CHAPTER SIX

The Calculus of Residues

§6-1 Singularities and Zeroes

♣ Laurent Series

$$f(z)=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n$$

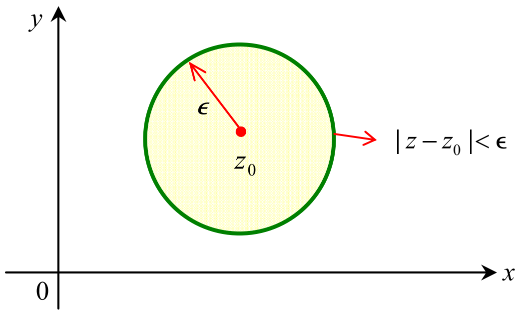
$$\Rightarrow \qquad f(z)=\cdots+c_{-2}(z-z_0)^{-2}+c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)^1+\cdots$$

♣ **Some Definitions:**

1) **Principal part and analytic part**
The portion of the Laurent series containing only the negative powers of $(z-z_0)$ is called principal part; the remainder of the series — the summation of the terms with zero and positive powers — is known as the analytic part.

2) **Pole of order N**
A function whose Laurent expansion about a singular point z_0 has a principal part, in which the most negative power of $(z-z_0)$ is $-N$, is said to have *a pole of order N* at z_0.
♦ A function has a pole of order one, at some point, is said to have a simple pole at that point.

1. Singular Point (Singularity)
A point z_0 is called a singular point of the function *f* (or called a singularity of the function *f*), if *f* is not analytic at $z=z_0$, but every neighborhood of z_0 contains at least one point at which *f* is analytic.



2. Isolated Singularity (Isolated Singular Point).
A point z_0 is an isolated singularity (or an isolated singular point) of the function *f*, if *f* is not analytic at $z=z_0$ but is analytic in a deleted neighborhood of z_0.
That is, *f* is analytic in
$$\Re'(z_0,\epsilon)=\{z\,|\,0<|z-z_0|<\epsilon\ ,\quad z \quad \text{and} \quad \epsilon\ \in C\}$$

♣ **Nonisolated singularity**
A singular point $z=z_0$ of a function $f(z)$ is **nomisolated** if *every* neighborhood of z_0 contains at least one singularity of $f(z)$ other than z_0.

Example 1
The function
$$f(z)=\frac{z}{z^2+4} \quad \Rightarrow \quad z=\pm 2i \ \text{ both are isolated singularities}$$
In other words, $f(z)$ is analytic in the deleted neighborhoods $0<|z-2i|<1$ and $0<|z+2i|<1$.
The function
$$f(z)=\operatorname{Ln} z$$
$\Rightarrow$ The branch point $z=0$ is a non-isolated singularity of $f(z)$ since every neighborhood of

$z=0$ contains points on the negative real axis.

Example 2

The function

$$f(z)=\frac{1}{\sin(\pi/z)}$$

has the singular points $z=0$ and $z=1/n$, $n=\pm1, \pm2, \cdots$, all lying on the segment of the real axis from $z=-1$ to $z=1$.

Each singular point except $z=0$ is isolated.

The singular point $z=0$ is not singular because every deleted ϵ neighborhood of the origin contains other singular points of the function $f(z)$.

3. Removable Singularity

Let $f(z)$ be analytic in the punctured disk

$$0<|z-z_0|<\epsilon$$

and let

$$\lim_{n\to\infty}(z-z_0)f(z)=0$$

Then $f(z)$ has a removable singularity at $z=z_0$.

In other words, $f(z)$ can be assigned a value $f(z)$ at z_0 so that the resulting function is analytic in $|z-z_0|<\epsilon$.

The proof of this theorem can be seen in the textbook " Complex Variable, Levinson / Redheffer" from p.154 to p.154.

Example 3

Consider the function

$$f(z)=\frac{\sin z}{z}$$

1) Because $0/0$ is undefined, the point $z=0$ is singular.

2) Since $\sin z = z - z^3/3! + z^5/5! - + \cdots$, we have

$$\frac{\sin z}{z}=1-\frac{z^2}{3!}+\frac{z^4}{5!}-\cdots \quad \Rightarrow$$ Taylor series on the right represents an analytic function

3) By defining $f(0)=1$, we obtain a function

$$f(z)=\begin{cases}\dfrac{\sin z}{z}, & z\neq 0\\ 1, & z=0\end{cases}$$

which is analytic for all z. The singularity of $f(z)$ at $z=0$ has been removed by an appropriate definition of $f(0)$.

$\Rightarrow$ $z=0$ is a removable singular point.

♣ $z=0$ is also a removable singular point of the function $f(z)=e^{\mathrm{Ln}\,z}$.

4. Pole and Essential Singularity

1) If the Laurent Series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

In the neighborhood of an isolated singular point $z=z_0$ contains only a finite number of negative powers of $z-z_0$, then $z=z_0$ is called a pole of $f(z)$.

If $(z-z_0)^{-m}$ is the highest negative power in the series (expansion), the pole is said to be of order m and the series

$$\sum_{n=1}^{m}\frac{b_n}{(z-z_0)^n}=\frac{b_1}{z-z_0}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_m}{(z-z_0)^m}$$

is called the principal part of $f(z)$ at $z=z_0$.

Then, the Laurent series is rewritten as

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{m}\frac{b_n}{(z-z_0)^n}$$

2) If the Laurent series of $f(z)$ in the neighborhood of an isolated singular point $z=z_0$ contains infinitely many negative powers of $z-z_0$, then

$$z=z_0$$

is called an essential singularity of $f(z)$.

$$\Rightarrow \qquad f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$$

Example 4

The function

$$f(z)=\frac{1}{z(z-1)^2}$$

has singularities at $z=0$ and $z=1$.

1) Laurent series about $z=0$:

$$f(z)=z^{-1}+2+3z+4z^2+\ldots,0<|z|<1$$

$\Rightarrow$ $z=0$ is a pole of order 1.

2) Laurent series about $z=1$:

$$f(z)=(z-1)^{-2}+(z-1)^{-1}+1-(z-1)+(z-1)^2+\cdots,\quad 0<|z-1|<1$$

$\Rightarrow$ $z=1$ is a pole of order 2.

Example 5

The function

$$\sin\frac{1}{z}=z^{-1}-\frac{z^{-3}}{3!}+\frac{z^{-5}}{5!}+\cdots,\quad z\neq 0$$

$\Rightarrow$ $z=0$ is an essential singularity

Example 6

Consider the function

$$\frac{e^{1/(z-1)}}{(z-1)^2}$$

Using $e^u=1+u+\dfrac{u^2}{2!}+\cdots,$ and putting $u=(z-1)^{-1}$, we find that

$$\frac{e^{1/(z-1)}}{(z-1)^2}=(z-1)^{-2}+(z-1)^{-3}+\frac{(z-1^{-4})}{2!}+\cdots,\quad z\neq 1$$

$\Rightarrow$ $z=1$ is an essential singularity

♣ **Three different kinds of principal part:**

1) A principal part with a positive finite number of nonzero terms.

Ex. $c_{-N}(z-z_0)^{-N}+c_{-(N-1)}(z-z_0)^{-(N-1)}+\cdots+c_{-1}(z-z_0)^{-1}$, where $c_{-N}\neq 0$

2) There are an infinite number of nonzero terms in the principal part.

3) There are functions possessing an isolated singular point z_0 such that a sought-after Laurent expansion about z_0 will be found to have no term in the principal part, that is, all the coefficients involving negative powers of $(z-z_0)$ are zero.

5. Some Examples about Pole and Singularity

1) Pole

If

a) $\lim_{z\to z_0}(z-z_0)^m f(z)$ exist and

b) $\lim_{z \to z_0} (z-z_0)^m f(z) \neq 0$

$\Rightarrow$ $z = z_0$ is a pole of order *m* of $f(z)$.

* A pole of order one is a simple pole.

2) The order of the pole for a function of the form $f(z) = g(z)/h(z)$:

If $g(z)$ and $h(z)$ are analytic at z_0, with $g(z_0) \neq 0$ and $h(z_0) = 0$, then $f(z)$ will have an isolated singularity at z_0.

Let

$$h(z) = a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots, \text{ where } a_N \neq 0 \text{ and } N \geq 1 .$$

Using the above equation, we have

$$f(z) = \frac{g(z)}{a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \cdots}$$

To show that this expression has a pole of order *N* at z_0, consider

$$\lim_{z \to z_0} \left[(z-z_0)^N f(z) \right] = \lim_{z \to z_0} \frac{(z-z_0)^N g(z)}{a_N (z-z_0)^N + a_{N+1}(z-z_0)^{N+1} + \ldots}$$

$$= \lim_{z \to z_0} \frac{g(z)}{a_N + a_{N+1}(z-z_0) + \ldots} = \frac{g(z_0)}{a_N} .$$

Since this limit is finite and nonzero, we may conclude the following rule:

Rule I (Quotients):

If $f(z) = g(z)/h(z)$, where $g(z)$ and $h(z)$ are analytic at z_0, and if $h(z_0) = 0$ and $g(z_0) \neq 0$, then the order of the pole of $f(z)$ at z_0 is identical to the order of the zero of $h(z)$ at this point.

