Chap. 11 Virtual Work

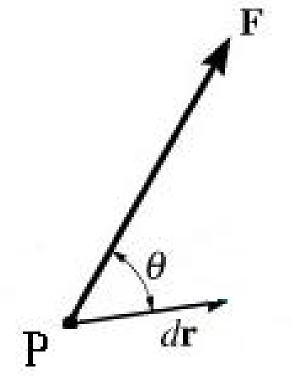
Virtual Work

簡化及解決變形材料的問題
(對於machine或由連桿組成的機構特別有效)

定義：

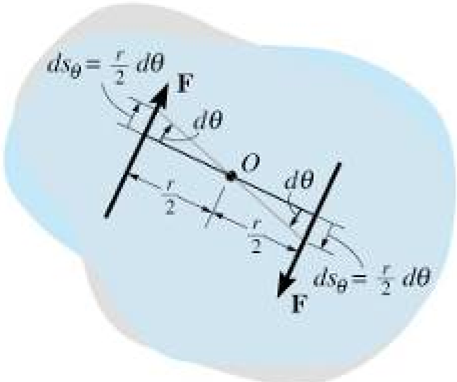
Work of a force： $dU = \underline{F} \cdot d\underline{r}$

$\underline{F}$ 在位移方向上的分量乘以位移，
爲$\underline{F}$ 對質點所做的功。



Virtual Work

Work of a couple (moment)



Rotation

(c)

$$M = F \cdot r$$

$$dU = F\left(\frac{r}{2}d\theta\right) + F\left(\frac{r}{2}d\theta\right) = F \cdot r d\theta = M d\theta$$

Virtual Work 虛功

虛位移 virtual displacement δs

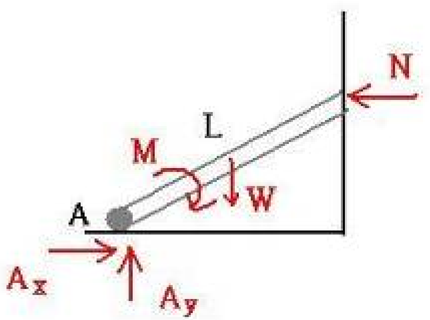
虛旋轉角 virtual rotation $\delta\theta$

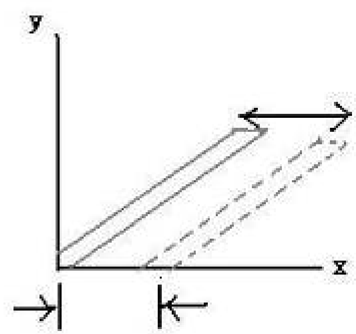
$$\begin{cases} \delta U = F\cos\theta\,\delta s \\ \delta U = M\delta\theta \end{cases}$$

Principle of virtual work 虛功原理

- 一物體平衡時，由外力和力偶所做的虛功總和爲0，即δU=0

Ex.





$$\delta U = A_x \cdot \delta x - N \cdot \delta x = (A_x - N)\delta x$$

虛功

EXAMPLE 11.1

Determine the angle θ for equilibrium of the two-member linkage shown in Fig. 11–7*a*. Each member has a mass of 10 kg.

Solution

Free-Body Diagram. The system has only one degree of freedom since the location of both links may be specified by the single independent coordinate $(q =)\ \theta$. As shown on the free-body diagram in Fig. 11–7*b*, when θ undergoes a *positive* (clockwise) virtual rotation $\delta\theta$, only the active forces, **F** and the two 98.1-N weights, do work. (The reactive forces $\mathbf{D}_x$ and $\mathbf{D}_y$ are fixed, and $\mathbf{B}_y$ does not move along its line of action.)

Virtual Displacements. If the origin of coordinates is established at the *fixed* pin support D, the location of **F** and **W** may be specified by the *position coordinates* x_B and y_w, as shown in the figure. In order to determine the work, note that these coordinates are parallel to the lines of action of their associated forces.

