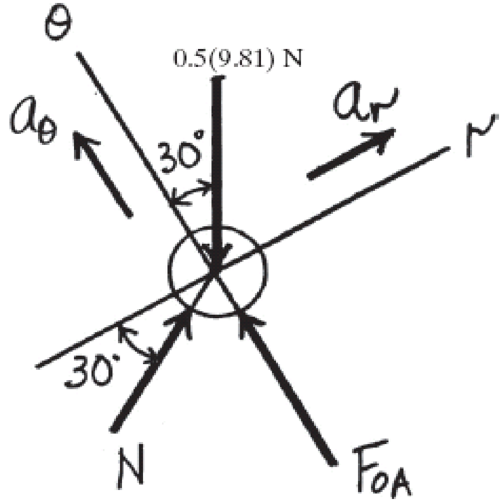
Ans.



p. 154, 13-112

The 0.5-kg ball is guided along the vertical circular path $r = 2r_C\cos\theta$ *using* the arm *OA.* If the arm has an angular velocity $\dot{\theta} = 0.4$ rad/s and an angular acceleration $\ddot{\theta} = 0.8$ rad/s^2 at the instant $\theta = 30°$, determine the force of the arm on the bal. Neglect friction and the size of the ball. Set $r_C = 0.12$ m.





$$r = 2(0.12)\cos\theta = 0.24\cos\theta$$

$$\dot{r} = -0.24\sin\theta\dot{\theta}$$

$$\ddot{r} = -0.24\cos\theta\dot{\theta}^2 - 0.24\sin\theta\ddot{\theta}$$

At $\theta = 30°, \dot{\theta} = 0.4$ rad/s, and $\ddot{\theta} = 0.8$ rad/s^2

$$r = 0.24\cos 30° = 0.2078\text{ m}$$

$$\dot{r} = -0.24\sin 30°(0.4) = -0.048\text{ m/s}$$

$$\ddot{r} = -0.24\cos 30°(0.4)^2 - 0.24\sin 30°(0.8) = -0.1293\text{ m/s}^2$$

$$a_r = \ddot{r} - r\dot{\theta}^2 = -0.1293 - 0.2078(0.4)^2 = -0.1625\text{ m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0.2078(0.8) + 2(-0.048)(0.4) = 0.1278\text{ m/s}^2$$

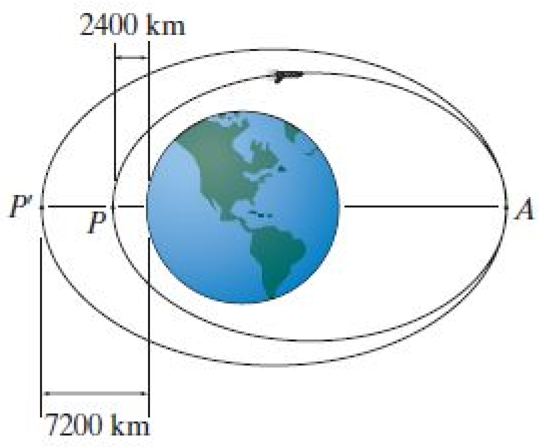
$+\nearrow\Sigma F_r = ma_r;\quad N\cos 30° - 0.5(9.81)\sin 30° = 0.5(-0.1625)\quad N = 2.738\text{ N}$

$\nwarrow+\Sigma F_\theta = ma_\theta;\quad F_{OA} + 2.738\sin 30° - 0.5(9.81)\cos 30° = 0.5(0.1278)$

$F_{OA} = 2.943\text{ N}$ **Ans.**

p. 163, 13-120

The space shuttle is launched with a velocity of 28 000 km/h parallel to the tangent of the earth's surface at point *P* and then travels around the elliptical orbit.When it reaches point *A*, its engines are turned on and its velocity is suddenly increased. Determine the required increase in velocity so that it enters the second elliptical orbit.



When the shuttle is travelling around the circular orbit of radius $r_o = 2(10^6) + 6378(10^3) = 8.378(10^6)$ m, its speed is

$$v_o = \sqrt{\frac{GM_e}{r_o}} = \sqrt{\frac{66.73(10^{-12})(5.976)(10^{24})}{8.378(10^6)}} = 6899.15 \text{ m/s}$$

When the shuttle enters the elliptical orbit, $r_p = r_o = 8.378(10^6)$ m and $r_a = 8(10^6) + 6378(10^3) = 14.378(10^6)$ m.

$$r_a = \frac{r_p}{\frac{2GM_e}{r_p v_p^2} - 1}$$

$$14.378(10^6) = \frac{8.378(10^6)}{\frac{2(66.73)(10^{-12})(5.976)(10^{24})}{8.378(10^6)v_p^2} - 1}$$