♣ For the case of $h(z_0) = 0$ and $g(z_0) = 0$, if

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left[g(z)/h(z) \right] = \infty$$

then $f(z)$ has a pole at z_0, whereas if

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \left[g(z)/h(z) \right] = \text{finite}$$

$f(z)$ has a removable singularity at z_0.

L'Hôspital rule is often useful in finding the limit.

Rule II (Quotients):

The order of the pole of $f(z) = g(z)/h(z)$ at z_0 is the order of the **zero** of $g(z)$ at this point **less** the order of the zero of $g(z)$ at the same point.

Example 7

The function *f* defined by

$$f(z) = \frac{2z+3}{(z-1)(z-2)^3}$$

Since

$$\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} \frac{2z+3}{(z-2)^3} = -5 \neq 0$$

$\Rightarrow$ $z = 1$ is a simple pole of $f(z)$.

and

$$\lim_{z \to 2} (z-2)^3 f(z) = \lim_{z \to 2} \frac{2z+3}{z-1} = 7 \neq 0$$

$\Rightarrow$ $z = 2$ is a pole of order 3 of $f(z)$.

♣ 在遇到三角函數時，不可直接冒然判斷其 order。

Example 8

The function *f* defined by

$$f(z) = \frac{\sin z}{z^4} \qquad \text{for all } |z| < 0$$

Hence $\lim_{z\to 0} z^4 f(z) = \lim_{z\to 0} \sin z = 0$

The limit of $\lim_{z\to 0} z^4 f(z)$ exists, but equal to 0. Therefore, we can not say that $z=0$ is a pole of order four of the function $f(z)$.

However, if we take

$$\lim_{z\to 0} z^3 f(z) = \lim_{z\to 0} \frac{\sin z}{z} = 1 \neq 0$$

$\Rightarrow$ $z=0$ is a pole of order 3 of the function $f(z)$.

We can also use following method to determine the order of $z=0$：

$$f(z) = \frac{\sin z}{z^4} = \frac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots}{z^4}$$

$$= \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{z}{5!} - \frac{z^3}{7!} + \cdots$$

因為在 $\frac{1}{z^3}$ 此項以前之項的係數均為 0，故知

$\Rightarrow$ $z=0$ is a pole of order 3 of $f(z)$.

Example 9

Discuss the singularity of

$$f(z) = \frac{z\cos z}{(z-1)(z^2+1)^2(z^2+3z+2).}$$

<Sol.>

Factoring the denominator, we have

$$f(z) = \frac{z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)}$$

There is a pole of order 1 at $z=1$ since

$$\lim_{z\to 1}\left[(z-1)f(z)\right] = \lim_{z\to 1} \frac{(z-1)z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)} = \frac{\cos 1}{24}$$

which is finite and nonzero.

There is a pole of order 1 at $z=-i$ since

$$\lim_{z\to -i}\left[(z+i)^2 f(z)\right] = \lim_{z\to 1-i} \frac{(z+i)^2 z\cos z}{(z-1)(z+i)^2(z-i)^2(z+2)(z+1)} = \frac{-i\cos(-i)}{8(2-i)}.$$

Similarly, there is a pole of order 2 at $z=i$ and poles of order 1 at $z=-2$ and $z=-1$.

Example 10

Discuss the singularity of

$$f(z) = \frac{e^z}{\sin z}.$$

<Sol.>

Wherever $\sin z = 0$, that is, for $z = k\pi$, $k = 0, \pm 1, \pm 2, \cdots$, $f(z)$ has isolated singularities.

Assuming these are simple poles, we evaluate

$$\lim_{z\to k\pi}\left[(z-k\pi)e^z/\sin z\right]$$

This indeterminate form is evaluated from L'Hôspital's rule and equals

$$\lim_{z\to k\pi} \frac{(z-k\pi)e^z + e^z}{\cos z} = \frac{e^{k\pi}}{\cos k\pi}.$$

Because this result is finite and nonzero, the pole at $z=k\pi$ is of first order.

Example 11

Find the order of the pole of $(z^2+1)/(e^z+1)$ at $z=i\pi$.

<Sol.>

With $g(z)=(z^2+1)$ and $h(z)=(e^z+1)$, we verify that

$$g(i\pi)=-\pi^2+1\neq 0 \quad \text{and} \quad h(i\pi)=(e^{i\pi}+1)=0$$

To find the order of zero of $h(z)=(e^z+1)$ at $z=i\pi$, we make the Taylor expansion

$$h(z)=e^z+1=c_0+c_1(z-i\pi)+c_2(z-i\pi)^2$$

Note that $c_0=0$ because at $h(i\pi)=0$.

Since

$$c_1=\frac{d}{dz}(e^z+1)\Big|_{z=i\pi}=-1\neq 0$$

We see that $h(z)$ has a zero of order 1 at $z=i\pi$. Thus, by **Rule I**, $f(z)$ has a pole of order 1 at $z=i\pi$.

Example 12

Find the order of the pole of

$$f(z)=\frac{\sinh z}{\sin^5 z} \quad \text{at} \quad z=0.$$

<Sol.>

With $g(z)=\sinh z$ and $h(z)=\sin^5 z$, we find that

$$g(0)=0 \quad \text{and} \quad h(0)=0$$

Since

$$g(z)=\sinh z=z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots$$

We see that $g(z)$ has a zero of order 1 at $z=0$.

Because

$$h(z)=\sin^5 z=\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)^5=z^5-\frac{5z^7}{3!}+\cdots$$

We see that the lowest power of z in the Maclaurin series for $h(z)=\sin^5 z$ is 5. Thus, $\sin^5 z$ has a zero of order 5 at $z=0$. The order of the pole of $f(z)$ at $z=0$ is, by Rule II, the order of the zero of $\sin^5 z$ less than the order of the zero of $\sinh z$, that is, $5-1=4$.

Example 13

Find the poles and establish their order for the function

$$f(z)=\frac{1}{(\operatorname{Ln} z-i\pi)(z^{1/2}-1)}.$$

<Sol.>

Both the principal branch of $z^{1/2}$ and $\operatorname{Ln} z$ are analytic in the cut plane defined by the branch cut

$$y=0, \quad -\infty\leq x\leq 0$$

This cut plane is the domain of analyticity of $f(z)$.

If $\operatorname{Ln} z-i\pi=0$, we have

$$\operatorname{Ln} z=i\pi \quad \text{or} \quad z=e^{i\pi}=-1$$

This condition can not occur in the domain. Alternatively, we can say that $z=-1$ is not an isolated singular point of $f(z)$ since it lies on the branch cut containing all nonisolated singular points of $f(z)$.

Consider $z^{1/2}-1=0$, or

$$z^{1/2}=1$$

Squaring, we get $z=1$. Since the principal value of $1^{1/2}$ is 1, we see that $f(z)$ has an isolated singular point at $z=1$. Now

$$z^{1/2}-1=\sum_{n=0}^{\infty}c_n(z-1)^n.$$

We readily find that

$$c_0 = 0 \quad \text{and} \quad c_1 = [(1/2)/z^{1/2}]\Big|_{z=1}$$

This shows that $z^{1/2} - 1$ has a zero of order 1 at $z = 1$.
Since

$$f(z) = \frac{\left(\dfrac{1}{\operatorname{Ln} z - i\pi}\right)}{z^{1/2} - 1}$$

and the numerator of this expansion is analytic at $z = 1$ while the denominator $z^{1/2} - 1$ has a zero of order 1 at the same point, the given $f(z)$ must, by **Rule I**, have a simple pole at $z = 1$.

♣ Comments on the term "pole"

1) If $f(z)$ has a pole at $z = z_0$, then $\lim_{z \to z_0} f(z) = \infty$.

2) Equivalently, $|f(z)| \to$ unbounded as $z \to z_0$.

3) Suppose $f(z)$ has a pole of order N at $z = z_0$. Then

$$f(z) = c_{-N}(z - z_0)^{-N} + c_{-(N-1)}(z - z_0)^{-(N-1)} + \cdots + c_0 + c_1(z - z_0) + \cdots,$$

$$\Rightarrow \qquad (z - z_0)^N f(z) = c_{-N} + c_{-(N-1)}(z - z_0) + \cdots + c_0(z - z_0)^{N+1} + \cdots$$

Dividing both sides of this equation by $(z - z_0)^N$, we have this representation of $f(z)$ in a neighborhood of $z = z_0$:

$$f(z) = \psi(z)/(z - z_0)^N$$

where

$$\psi(z) = c_{-N} + c_{-(N-1)}(z - z_0) + c_{-(N-2)}(z - z_0) + \cdots$$

and $c_{-N} \neq 0$.

Thus, as $z \to z_0$, we have

$$f(z) \approx \frac{\psi(z_0)}{(z - z_0)^N} = \frac{c_{-N}}{(z - z_0)^N},$$

and so

$$|f(z)| \approx \frac{|c_{-N}|}{|(z - z_0)|^N}.$$

We see that the higher the order N of the pole, the more steeply will be the surface depicting $|f(z)|$ rise as $z \to z_0$.