Expressing the position coordinates in terms of the independent coordinate θ and taking the derivatives yields

$$x_B = 2(1 \cos\theta)\ \text{m} \qquad \delta x_B = -2 \sin\theta\ \delta\theta\ \text{m} \tag{1}$$

$$y_w = \tfrac{1}{2}(1 \sin\theta)\ \text{m} \qquad \delta y_w = 0.5 \cos\theta\ \delta\theta\ \text{m} \tag{2}$$

It is seen by the *signs* of these equations, and indicated in Fig. 11–7*b*, that an *increase* in θ (i.e., $\delta\theta$) causes a *decrease* in x_B and an *increase* in y_w.

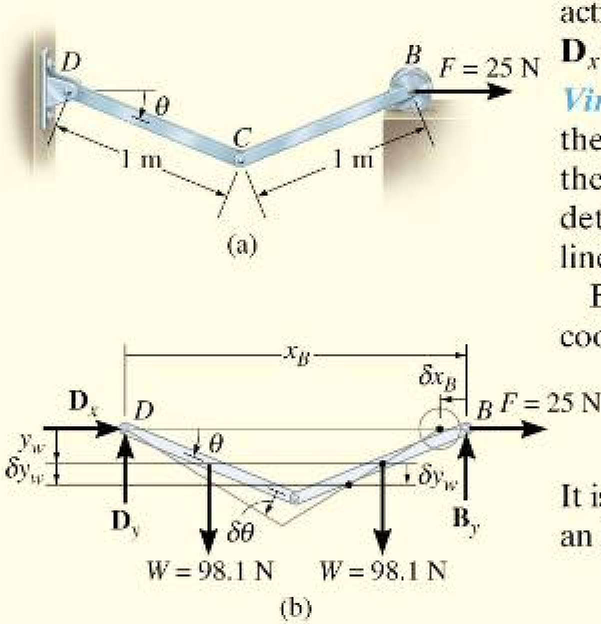


Fig. 11–7

EXAMPLE 11.1

Virtual-Work Equation. If the virtual displacements δx_B and δy_w were *both positive*, then the forces **W** and **F** would do positive work since the forces and their corresponding displacements would have the same sense. Hence, the virtual-work equation for the displacement $\delta\theta$ is

$$\delta U = 0; \qquad W\,\delta y_w + W\,\delta y_w + F\,\delta x_B = 0 \tag{3}$$

Substituting Eqs. 1 and 2 into Eq. 3 in order to relate the virtual displacements to the common virtual displacement $\delta\theta$ yields

$$98.1(0.5\cos\theta\,\delta\theta) + 98.1(0.5\cos\theta\,\delta\theta) + 25(-2\sin\theta\,\delta\theta) = 0$$

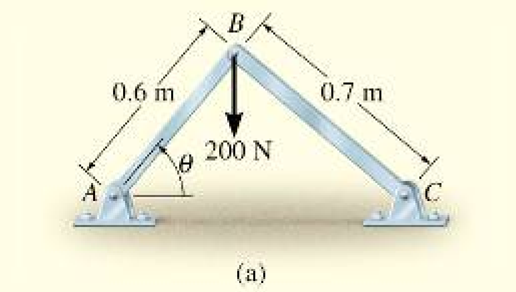
Notice that the "negative work" done by **F** (force in the opposite sense to displacement) has been *accounted for* in the above equation by the "negative sign" of Eq. 1. Factoring out the *common displacement* $\delta\theta$ and solving for θ, noting that $\delta\theta \neq 0$, yields

$$(98.1\cos\theta - 50\sin\theta)\,\delta\theta = 0$$

$$\theta = \tan^{-1}\frac{98.1}{50} = 63.0^\circ \qquad \textit{Ans.}$$

If this problem had been solved using the equations of equilibrium, it would have been necessary to dismember the links and apply three scalar equations to *each* link. The principle of virtual work, by means of calculus, has eliminated this task so that the answer is obtained directly.