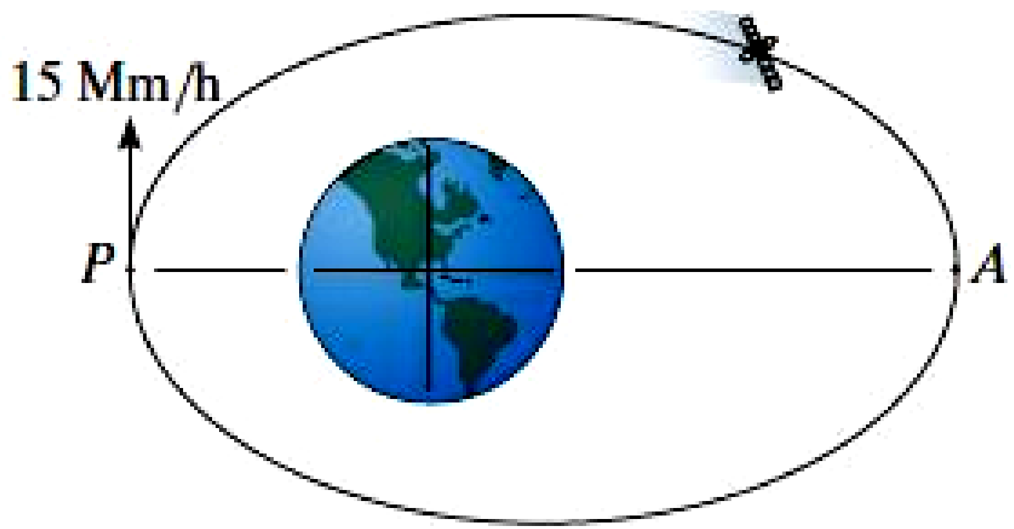
$$v_p = 7755.54 \text{ m/s}$$

Thus, the required increase in speed for the shuttle at point P is

$$\Delta v_p = v_p - v_o = 7755.54 - 6899.15 = 856.39 \text{ m/s} = 856 \text{ m/s}$$ **Ans.**

p. 165, 13-132

The satellite is in an elliptical orbit having an eccentricity of e = 0.5. If its velocity at perigee is $v_P = 15$ Mm/hr , determine its velocity at apogee A and the period of the satellite.



Here, $v_P = \left[15(10^6)\,\frac{\text{m}}{\text{h}}\right]\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 4166.67$ m/s.

$$h = r_P v_P$$

$$h = r_P\,(4166.67) = 4166.67 r_p \qquad (1)$$

and

$$C = \frac{1}{r_P}\left(1 - \frac{GM_e}{r_P\,v_P^{\,2}}\right)$$

$$C = \frac{1}{r_p}\left[1 - \frac{66.73(10^{-12})(5.976)(10^{24})}{r_p(4166.67^2)}\right]$$

$$C = \frac{1}{r_P}\left[1 - \frac{22.97(10^6)}{r_P}\right]$$

$$e = \frac{Ch^2}{GM_e}$$

$$0.15 = \frac{\frac{1}{r_P}\left[1 - \frac{22.97(10^6)}{r_P}\right](4166.67\,r_P)^2}{66.73(10^{-12})(5.976)(10^{24})}$$

$$r_P = 26.415(10^6)\text{ m}$$

Using the result of r_p

$$r_A = \frac{r_P}{\frac{2GM_e}{r_P\,v_P^{\,2}} - 1}$$

$$= \frac{26.415(10^6)}{\frac{2(66.73)(10^{-12})(5.976)(10^{24})}{26.415(10^6)(4166.67^2)} - 1}$$

$$= 35.738(10^6)\text{ m}$$

Since $h = r_P v_P = 26.415(10^6)(4166.67^2) = 110.06(10^9)\ \text{m}^2/\text{s}$ is constant,

$$r_A v_A = h$$

$$35.738(10^6) v_A = 110.06(10^9)$$

$$v_A = 3079.71\ \text{m/s} = 3.08\ \text{km/s}$$ **Ans.**

Using the results of h, r_A, and r_P,

$$T = \frac{\pi}{6}\left(r_P + r_A\right)\sqrt{r_P\, r_A}$$

$$= \frac{\pi}{110.06(10^9)}\left[26.415(10^6) + 35.738(10^6)\right]\sqrt{26.415(10^6)(35.738)(10^6)}$$

$$= 54\,508.43\ \text{s} = 15.1\ \text{hr}$$ **Ans.**