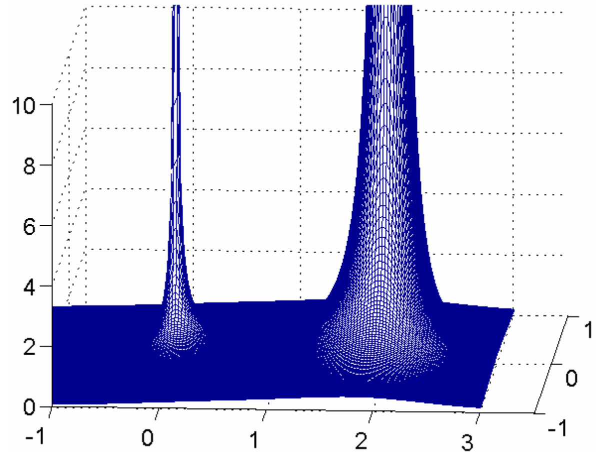
Example 14

Using MATLAB, obtain a plot of $|f(z)| = |1/[z(z-2)^2]|$ and make comments on the poles of $f(z)$.

<Sol.>

MATLAB commands:

```
% plot for f(z)
x=linspace(-3,3,200);
y=linspace(-1,1,200);
[X,Y]=meshgrid(x,y);
z=X+i*Y;
f=z.*(z-2).^2;
f=1./f;
f=abs(f);mesh(X,Y,f);AXIS([-1.0 3.5 -1 1 0 10]);
view(10,15)
```



H.W. 1 (a) Using MATLAB, obtain a plot similar to that of **Example 14** but use the function

$$f(z)=\frac{1}{(z+2)^2(z-2)^2 z}$$

which has two poles of order 2, and a simple pole.

(b) We have observed that the magnitude of a function does not have a limit of infinity at an essential singularity. Illustrate this by generating a MATLAB plot of the magnitude of $e^{1/z}$ in the region $0<|z|\le 1$. For a useful plot, it is best not to let $|z|$ to get too close to zero — as you will see after some experimentation.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 7, Exercise 6.2, Pearson Education, Inc., 2005.】

H.W. 2 A nonisolated essential singular point of a function is a singular point whose every neighborhood contains an infinite number of isolated singular points of the function. An example is $f(z)=1/\sin(1/z)$, whose nonisolated essential singular point is at $z=0$.

(a) Show that in the domain $|z|<\epsilon$ $(\epsilon>0)$, there are plots at

$$z=\pm 1/n\pi,\ \pm 1/(n+1)\pi,\ \pm 1/(n+2)\pi,\cdots$$

where n is an integer such that $n>1/(\pi\epsilon)$.

(b) Is there a Laurent expansion of $f(z)$ in a deleted neighborhood of $z=0$?

(c) Find a different function with nonisolated essential singular point at $z=0$. Prove that is has an infinite number of isolated singular points inside $|z|=\epsilon$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 36, Exercise 6.2, Pearson Education, Inc., 2005.】

§6-2 Residue Theorem

♣ **Definition of Residue**

Let $f(z)$ be analytic on a simple closed contour C and at all points interior to C except for the point z_0. Then, the residue of $f(z)$ at z_0, written $\operatorname*{Res}_{z=z_0} f(z)$ or $\operatorname{Res}[f(z),z_0]$, is defined by

$$\operatorname*{Res}_{z=z_0} f(z)=\operatorname{Res}[f(z),z_0]=\frac{1}{2\pi i}\oint_C f(z)dz$$

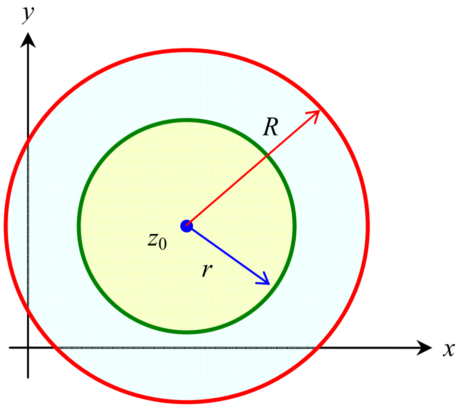
♣ Connection between $\operatorname*{Res}_{z=z_0} f(z)$ and the Laurent series of $f(z)$

1) Laurent expansion of $f(z)$:

$$f(z)=\cdots+c_{-2}(z-z_0)^{-2}+c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)^1+\cdots,\quad 0<|z-z_0|<R$$

2) Residue of $f(z)$ at $z=z_0$:

$$\mathrm{Res}[f(z),z_0] = \frac{1}{2\pi i}\oint_C f(z)dz$$
$$= \frac{1}{2\pi i}\oint_{|z-z_0|=r} \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n dz$$
$$= \frac{1}{2\pi i}\sum_{n=-\infty}^{\infty} c_n \oint_{|z-z_0|=r} (z-z_0)^n dz$$



Note that

$$\oint_{|z-z_0|=r} (z-z_0)^n dz = \begin{cases} 0, & n \neq -1 \\ 2\pi i, & n = -1 \end{cases}$$

Thus, we have

$$\mathrm{Res}[f(z),z_0] = c_{-1}$$

♣ **Theorem**

The residue of function $f(z)$ at the isolated singular point z_0 is the coefficient of $(z-z_0)^{-1}$ in the Laurent series representing $f(z)$ in an annulus $0<|z-z_0|<R$.

Example 1

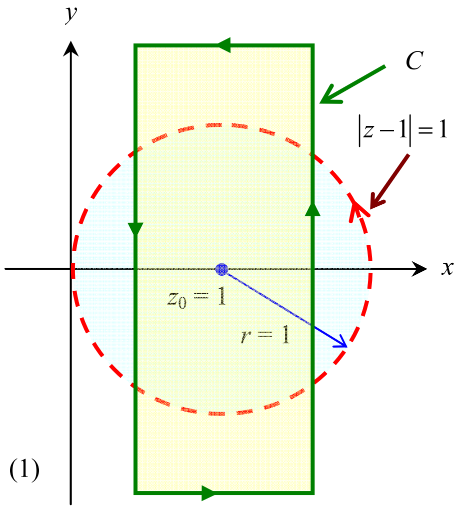
Let

$$f(z) = \frac{1}{z(z-1)}$$

Using the above theorem, find

$$\frac{1}{2\pi i}\oint_C f(z)dz$$

where C is the contour shown in the left figure.



<Sol.>

From the shown figure, we know that $z=1$ is an isolated singularity. We consider two Laurent series expansions of $f(z)$ that converge in annular regions centered at $z=1$:

$$\frac{1}{z(z-1)} = (z-1)^{-1} - 1 + (z-1) - (z-1)^2 + \cdots, \quad 0<|z-1|<1 \quad (1)$$

$$\frac{1}{z(z-1)} = (z-1)^{-2} - (z-1)^{-3} + (z-1)^{-4} - \cdots, \quad |z-1|>1 \quad (2)$$

The series of Eq. (1), which applies around the point $z=1$, is of use to us here. The coefficient of $(z-1)^{-1}$ is 1, which means that

$$\mathrm{Res}\left[\frac{1}{z(z-1)},1\right] = 1$$

Thus, we have

$$\frac{1}{2\pi i}\oint_C f(z)dz = 1$$

♣ If the contour C now encloses only the singular point $z=0$ (for example, C: $|z|=1/2$), then our solution would require that we extract the residue at $z=0$ from the expansion:

$$\frac{1}{z(z-1)} = -z^{-1} - 1 - z - z^2 - \cdots, \quad 0<|z|<1$$

$$\Rightarrow \quad \mathrm{Res}[f(z),0] = -1$$

Thus, for this new contour we have

$$\frac{1}{2\pi i}\oint_C \frac{dz}{z(z-1)} = -1$$

Example 2

Find $\frac{1}{2\pi i}\oint_C z\sin(1/z)dz$ integrated around $|z|=2$.

<Sol.>

The point $z=0$ is an isolated singularity of $\sin(1/z)$ and lies inside the given contour of integration.

Laurent expansion of $z\sin(1/z)$ is

$$z\sin(\frac{1}{z})=1-\frac{(1/z)^2}{3!}+\frac{(1/z)^4}{5!}-\cdots,\quad |z|>0$$

Since the coefficient of the term $1/z$ is zero, we have

$$\text{Res}\left[\left(z\sin(\frac{1}{z})\right),0\right]=0$$

Thus, we have

$$\frac{1}{2\pi i}\oint_C z\sin(1/z)dz=0$$

1. Three Theorems for finding Residue

1) A formula for the residue in the case of a simple pole.

If $z=z_0$ is a simple pole of $f(z)$

$\Rightarrow$ $\underset{z=z_0}{\text{Res}}\, f(z)=\lim_{z\to z_0}(z-z_0)f(z)$

<pf.> Since $z=z_0$ is a simple pole of $f(z)$, hence the corresponding Laurent series is

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{(z-z_0)}$$

$$=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots+\frac{b_1}{(z-z_0)}$$

where $b_1\neq 0$.

Multiplying both sides of the above equation by $z-z_0$, we have

$$(z-z_0)f(z)=b_1+(z-z_0)[a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots]$$

If we let z approach z_0, then

$$\lim_{z\to z_0}(z-z_0)f(z)$$

$$=\lim_{z\to z_0}\left[\sum_{n=0}^{\infty}a_n(z-z_0)^{n+1}+b_1\right]$$

$$=b_1=\underset{z=z_0}{\text{Res}}\, f(z)$$

2) Let the function f be defined by

$$f(z)=\frac{p(z)}{q(z)}$$

where the function $p(z)$ and $q(z)$ are both analytic at $z=z_0$, and $p(z_0)\neq 0$.