EXAMPLE 11.3 1/2



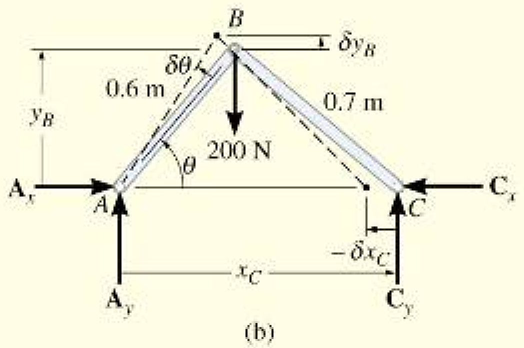


Fig. 11–9

Determine the horizontal force C_x that the pin at C must exert on BC in order to hold the mechanism shown in Fig. 11–9a in equilibrium when $\theta = 45°$. Neglect the weight of the members.

Solution

Free-Body Diagram. The reaction $\mathbf{C}_x$ can be obtained by *releasing* the pin constraint at C in the x direction and allowing the frame to be displaced in this direction. The system then has only one degree of freedom, defined by the independent coordinate θ, Fig. 11–9b. When θ undergoes a *positive* virtual displacement $\delta\theta$, only $\mathbf{C}_x$ and the 200-N force do work.

Virtual Displacements. Forces $\mathbf{C}_x$ and 200 N are located from the fixed origin A using position coordinates y_B and x_C. From Fig. 11–9b, x_C can be related to θ by the "law of cosines." Hence,

$$(0.7)^2 = (0.6)^2 + x_C^2 - 2(0.6)x_C \cos\theta \qquad (1)$$

$$0 = 0 + 2x_C\,\delta x_C - 1.2\,\delta x_C \cos\theta + 1.2x_C \sin\theta\,\delta\theta$$

$$\delta x_C = \frac{1.2x_C \sin\theta}{1.2\cos\theta - 2x_C}\delta\theta \qquad (2)$$

Also,

$$y_B = 0.6\sin\theta$$

$$\delta y_B = 0.6\cos\theta\,\delta\theta \qquad (3)$$

EXAMPLE 11.3 2/2

Virtual-Work Equation. When y_B and x_C undergo *positive* virtual displacements δy_B and δx_C, $\mathbf{C}_x$ and 200 N do *negative work* since they both act in the opposite sense to $\delta \mathbf{y}_B$ and $\delta \mathbf{x}_C$. Hence,

$$\delta U = 0; \qquad -200\,\delta y_B - C_x\,\delta x_C = 0$$

Substituting Eqs. 2 and 3 into this equation, factoring out $\delta\theta$, and solving for C_x yields

$$-200(0.6\cos\theta\,\delta\theta) - C_x\frac{1.2x_C\sin\theta}{1.2\cos\theta - 2x_C}\delta\theta = 0$$

$$C_x = \frac{-120\cos\theta\,(1.2\cos\theta - 2x_C)}{1.2x_C\sin\theta} \tag{4}$$

At the required equilibrium position $\theta = 45°$, the corresponding value of x_C can be found by using Eq. 1, in which case