Then, f has a pole of order m at $z=z_0$ if and only if z_0 is a pole of order m of q.

In particular, if $z=z_0$ is a simple pole of $f(z)$, we have

$\Rightarrow$ $\underset{z=z_0}{\text{Res}}\, f(z)=\frac{p(z)}{q'(z)}$

<pf.> Since $z=z_0$ is a simple pole of $f(z)$

$\Rightarrow$ $q(z_0)=0$ and $q'(z_0)\neq 0$

Consequently, $q(z)$ can be expanded in a Taylor's series of the form：

$$q(z)=q(z_0)+q'(z_0)(z-z_0)+\frac{q''(z_0)}{2!}(z-z_0)^2+\cdots\cdots$$
$$=q'(z_0)(z-z_0)+\frac{q''(z_0)}{2!}(z-z_0)^2+\cdots\cdots$$

Then, we can see that

$$\operatorname*{Res}_{z=z_0} f(z)=\lim_{z\to z_0}(z-z_0)f(z)$$
$$=\lim_{z\to z_0}(z-z_0)\frac{p(z)}{q'(z_0)(z-z_0)+\frac{q''(z_0)}{2!}(z-z_0)^2+\cdots\cdots}$$
$$=\frac{p(z_0)}{q(z_0)}$$

3) Suppose that function f is analytic in a deleted neighborhood of z_0, and if $z=z_0$ is a pole of order m of $f(z)$.

Then, we have

$$\operatorname*{Res}_{z=z_0} f(z)=\lim_{z\to z_0}\frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

<pf.> Since $z=z_0$ is a pole of order m of $f(z)$. Then the corresponding Laurent expansion is of the form

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n+\frac{b_1}{(z-z_0)}+\frac{b_2}{(z-z_0)^2}+\cdots+\frac{b_m}{(z-z_0)^m}$$

where $b_m\neq 0$, and the series converges in some neighborhood of $z=z_0$, except at the point z_0 itself.

By multiply both sides of the above equation by $(z-z_0)^m$, we obtain

$$(z-z_0)^m f(z)$$
$$=\sum_{n=0}^{\infty}a_n(z-z_0)^{n+m}+b_1(z-z_0)^{m-1}+b_2(z-z_0)^{m-2}+\cdots+b_m$$

This shows that the residue b_1 of $f(z)$ at $z=z_0$ is now the coefficient of $(z-z_0)^{m-1}$ in the series of $(z-z_0)^m f(z)$.

Then, we have

$$\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$
$$=\sum_{n=0}^{\infty}a_n(n+m)(n+m-1)\cdots(n+2)(z-z_0)^{n+1}+(m-1)!\ b_1$$

Hence, we obtain that

$$\lim_{z\to z_0}\left[\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]\right]$$
$$=(m-1)!\ b_1$$
$$=(m-1)!\ b_1\operatorname*{Res}_{z=z_0} f(z)$$

That is,

$$\operatorname*{Res}_{z=z_0} f(z)=\frac{1}{(m-1)!}\lim_{z\to z_0}\frac{d^{m-1}}{dz^{m-1}}[(z-z_0)^m f(z)]$$

Example 1

Find the residue of the function

$$f(z)=\frac{4-3z}{z^2-z}=\frac{4-3z}{z(z-1)}$$

<Sol.> Since the function $f(z)$ has simple poles at $z=0$ and $z=1$

$$\Rightarrow \quad \operatorname*{Res}_{z=0} f(z) = \left.\frac{4-3z}{2z-1}\right|_{z=0} = -4 \quad \text{and} \quad \operatorname*{Res}_{z=1} f(z) = \left.\frac{4-3z}{2z-1}\right|_{z=1} = 1$$

Example 2

Find the residue of $f(z) = \dfrac{e^z}{(z^2+1)z^2}$ at all poles.

<Sol.>

We rewrite $f(z)$ with a factored denominator

$$f(z) = \frac{e^z}{(z+i)(z-i)z^2}$$

$\Rightarrow$ simple pole: $z = \pm i$; pole of order 2: $z = 0$

Residue at $z = i$:

$$\operatorname*{Res}_{z=i} f(z) = \operatorname{Res}[f(z), i] = \lim_{z\to i} \frac{(z-i)e^z}{(z+i)(z-i)z^2} = \frac{e^i}{(2i)(-1)}$$

Residue at $z = -i$:

$$\operatorname*{Res}_{z=-i} f(z) = \operatorname{Res}[f(z), -i] = \lim_{z\to -i} \frac{(z+i)e^z}{(z+i)(z-i)z^2} = \frac{e^{-i}}{(2i)}$$

Residue at $z = 0$: Let $g(z) = e^z$ and $h(z) = z^2(z^2+1)$, then

$$\operatorname*{Res}_{z=0} f(z) = \operatorname{Res}[f(z), 0] = \lim_{z\to i} \frac{d}{dz} \frac{z^2 e^z}{(z^2+1)z^2} = \lim_{z\to i} \frac{(z^2+1)e^z - 2ze^z}{(z^2+1)^2} = 1$$

Example 3

Find the residue of $f(z) = \dfrac{\tan z}{z^2+z+1}$ at all singularities of $\tan z$.

<Sol.>

We rewrite $f(z)$ as

$$f(z) = \frac{\sin z}{\left(z^2+z+1\right)\cos z}$$

There are poles of $f(z)$ for z satisfying $\cos z = 0$, that is,

$$z = \pi/2 + k\pi, \quad k = 0, \ \pm 1, \ \pm 2, \ \cdots$$

By expanding $\cos z$ in a Taylor series about the point $z_0 = \pi/2 + k\pi$, we have

$$\cos z = a_1(z-z_0) + a_2(z-z_0)^2 + \cdots, \text{ where } \ a_1 = -\cos k\pi \neq 0$$

Since $\cos z$ has a zero of order 1 at $z = z_0$, $f(z)$ must have a pole of order 1 there.

Taking $g(z) = \sin z/(z^2+z+1)$ and $h(z) = \cos z$, we have $h'(z) = -\sin z$. Thus,

$$\operatorname*{Res}_{z_0} f(z) = \operatorname{Res}[f(z), \pi/2 + k\pi] = -\left.\frac{1}{z^2+z+1}\right|_{\pi/2+k\pi}$$

$$= \frac{-1}{(\pi/2+k\pi)^2 + (\pi/2+k\pi) + 1}, \quad k = 0, \ \pm 1, \ \pm 2, \ \cdots$$

♣ There are also poles of $f(z)$, for z satisfying $z^2 + z + 1 = 0$, i.e.,

$$z_1 = -1/2 + i\sqrt{3}/2 \quad \text{and} \quad z_2 = -1/2 - i\sqrt{3}/2$$

Residue at z_1 :

$$\operatorname*{Res}_{z_1} f(z) = \operatorname{Res}\left[f(z), -1/2 + i\sqrt{3}/2\right] = \tan(-1/2 + i\sqrt{3}/2)/(i\sqrt{3})$$

Residue at z_2 :

$$\operatorname*{Res}_{z_2} f(z) = \operatorname{Res}\left[f(z), -1/2 - i\sqrt{3}/2\right] = \tan(-1/2 - i\sqrt{3}/2)/(-i\sqrt{3})$$

Example 4

Find the residue of $f(z) = \dfrac{z^{1/2}}{z^3 - 4z^2 + 4z}$ at all poles. Use the principal branch of $z^{1/2}$.

<Sol.>

We factor the denominator of $f(z)$ and obtain

$$f(z)=\frac{z^{1/2}}{z(z-2)^2}$$

There is no simple pole at $z=0$ since the factor $z^{1/2}$ has a branch point at this value of z that in turn causes $f(z)$ to have a branch point there.

However, $f(z)$ does have a pole of order 2 at $z=2$.

Residue at $z=2$:

$$\operatorname*{Res}_{z=2} f(z)=\operatorname{Res}\left[f(z),2\right]=\lim_{z\to i}\frac{d}{dz}\left[\frac{(z-2)^2 z^{1/2}}{z(z-2)^2}\right]=\frac{-1}{4(2)^{1/2}}$$

where, because we are using the principal branch of the square root, $2^{1/2}$ is chosen positive.

Example 5

Find the residue of $f(z)=\dfrac{e^{1/z}}{1-z}$ at all singularities.

<Sol.>

There is a simple pole at $z=1$. The residue at $z=1$ is

$$\operatorname*{Res}_{z=1} f(z)=\operatorname{Res}\left[f(z),1\right]=e$$

Since

$$e^{1/z}=1+z^{-1}+\frac{z^{-2}}{2!}+\cdots$$

has an essential singularity at $z=0$, this will also be true of $f(z)=e^{1/z}/(1-z)$.

Reside of $f(z)$ at $z=0$:

Laurent expansion:

$$\frac{1}{1-z}=1+z+z^2+\cdots,\quad |z|<1$$

we have

$$f(z)=\frac{e^{1/z}}{1-z}=\left(1+z^{-1}+\frac{z^{-2}}{2!}+\cdots\right)\left(1+z+z^2+\cdots\right)$$

$$=\cdots+c_{-2}z^{-2}+c_{-1}z^{-1}+c_0+\cdots$$

If we multiply the two series together and confine our attention to products resulting in z^{-1}, we have

$$c_{-1}z^{-1}=\left[1+\frac{1}{2!}+\frac{1}{3!}+\cdots\right]z^{-1}$$

Thus, we see that

$$\operatorname*{Res}_{z=0} f(z)=\operatorname{Res}\left[f(z),0\right]=c_{-1}=1+\frac{1}{2!}+\frac{1}{3!}+\cdots=e-1$$

where recalling the definition $e^1=1+\dfrac{1}{1!}+\dfrac{1}{2!}+\dfrac{1}{3!}+\cdots$.

Example 6

Find the residue of $f(z)=\dfrac{e^z-1}{\sin^3 z}$ at $z=0$.

<Sol.>

To establish the order of the pole, we will expand both these expressions in Maclaurin series by the usual means

$$e^z-1=z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots,\quad \sin^3 z=z^3-\frac{z^5}{2}+\cdots$$

Thus,

$$f(z)=\frac{e^z-1}{\sin^3 z}=\frac{z+\dfrac{z^2}{2!}+\dfrac{z^3}{3!}+\cdots}{z^3-\dfrac{z^5}{2}+\cdots}$$

Since the numerator has a zero of order 1 and the denominator has a zero of order 3, the quotient has a pole of order 2.

Reside of $f(z)$ at $z=0$:

Method I: Successive applications of L'Hôspital's Rule, we see that

$$\operatorname*{Res}_{z=0} f(z)=\operatorname{Res}\left[f(z),0\right]=\frac{1}{2}$$

Method II: Using long division, we have

$$\begin{array}{r|l} & z^{-2}+\dfrac{z^{-1}}{2}+\cdots \\ \hline z^{3}-\dfrac{z^{5}}{2}+\cdots & z+\dfrac{z^{2}}{2!}+\dfrac{z^{3}}{3!}+\cdots \end{array}$$

$$\Rightarrow \qquad \operatorname*{Res}_{z=0} f(z)=\operatorname{Res}\left[f(z),0\right]=\text{coefficient} \quad \text{of} \quad z^{-1}=\frac{1}{2}$$

H.W. 1 For each of the following functions state the location and order of each pole and find the corresponding residue.

(a) $f(z)=\dfrac{\sin z-z}{z\sinh z}$

(b) $f(z)=\dfrac{1}{\sin z^{2}}$

(c) $f(z)=\dfrac{\cos(1/z)}{\sin z}$

(d) $f(z)=\dfrac{1}{e^{2z}+e^{z}+1}$

(e) $f(z)=\dfrac{1}{\left[\operatorname{Ln}(z/e)-1\right]^{2}}$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 8, 10, 11, 13, and 14, Exercise 6.3, Pearson Education, Inc., 2005.】

H.W. 2 Find the residue of the following functions at the indicated point:

(a) $f(z)=\dfrac{1}{\cos\left(\dfrac{\pi}{2}e^{z}+\sin z\right)}$, at $z=0$

(b) $f(z)=\dfrac{1}{\sin\left[z\left(e^{z}-1\right)\right]}$, at $z=0$

(c) $f(z)=\dfrac{\cos\left(z-1\right)}{z^{10}}+\dfrac{2}{z-1}$, at $z=1$ and $z=0$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 22-25, Exercise 6.3, Pearson Education, Inc., 2005.】

H.W. 3 (a) Consider the analytic function $f(z)=g(z)/h(z)$, having a pole at z_0. Let $g(z_0)\neq 0$, $h(z_0)=h'(z_0)=0$, $h''(z_0)\neq 0$. Thus, $f(z)$ has a pole of second order at $z=z_0$. Show that

$$\operatorname*{Res}_{z=z_0} f(z)=\operatorname{Res}\left[f(z),z_0\right]=\frac{2g'(z_0)}{h'(z_0)}-\frac{2}{3}\frac{g(z_0)h''(z_0)}{\left[h'(z_0)\right]^{2}}$$

(b) Use the formula of part (a) to obtain

$$\operatorname{Res}\left[\frac{\cos z}{\left(\operatorname{Ln} z-1\right)^{2}},\ e\right]$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 15, Exercise 6.3, Pearson Education, Inc., 2005.】

H.W. 4 Consider the function $f(z)=1/\sin z$. Here we use residues to obtain

$$\frac{1}{\sin z} = \sum_{n=-\infty}^{\infty} c_n z^n \quad \text{for} \quad \pi < |z| < 2\pi \quad \text{(an annular domain)}$$

(a) Use the Laurent's theorem to show that

$$c_n = \frac{1}{2\pi i} \oint \frac{1}{z^{n+1} \sin z} dz$$

where the integral can be around $|z| = R$, $\pi < R < 2\pi$.

(b) Show that

$$c_n = d_n + e_n + f_n$$

where

$$d_n = \text{Res}\left[\frac{1}{z^{n+1}\sin z},\ 0\right], \quad e_n = \text{Res}\left[\frac{1}{z^{n+1}\sin z},\ \pi\right], \quad f_n = \text{Res}\left[\frac{1}{z^{n+1}\sin z},\ -\pi\right]$$

(c) Show that for $n \le -2$, $d_n = 0$.

(d) Show that $e_n + f_n = 0$ when n is even, and $e_n + f_n = -2/\pi^{n+1}$ for n odd.

(e) Show that $c_n = 0$ when n is even, and that $c_{-1} = -1$, $c_1 = -2/\pi^2 + 1/6$, $c_3 = -2/\pi^4 + 7/360$, and $c_{nn} = -2/\pi^{n+1}$ for $n \le -3$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 41, Exercise 6.3, Pearson Education, Inc., 2005.】

H.W. 5 If a function $f(z)$ is analytic everywhere in the finite plane except for a finite number of singular points interior to a positively oriented simple closed contour *C*, show that

$$\oint_C f(z)dz = 2\pi i \operatorname*{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

【本題摘自：James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications*, 7rd ed., Sec. 64, pp. 228-229, McGraw Hill, Inc., 2003.】

H.W. 6 (a) Let $n \ge 1$ be an integer. Show that the n poles of

$$\frac{1}{z^n + z^{n-1} + z^{n-2} + \cdots + 1}$$

are at $\cos(2k\pi/(n+1)) + i\sin(2k\pi/(n+1))$, $k = 1, 2, \cdots, n$.

(b) Show that the poles are simple.

(c) Show that the residue at $\cos(2k\pi/(n+1)) + i\sin(2k\pi/(n+1))$ is

$$\frac{\cos(2k\pi/(n+1)) + i\sin(2k\pi/(n+1)) - 1}{(n+1)\left[\cos(2k\pi n/(n+1)) + i\sin(2k\pi n/(n+1))\right]}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 26, Exercise 6.3, Pearson Education, Inc., 2005.】

§6-3 Cauchy's Residue Theorem

1. **Cauchy's Residue Theorem**

If

1) *C* is a simple closed curve (or contour) in a domain *D*.

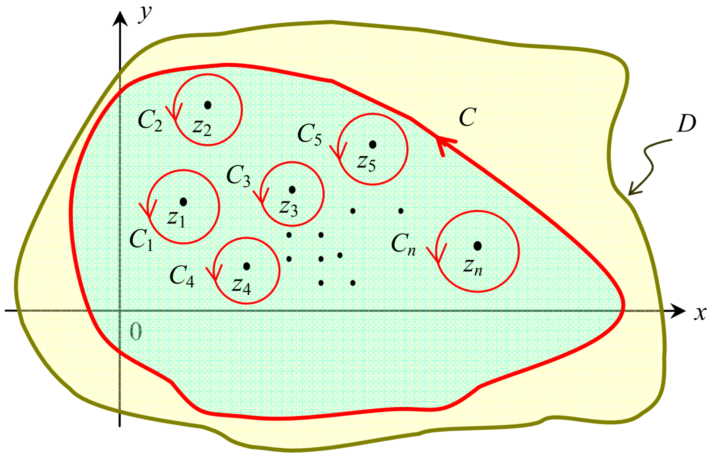
2) $f(z)$ is analytic in *D* except at finite many points $z_1, z_2, z_3, \cdots, z_n$ in the interior of *C*. Then, we have

$$\Rightarrow \quad \oint_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname*{Res}_{z=z_j} f(z)$$

<pf.> Notice that the integral along *C* is taken in positive direction.

Let $C_1, C_2, C_3, \cdots, C_n$ be *n* circles having centers at $z_1, z_2, z_3, \cdots, z_n$ respectively such that they all lie interior to *C* and exterior to each other, as shown in above Figure.

Hence, we have



$$\begin{aligned}
&\oint_C f(z)dz \\
&= \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz + \cdots + \oint_{C_n} f(z)dz \\
&= 2\pi i \operatorname*{Res}_{z=z_1} f(z) + 2\pi i \operatorname*{Res}_{z=z_2} f(z) + \cdots + 2\pi i \operatorname*{Res}_{z=z_n} f(z) \\
&= 2\pi i \sum_{j=1}^{\infty} \operatorname*{Res}_{z=z_j} f(z)
\end{aligned}$$

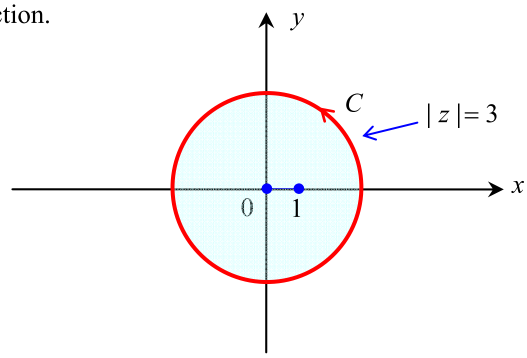
Example 1

Find $\oint_C \frac{5z-2}{z(z-1)}dz$, $C: |z| = 3$

where the integral along C is taken in the positive direction.