$$x_C^2 - 1.2\cos 45°\,x_C - 0.13 = 0$$

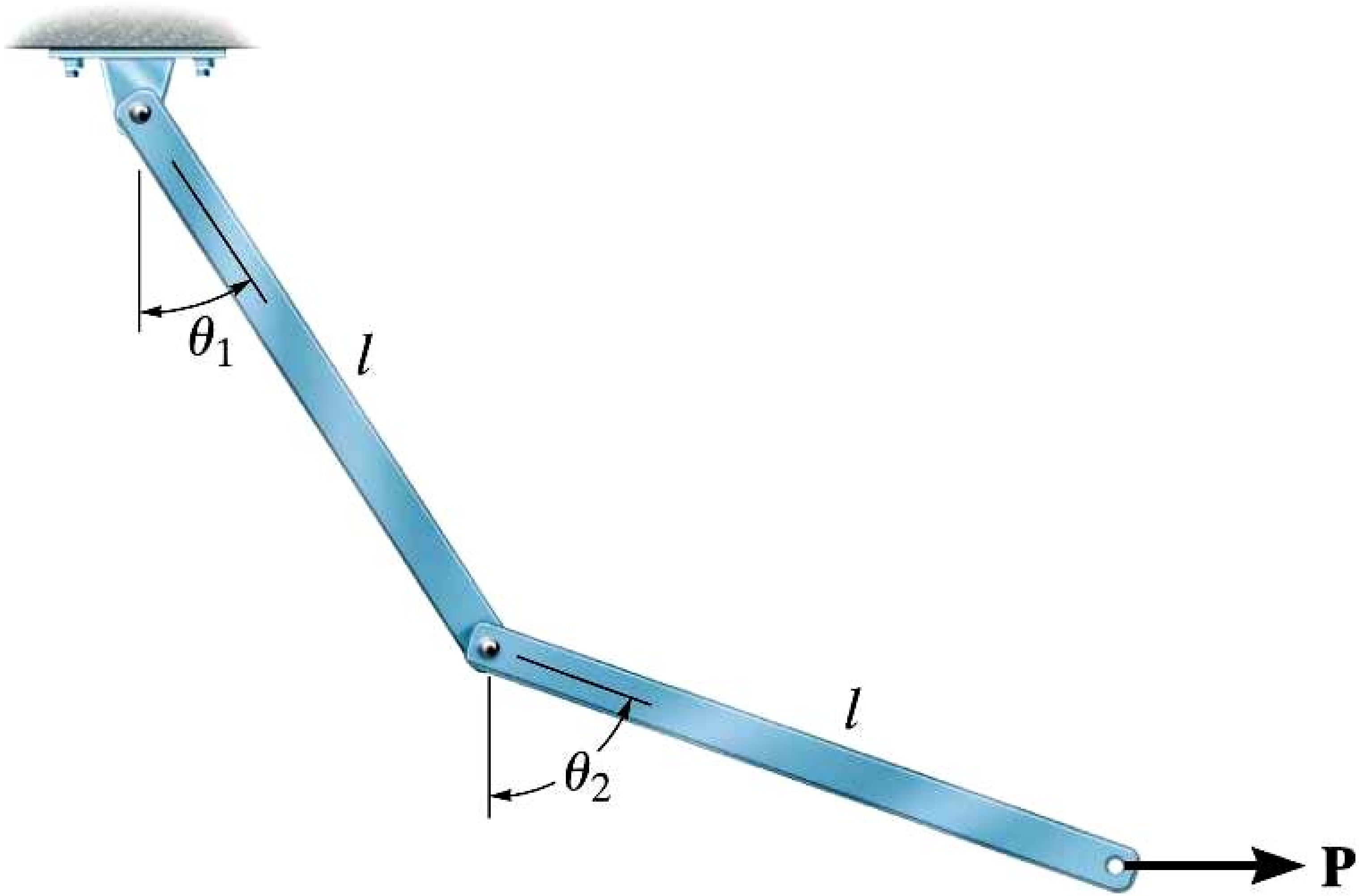
Solving for the positive root yields

$$x_C = 0.981\text{ m}$$

Thus, from Eq. 4,

$$C_x = 114\text{ N}$$ *Ans.*

A horizontal force acts on the end of the link as shown. Determine the angles θ_1 and θ_2 for equilibrium of the two links. Each link is uniform and has a mass m.