<Sol.>

$$\begin{aligned}
&\oint_C \frac{5z-2}{z(z-1)}dz \\
&= 2\pi i \left[\operatorname*{Res}_{z=0} \frac{5z-2}{z(z-1)} + \operatorname*{Res}_{z=1} \frac{5z-2}{z(z-1)} \right] \\
&= 2\pi i \left. \frac{5z-2}{(z-1)} \right|_{z=0} + \left. \frac{5z-2}{z} \right|_{z=1} \\
&= 2\pi i [2+3] \\
&= 10\pi i
\end{aligned}$$



Example 2

Let us show that

$$\oint_C \frac{\sin z}{z^4} dz = -\frac{\pi}{3} i$$

where $C: |z| = 1$, described in the positive directions.

<pf.> Here we used three different methods to demonstrate this problem as following：

Method I：

Using Derivatives of Analytic Functions, we know that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(z_0)$$

Then,

$$\begin{aligned}
\oint_C \frac{\sin z}{z^4} dz &= \frac{2\pi i}{3!} \frac{\partial^3}{\partial z^3} [\sin z] \bigg|_{z=0} \\
&= -\frac{\pi i}{3} (-\cos z) \bigg|_{z=0} \\
&= -\frac{\pi}{3} i
\end{aligned}$$

Method II：

Since $$\frac{\sin z}{z^4}=\frac{1}{z^4}[z-\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots]$$

$$=\frac{1}{z^3}-\frac{1}{3!}\frac{1}{z}+\frac{z}{5!}-\frac{z^3}{7!}+\cdots$$

$\Rightarrow$ $$\operatorname*{Res}_{z=0}\frac{\sin z}{z^4}=\operatorname*{Res}_{z=0}\left[\frac{1}{z^3}-\frac{1}{3!}\frac{1}{z}+\frac{z}{5!}-\frac{z^3}{7!}+\cdots\right]$$

$$=-\frac{1}{3!}$$

$$=-\frac{1}{6}$$

Thus, we have

$$\oint_C\frac{\sin z}{z^4}dz=2\pi i\operatorname*{Res}_{z=0}\frac{\sin z}{z^4}$$

$$=2\pi i\left(-\frac{1}{6}\right)$$

$$=-\frac{\pi}{3}i$$

Method III：

Since $z=0$ is a pole of order 3 of $f(z)=\dfrac{\sin z}{z^4}$

$\Rightarrow$ $$\operatorname*{Res}_{z=0}\frac{\sin z}{z^4}=\frac{1}{2!}\lim_{z\to 0}\left[\frac{d^2}{dz^2}(z^3\cdot\frac{\sin z}{z^4})\right]$$

$$=\frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\frac{\sin z}{z}$$

$$=\frac{1}{2}\lim_{z\to 0}\left[\frac{2\sin z}{z^3}-\frac{2\cos z}{z^2}-\frac{\sin z}{z^3}\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\left[\frac{2\sin z-2z\cos z-z^2\sin z}{z^3}\right]$$

Then, we use L'Hôspital's Rule,

$\Rightarrow$ $$\operatorname*{Res}_{z=0}\frac{\sin z}{z^4}=\frac{1}{2}\lim_{z\to 0}\left[\frac{2\cos z-2\cos z+2z\sin z-2z\sin z-z^2\cos z}{3z^3}\right]$$

$$=\frac{1}{2}\lim_{z\to 0}\left[-\frac{\cos z}{3}\right]$$

$$=\frac{1}{2}\left(-\frac{1}{3}\right)$$

$$=-\frac{1}{6}$$

Hence, we have

$$\oint_C z^{-4}\sin z\,dz$$

$$=2\pi i\operatorname*{Res}_{z=0}\frac{\sin z}{z^4}$$

$$=2\pi i\cdot\left(-\frac{1}{6}\right)=-\frac{\pi}{3}i$$

Example 3

Let us show that

$$\oint_C\frac{(1+z^5)\sinh z}{z^6}dz=-\frac{\pi i}{60}$$

where C is the unit circle $|z|=1$ described in the positive direction.

<pf.> **Method I：**

Using the theorem for the derivatives of analytic functions, we know that

$$\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz=\frac{2\pi i}{n!}f^{(n)}(z_0)$$

Also, from the Leibniz's Formula：

$$[f(x)\cdot g(x)]^{[n]}$$

$$=\sum_{K=0}^{n}\binom{n}{K}f^{(K)}(x)g^{(n-K)}(x)$$

Then, we have

$$\frac{d^5}{dz^5}\left[(1+z^5)\sinh z\right]\Big|_{z=0}$$

$$=\left[(1+z^5)\cosh z+5\cdot 5z^4\sinh z+10\cdot 20z^3\cosh z+10\cdot 60z^2\sinh z\right.$$

$$\left.+5\cdot 120z\cosh z+1\cdot 120z\sinh z\ \right]\Big|_{z=0}$$

$$=1$$

Hence, we have

$$\oint_C \frac{(1+z^5)\sinh z}{z^6}dz$$

$$=\frac{2\pi i}{5!}\frac{d^5}{dz^5}\left[(1+z^5)\sinh z\right]\Big|_{z=0}$$

$$=-\frac{2\pi i}{120}\cdot 1=\frac{\pi i}{60}$$

Method II：

Since

$$\frac{(1+z^5)\sinh z}{z^6}$$

$$=\frac{1+z^5}{z^6}\left[z+\frac{z^3}{3!}+\frac{z^5}{5!}-\frac{z^7}{7!}+\cdots\right]$$

$$=(1+z^5)\left[z^{-5}+\frac{z^{-2}}{3!}+\frac{1}{5!}-\frac{z}{7!}+\cdots\right]$$

Then, we have

$$\operatorname*{Res}_{z=0}\frac{(1+z^5)\sinh z}{z^6}$$

$$=\operatorname*{Res}_{z=0}(1+z^5)\left[z^{-5}+\frac{z^{-2}}{3!}+\frac{1}{5!}-\frac{z}{7!}+\cdots\right]=\frac{1}{5!}$$

Hence

$$\oint_C \frac{(1+z^5)\sinh z}{z^6}dz=2\pi i\operatorname*{Res}_{z=0}\frac{(1+z^5)\sinh z}{z^6}$$

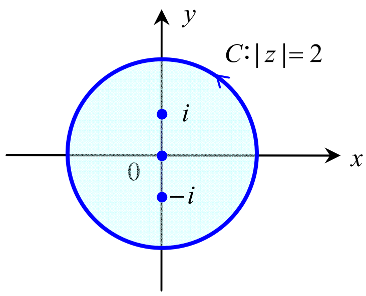
$$=2\pi i\cdot\frac{1}{5!}=-\frac{\pi i}{60}$$

Example 4

Let us show that

$$\oint_C \frac{z^2+4}{(z-i)(z+i)}dz=0$$

where C is the circle $|z|=2$ described in the positive directions.



<Sol.>

$$\oint_C \frac{z^2+4}{(z-i)(z+i)}dz$$

$$= 2\pi i\left[\operatorname*{Res}_{z=i}\frac{z^2+4}{(z-i)(z+i)}+\operatorname*{Res}_{z=-i}\frac{z^2+4}{(z-i)(z+i)}\right]$$

$$= 2\pi i\,\frac{z^2+4}{z+i}\bigg|_{z=i}+\frac{z^2+4}{z-i}\bigg|_{z=-i}$$

$$= 2\pi i\left[\frac{3}{2i}+\frac{3}{-2i}\right]$$

$$= 10\pi i\cdot 0 = 0$$

H.W. 1 Evaluate the following integrals by using the method of residue:

(a) $\oint \sum_{n=-5}^{\infty}\dfrac{1}{(z+i)^n(n+6)!}dz$ around $|z-i|=3$

(b) $\oint z\sin\left(\dfrac{1}{z-1}\right)dz$ around $|z|=2$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 5 and 7, Exercise 6.1, Pearson Education, Inc., 2005.】

H.W.2 Use residue calculus to show that if $n\ge 1$ is an integer, then

$$\oint_C\left(z+\frac{1}{z}\right)dz=\begin{cases}\dfrac{2\pi i n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}$$

where C is any simple closed contour encircling the origin.

<Hint> Use the binomial theorem.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 10, Exercise 6.1, Pearson Education, Inc., 2005.】

H.W.3 Use residues to evaluate the following integrals. Use the principal branch of multivalued functions.

(a) $\oint_C \dfrac{1}{\left[\operatorname{Ln}(\operatorname{Ln}z)-1\right]}dz$ around $C:|z-16|=5$

(b) $\oint_C \dfrac{dz}{\bar{z}-b}$ around $C:|z|=a>0$. Note that the integrand is not analytic. Consider $a>|b|$ and $a<|b|$. <Hint> Multiply numerator and denominator by z.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 30 and 34, Exercise 6.3, Pearson Education, Inc., 2005.】

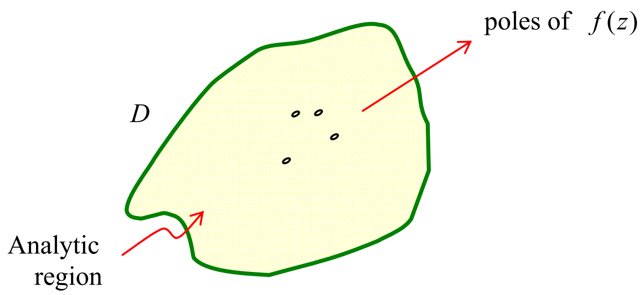
2. **Special Topics**

Poles and Zeros of Meromorphic Functions

Here, we want to discuss Meromorphic functions. Before we discuss Meromorphic functions, we must define it first.

Definition：

A function f which is analytic in a domain D, except at some points of D, at which it has poles, is said to be meromorphic in D.



* We know that the non-analytic points is called singularities, it involves：

A) Essential Singularities, it defines that $f(z)$ can be expansion as

$$f(z)=\sum_{n=-\infty}^{n} a_n (z-a)^n$$

B) Poles, it defines that $f(z)$ can be expanded by

$$f(z)=\sum_{n=-m}^{n} a_n (z-a)^n$$

* A meromorphic function may have an essential singularity at $z=\infty$.

Theorem：

Let $f(z)$ be a meromorphic function in a simply connected domain *D*, in which contains a simple closed contour *C*.

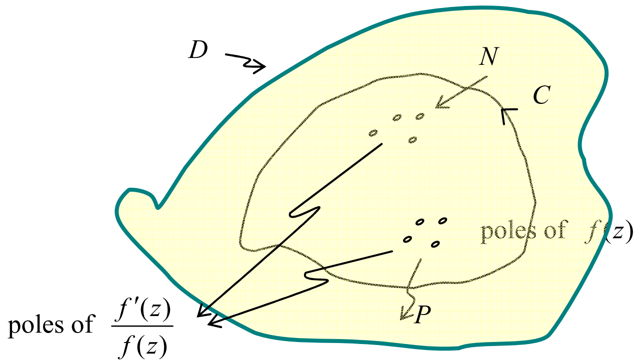
Suppose that $f(z)$ has no zeros on *C*.

Let *N* be the number of zeros and *P* be the number of poles of the function $f(z)$ in the interior of *C*, where a multiple zero or pole is counted according to its multiplicity. Then,

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N-P$$

where the integral is taken in the positive direction.

<pf.>



First of all, observe that the number of zeros and poles of $f(z)$ interior of *C* is finite.

Suppose that $z=\alpha$ is a zero of order *K* of $f(z)$.

Then, we can express $f(z)$ as

$$f(z)=(z-\alpha)^K\lambda(z)$$

where $\lambda(z)$ is analytic at $z=\alpha$ and $\lambda(z)\neq 0$.

** Here, if $\lambda(z)=0$, it implies that there is another zero in $f(z)$, then $z=\alpha$ is a zero of order $K+1$ of $f(z)$.

Hence, we have

$$f'(z)=(z-\alpha)^K\lambda'(z)+K(z-\alpha)^{K-1}\lambda(z)$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)}=\frac{K}{z-\alpha}+\frac{\lambda'(z)}{\lambda(z)}$$

Since $\frac{\lambda'(z)}{\lambda(z)}$ is analytic at $z=\alpha$, it means that $\frac{f'(z)}{f(z)}$ has a simple pole at $z=\alpha$, which is the only one pole that can't be reproduced and $\frac{f'(z)}{f(z)}$ is with residue *K*.

Suppose that $z=\beta$ is a pole of order *m* of $f(z)$, then

$$f(z)=\frac{1}{(z-\beta)^m}v(z)$$

where $v(z)$ is analytic at $z=\beta$, and $v(\beta)\neq 0$.

Since

$$f'(z)=\frac{-m}{(z-\beta)^{m+1}}v(z)+\frac{1}{(z-\beta)^m}v'(z)$$

we have

$$\frac{f'(z)}{f(z)} = \frac{-m}{z-\beta} + \frac{v'(z)}{v(z)}$$

Since $v(z)$ is analytic, then $v'(z)$ is analytic, too.

This implies that $\dfrac{v'(z)}{v(z)}$ is analytic.

Since $\dfrac{v'(z)}{v(z)}$ is analytic at $z=\beta$, hence $\dfrac{f'(z)}{f(z)}$ has a simple pole at $z=\beta$ with residue $(-m)$.

Moreover, the only pole of $\dfrac{f'(z)}{f(z)}$ interior to *C* is precisely the pole or zero of $f(z)$.

If α_j and β_j are the zeros and poles of $f(z)$ interior to *C*, and K_j and m_j are their orders, respectively.

Then, by Cauchy's Residue Theorem, we have

$$\oint_C \frac{f'(z)}{f(z)}dz = 2\pi i\left[\sum K_j - \sum m_j\right]$$

$$= 2\pi i(N-P)$$

$$\Rightarrow \qquad \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz = N-P$$

§6-4 The Evaluation of the Integrals of Certain Periodic Functions Taken Between the limits 0 and 2π —— By complex variable analysis

1. 利用複變分析法解實變數積分時，通常有底下幾種型式：

1) $\displaystyle\int_0^{2\pi} F(\cos\theta,\sin\theta)d\theta$

2) $\displaystyle\int_{-\pi}^{\pi} F(\cos\theta,\sin\theta)d\theta$

$F(\cos\theta,\sin\theta)$ is rational function of $\cos\theta$ and $\sin\theta$

3) $\displaystyle\int_0^{\pi} F(\cos\theta,\sin\theta)d\theta = \frac{1}{2}\int_{-\pi}^{\pi} F(\cos\theta,\sin\theta)d\theta$

where $F(\cos\theta,\sin\theta)$ is even function of θ.

In dealing with the above problems, we can change them (variable; real variable) into complex variable.

Hence, we consider a unit circle $|z|=1$ as an integral path described in the positive direction.

Thus, we have

$$z(\theta) = \cos\theta + i\sin\theta, \qquad -\pi \le \theta \le \pi$$

$$\Rightarrow \qquad \frac{1}{z} = \frac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$$

Hence,

$$\begin{cases} \cos\theta = \dfrac{1}{2}\left(z+\dfrac{1}{z}\right) \\ \sin\theta = \dfrac{1}{2i}\left(z-\dfrac{1}{z}\right) \end{cases}$$

and

$$dz = ie^{i\theta}\,d\theta = iz\,d\theta \qquad \Rightarrow \qquad d\theta = \frac{dz}{iz}$$

Similarly,

$$z^2 = (\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

$$\frac{1}{z^2} = \cos 2\theta - i\sin 2\theta$$

hence we have

$$\begin{cases} \cos 2\theta = \dfrac{1}{2}\left(z^2 + \dfrac{1}{z^2}\right) \\ \sin 2\theta = \dfrac{1}{2i}\left(z^2 - \dfrac{1}{z^2}\right) \end{cases}$$

Using the same method, we obtain

$$z^3 = \cos 3\theta + i\sin 3\theta \qquad \text{and} \qquad \frac{1}{z^3} = \cos 3\theta - i\sin 3\theta$$

that is

$$\begin{cases} \cos 3\theta = \dfrac{1}{2}\left(z^3 + \dfrac{1}{z^3}\right) \\ \sin 2\theta = \dfrac{1}{2i}\left(z^3 - \dfrac{1}{z^3}\right) \end{cases}$$

以下依此類推，將上列之結果代入 1), 2) 及 3)式即可使其變成複變數分析型式。

2. Some Examples

Example 1

Let us show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2 - b^2}} \;,\quad a > |b|$$

<pf.> Let C: $z(\theta) = \cos\theta + i\sin\theta$, $-\pi \le \theta \le \pi$

Then, we have

$$\Rightarrow \qquad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right) \quad \text{and} \quad d\theta = \frac{dz}{iz}$$

Hence, the integral becomes

$$\int_0^{2\pi} \frac{d\theta}{a + b\sin\theta}$$

$$= \oint_C \frac{\dfrac{dz}{iz}}{a + b\cdot\dfrac{1}{2i}\left(z - \dfrac{1}{z}\right)}$$

$$= \oint_C \frac{2}{bz^2 + 2aiz - b} dz$$

$$= \oint_C \frac{2dz}{b\left(z - \dfrac{-a+\sqrt{a^2-b^2}}{b}i\right)\left(z - \dfrac{-a-\sqrt{a^2-b^2}}{b}i\right)}$$

$$= 2\pi i \operatorname*{Res}_{z=\frac{-b+\sqrt{a^2-b^2}}{b}} \left[\frac{2}{b\left(z - \dfrac{-a+\sqrt{a^2-b^2}}{b}i\right)\left(z - \dfrac{-a-\sqrt{a^2-b^2}}{b}i\right)}\right]$$

$$= 2\pi i\left(\frac{2}{b}\right)\left[\frac{1}{z - \dfrac{-a+\sqrt{a^2-b^2}}{b}}\right]\Bigg|_{z = \frac{-a+\sqrt{a^2-b^2}}{b}}$$

$$= \frac{4\pi i}{b}\cdot\frac{1}{\dfrac{2\sqrt{a^2-b^2}}{b}i}$$

$$= \frac{2\pi}{\sqrt{a^2-b^2}}$$

♣ **Similar integral**

$$I=\int_0^{2\pi}\frac{d\theta}{k+\sin\theta}=\frac{2\pi}{\sqrt{k^2-1}},\quad k>1$$