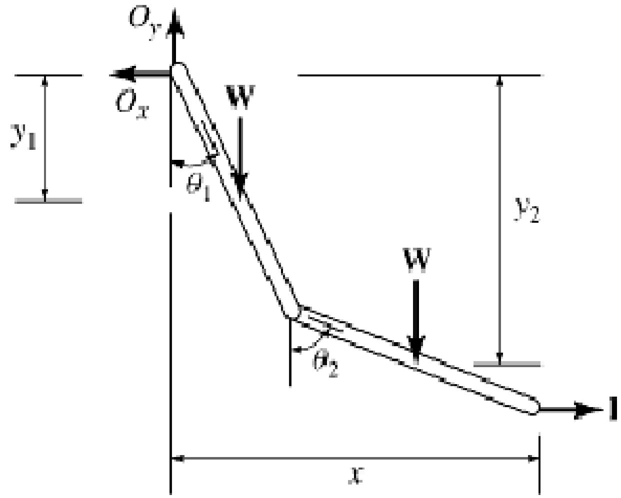


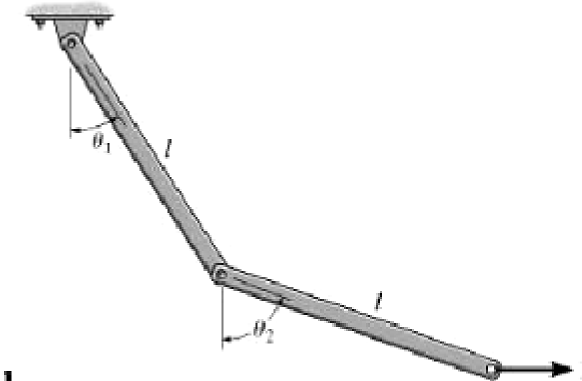
Solution:

$$x = l\sin(\theta_1) + l\sin(\theta_2)$$

$$y_1 = \frac{l}{2}\cos(\theta_1)$$

$$y_2 = l\cos(\theta_1) + \frac{l}{2}\cos(\theta_2)$$





$$\delta x = l\cos(\theta_1)\delta\theta_1 + l\cos(\theta_2)\delta\theta_2$$

$$\delta y_1 = -\frac{l}{2}\sin(\theta_1)\delta\theta_1$$

$$\delta y_2 = -l\sin(\theta_1)\delta\theta_1 - \frac{l}{2}\sin(\theta_2)\delta\theta_2$$

$$\delta U = P\delta_x + mg\delta y_1 + mg\delta y_2 = 0$$

$$\delta U = P\,l\left(\cos(\theta_1)\,\delta\theta_1 + \cos(\theta_2)\,\delta\theta_2\right) - mg\left(\frac{l}{2}\right)\left(3\sin(\theta_1)\delta\theta_1 + \sin(\theta_2)\,\delta\theta_2\right) = 0$$

$$\delta U = \left(P\,l\cos(\theta_1) - \frac{3}{2}mg\,l\sin(\theta_1)\right)\delta\theta_1 + \left(P\,l\cos(\theta_2) - \frac{l}{2}mg\,l\sin(\theta_2)\right)\delta\theta_2 = 0$$

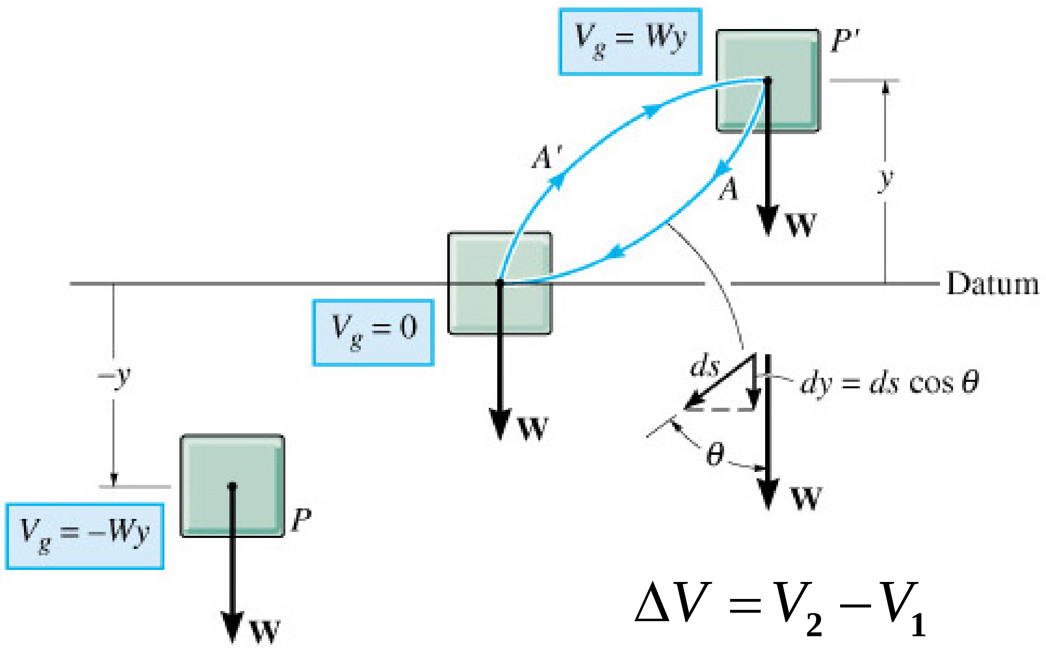
Thus we have 2 equations:

$$P\ l\ \cos(\theta_1) - \frac{3}{2}\ m\ g\ l\ \sin(\theta_1) = 0 \qquad \theta_1 = \operatorname{atan}\left(\frac{2\ P}{3\ m\ g}\right)$$

$$P\ l\ \cos(\theta_2) - \frac{1}{2}\ m\ g\ l\ \sin(\theta_2) = 0 \qquad \theta_2 = \operatorname{atan}\left(\frac{2\ P}{m\ g}\right)$$

Potential Energy

- Gravitational PE　重力位能

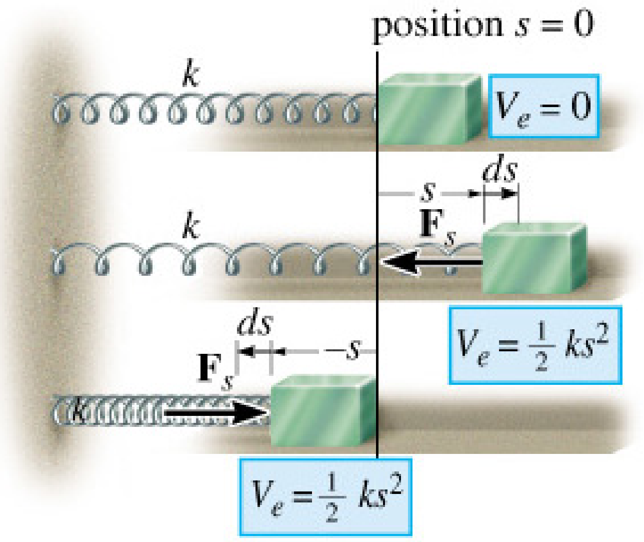


$$\Delta V = V_2 - V_1$$

$$\Delta V_g = -Wy - Wy = -2Wy$$

Potential Energy

- Elastic PE　彈力位能



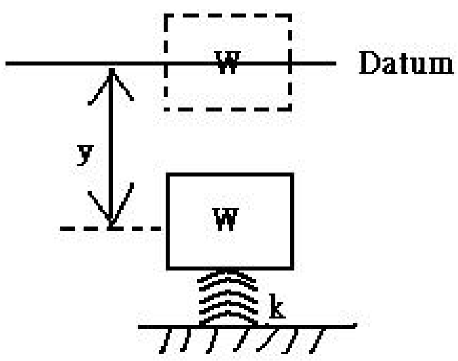
$$F_s = -ks$$

Potential Energy

- Potential Function 位能函數

$$V = V_g + V_e$$

Ex.



$$V = V_g + V_e = -Wy + \frac{1}{2}ky^2$$

Potential Energy

平衡條件

一個自由度：

$$\frac{dV}{dy} = \mathbf{0}$$

如前例.

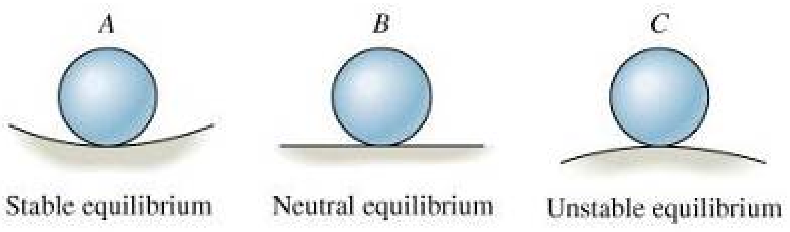
$$\frac{dV}{dy} = -W + ky = \mathbf{0} \qquad y_{eq} = \frac{W}{k}$$

多個自由度：

$$\frac{\partial V}{\partial y_{\mathbf{1}}} = \frac{\partial V}{\partial y_{\mathbf{2}}} = ... = \frac{\partial V}{\partial y_n} = \mathbf{0}$$

Potential Energy

穩定性



$$\begin{cases} \textit{Stable} & \dfrac{dV}{dy}=0,\dfrac{d^2V}{dy^2}>0(\text{在}y_{eq}\text{時有極小值}) \\ \textit{Neutral} & \dfrac{dV}{dy}=\dfrac{d^2V}{dy^2}=.....=0 \\ \textit{Unstable} & \dfrac{dV}{dy}=0,\dfrac{d^2V}{dy^2}<0(\text{在}y_{eq}\text{時有極大值}) \end{cases}$$

Potential Energy

穩定性

Maclaurin series; Taylor's expansion ：

$$V = V_0 + \cancel{\left(\frac{\partial V}{\partial \theta}\right)_0 \theta} + \frac{1}{2}\left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 \theta^2 + \ldots\ldots \qquad \text{(when equilibrium)}$$

$$\therefore V - V_0 = \delta V = \frac{1}{2}\left(\frac{\partial^2 V}{\partial \theta^2}\right)_0 \theta^2 \qquad \leftarrow \text{dominant term}$$

$(= -\delta U)$

Potential Energy

穩定性

兩個自由度：

Stable $\dfrac{\partial V}{\partial q_1} = \dfrac{\partial V}{\partial q_2} = 0$

$$\left[\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)^2 - \left(\frac{\partial^2 V}{\partial q_1{}^2}\right)\left(\frac{\partial^2 V}{\partial q_2{}^2}\right)\right] < 0$$

$$\left(\frac{\partial^2 V}{\partial q_1{}^2} + \frac{\partial^2 V}{\partial q_2{}^2}\right) > 0$$

$$\therefore \frac{\partial^2 V}{\partial q_1{}^2} > 0 \qquad \frac{\partial^2 V}{\partial q_2{}^2} > 0$$

Potential Energy

穩定性

兩個自由度：

Unstable $\dfrac{\partial V}{\partial q_1} = \dfrac{\partial V}{\partial q_2} = 0$

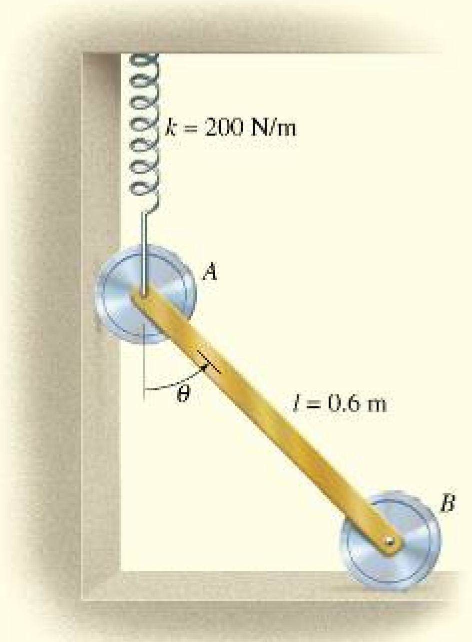
$$\left[\left(\frac{\partial^2 V}{\partial q_1 \partial q_2}\right)^2 - \left(\frac{\partial^2 V}{\partial {q_1}^2}\right)\left(\frac{\partial^2 V}{\partial {q_2}^2}\right)\right] < 0$$

$$\left(\frac{\partial^2 V}{\partial {q_1}^2} + \frac{\partial^2 V}{\partial {q_2}^2}\right) < 0$$

$$\therefore \frac{\partial^2 V}{\partial {q_1}^2} < 0 \qquad \frac{\partial^2 V}{\partial {q_2}^2} < 0$$

EXAMPLE 11.5

The uniform link shown in Fig. 11–16*a* has a mass of 10 kg. The spring is unstretched when $\theta = 0°$. Determine the angle θ for equilibrium and investigate the stability at the equilibrium position.



(a)

Fig. 11–16

Solution

Potential Function. The datum is established at the top of the link when the *spring is unstretched*, Fig. 11–16*b*. When the link is located at the arbitrary position θ, the spring increases its potential energy by stretching and the weight decreases its potential energy. Hence,

$$V = V_e + V_g = \frac{1}{2}ks^2 - W\left(s + \frac{l}{2}\cos\theta\right)$$

Since $l = s + l\cos\theta$ or $s = l(1 - \cos\theta)$, then

$$V = \frac{1}{2}kl^2(1 - \cos\theta)^2 - \frac{Wl}{2}(2 - \cos\theta)$$

Equilibrium Position. The first derivative of V gives

$$\frac{dV}{d\theta} = kl^2(1 - \cos\theta)\sin\theta - \frac{Wl}{2}\sin\theta = 0$$

or

$$l\left[kl(1 - \cos\theta) - \frac{W}{2}\right]\sin\theta = 0$$

EXAMPLE 11.5 2/2

This equation is satisfied provided

$$\sin \theta = 0 \qquad \theta = 0° \qquad \textit{Ans.}$$

$$\theta = \cos^{-1}\left(1 - \frac{W}{2kl}\right) = \cos^{-1}\left[1 - \frac{10(9.81)}{2(200)(0.6)}\right] = 53.8° \qquad \textit{Ans.}$$

Stability. Determining the second derivative of V gives

$$\frac{d^2V}{d\theta^2} = kl^2(1 - \cos\theta)\cos\theta + kl^2 \sin\theta \sin\theta - \frac{Wl}{2}\cos\theta$$

$$= kl^2(\cos\theta - \cos 2\theta) - \frac{Wl}{2}\cos\theta$$

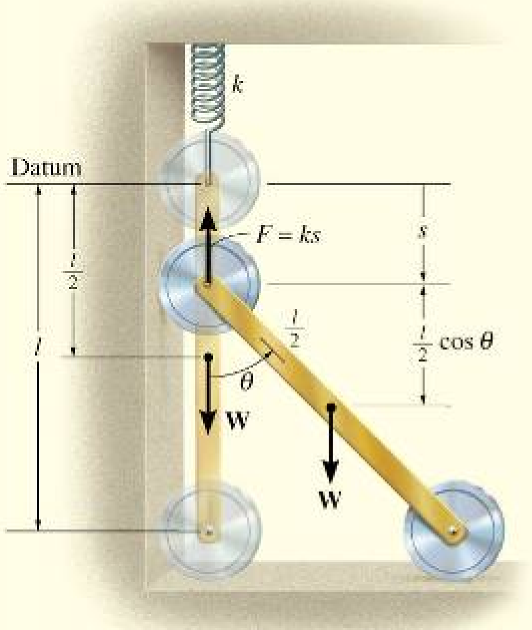
Substituting values for the constants, with $\theta = 0°$ and $\theta = 53.8°$, yields

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=0°} = 200(0.6)^2(\cos 0° - \cos 0°) - \frac{10(9.81)(0.6)}{2}\cos 0° \qquad \textit{Ans}$$

$$= -29.4 < 0 \qquad \text{(unstable equilibrium at } \theta = 0°\text{)}$$

$$\left.\frac{d^2V}{d\theta^2}\right|_{\theta=53.8°} = 200(0.6)^2(\cos 53.8° - \cos 107.6°) - \frac{10(9.81)(0.6)}{2}\cos 53.8°$$

$$= 46.9 > 0 \qquad \text{(stable equilibrium at } \theta = 53.8°\text{)} \qquad \textit{Ans.}$$



(b)

Fig. 11–16