Example 2

Find

$$I=\int_0^{2\pi}\frac{\cos 2\theta d\theta}{5-4\sin\theta}$$

<Sol.>

With the substitutions

$$\cos 2\theta=\frac{1}{2}\left(z^2+\frac{1}{z^2}\right),\quad \sin 2\theta=\frac{1}{2i}\left(z^2-\frac{1}{z^2}\right),\quad d\theta=\frac{dz}{iz}$$

we have

$$I=\oint_{|z|=1}\frac{\frac{z^2+z^{-2}}{2}}{5-\frac{2}{i}\left(z+z^{-1}\right)}\left(\frac{dz}{iz}\right)=\oint_{|z|=1}\frac{\left(z^4+1\right)dz}{2iz^2\left(2iz^2+5z-2i\right)}\left(\frac{dz}{iz}\right)$$

There is a second-order pole at $z=0$. Solving $2iz^2+5z-2i=0$, we find simple poles at $z=i/2$ and $z=2i$. The pole at $z=2i$ is outside the circle $|z|=1$ and can be ignored.

Thus, we have

$$I=2\pi i\sum\text{Res}\left[\frac{\left(z^4+1\right)dz}{2iz^2\left(2iz^2+5z-2i\right)}\right],\text{ at } z=0 \text{ and } z=i/2$$

From previous sections, we find the residue at $z=0$,

$$\frac{1}{2i}\frac{d}{dz}\left[\frac{\left(z^4+1\right)}{\left(2iz^2+5z-2i\right)}\right]\Bigg|_{z=0}=\frac{-5}{8}i$$

and the residue at $z=i/2$,

$$\frac{1}{(2i)\left(\frac{i}{2}\right)^2}\left[\frac{\left(\frac{i}{2}\right)^4+1}{\frac{d}{dz}\left(2iz^2+5z-2i\right)}\right]\Bigg|_{z=i/2}=\frac{17}{24}i$$

Thus, we obtain

$$I=\int_0^{2\pi}\frac{\cos 2\theta d\theta}{5-4\sin\theta}=2\pi i\left(\frac{-5}{8}i+\frac{17}{24}i\right)=-\frac{\pi}{6}$$

Example 3

Let us show that

$$\int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta-n\theta)d\theta=\frac{2\pi}{n!},\quad a<|b|$$

where $n=0,\ 1,\ 2,\ 3,\ \cdots$.

<pf.> Consider the integral

$$\int_0^{2\pi} e^{\cos\theta}\cdot\left[\cos(\sin\theta-n\theta)+i\sin(\sin\theta-n\theta)\right]d\theta$$

$$=\int_0^{2\pi} e^{\cos\theta}\cdot e^{i(\sin\theta-n\theta)}d\theta$$

$$=\int_0^{2\pi} e^{\cos\theta}\cdot e^{i\sin\theta}\cdot e^{in\theta}d\theta$$

$$=\int_0^{2\pi} e^{\cos\theta+i\sin\theta}\cdot(e^{i\theta})^{-n}d\theta \qquad ------ \quad (1)$$

Let $C: z(\theta)=\cos\theta+i\sin\theta=e^{i\theta}$, where $-\pi\le\theta\le\pi$.

Then, equation (1) becomes

$$\Rightarrow \oint_C e^z\cdot z^{-n}\cdot\frac{dz}{iz}$$

$$=\oint_C \frac{-ie^z}{z^{n+1}}dz$$

$$=2\pi i\cdot(-i)\cdot\frac{1}{n!}\cdot e^z\Big|_{z=0}$$

$$=\frac{2\pi}{n!}$$

Since Im = Im, Re = Re, we have

$$\Rightarrow \int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta-n\theta)d\theta$$

$$=\mathrm{Re}\left[\frac{2\pi}{n!}\right]=\frac{2\pi}{n!}$$

and

$$\int_0^{2\pi} e^{\cos\theta}\cdot\cos(\sin\theta-n\theta)d\theta \ =\mathrm{Im}\left[\frac{2\pi}{n!}\right]=0$$

Example 4

Show that

$$\int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2}=\frac{2\pi}{1-a^2},$$

where $|a|<1$.

<pf.> Let $C: z=\cos\theta+i\sin\theta$, $0\le\theta\le 2\pi$

$$\Rightarrow \quad \cos\theta=\frac{1}{2}\left(z+\frac{1}{z}\right)$$

$$d\theta=\frac{dz}{iz}$$

$$\Rightarrow \int_0^{2\pi}\frac{d\theta}{1-2a\cos\theta+a^2}$$

$$=\oint_C\frac{1}{1-2a\cdot\frac{1}{2}\left(z+\frac{1}{z}\right)+a^2}\cdot\frac{dz}{iz}$$

$$=\oint_C\frac{idz}{az^2-a^2z-z+a}$$

$$=\oint_C\frac{idz}{(z-a)(az-1)}$$

$$=2\pi i\cdot i\cdot\underset{z=a}{\mathrm{Res}}\frac{1}{(z-a)(az-1)}$$

$$=2\pi\cdot\frac{-1}{a^2-1}$$

$$=\frac{2\pi}{1-a^2}$$

H.W. 1 Using residues, establish the following identities:

(a) $\displaystyle\int_{-\pi/2}^{3\pi/2}\frac{\cos\theta}{a+b\cos\theta}d\theta=\frac{2\pi}{b}\left[1-\frac{1}{\sqrt{a^2-b^2}}\right]$, for $a>b>0$

(b) $\int_0^{2\pi} \cos^m \theta \, d\theta = \frac{2\pi}{2^m} \frac{m!}{\left[\left(\frac{m}{2}\right)!\right]^2}$, for $m \geq 0$ even. Show that the preceding integral is zero when m is odd.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 4 and 6, Exercise 6.4, Pearson Education, Inc., 2005.】

H.W. 2 Using residues, establish the following identities:

(a) $\int_0^{2\pi} \frac{d\theta}{(a + b\sin\theta)^2} = \frac{2\pi a}{\left(\sqrt{a^2 - b^2}\right)^2}$, for $a > b \geq 0$

(b) $\int_0^{2\pi} \frac{d\theta}{a + \sin^2\theta} = \frac{2\pi}{\sqrt{a(a+1)}}$, for $a > 0$

(c) $\int_0^{2\pi} \frac{\cos\theta \, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi}{a(a^2 - 1)}$, for a real, $|a| > 1$

(d) $\int_0^{2\pi} \frac{\cos n\theta \, d\theta}{\cosh a + \cos\theta} = \frac{2\pi(-1)e^{-na}}{\sinh a}$, $n \geq 0$ is an integer, $a > 0$

(e) $\int_0^{2\pi} \frac{d\theta}{a^2\sin^2\theta + b^2\cos^2\theta + a^2} = \frac{2\pi}{ab}$, for a, b real, $ab > 0$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 7-9 and 11-12, Exercise 6.4, Pearson Education, Inc., 2005.】

§6-5 The Evaluation of Certain Types of Integrals Taken Between The Limits $-\infty$ and ∞

♣ **Improper integrals**

1. The improper integral is an integral in which one or both limits are infinite, that is expressions of the form

$$\int_k^{\infty} f(x)dx, \quad \int_{-\infty}^{k} f(x)dx, \quad \int_{-\infty}^{\infty} f(x)dx,$$

where $f(x)$ is a real function of x, and k is a real constant.

2. The improper integral

$$\int_{-\infty}^{\infty} f(x)dx$$

can be defined as following form:

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{t_1 \to -\infty} \int_{t_1}^{0} f(x)dx + \lim_{t_2 \to \infty} \int_0^{t_2} f(x)dx$$

and we let

$$I = \int_{-\infty}^{\infty} f(x)dx$$

$$I_1 = \lim_{t_1 \to -\infty} \int_{t_1}^{0} f(x)dx$$

$$I_2 = \lim_{t_2 \to \infty} \int_0^{t_2} f(x)dx$$

If both of I_1 and I_2 exist,

$\Rightarrow \int_{-\infty}^{\infty} f(x)dx$ is convergent.

*** In advanced calculus, we know that:
if the function $f(x)$ is continuous in $[a, \infty)$,

1) $\lim_{x \to \infty} x f(x) = A \neq 0$

$\Rightarrow \quad \int_a^\infty f(x)dx$ is divergent.

2) $\lim_{x\to\infty} x^P f(x) = A$

where A is finite and $P > 1$,

$\Rightarrow \quad \int_a^\infty f(x)dx$ is convergent.

*** In the above two cases, a can be equal to $-\infty$, too.

Example 1

(a) $\int_1^\infty \frac{1}{1+x^2}dx = \lim_{R\to\infty}\int_1^R \frac{1}{1+x^2}dx = \lim_{R\to\infty}\left(\tan^{-1}R - \tan^{-1}1\right) = \frac{\pi}{2} - \frac{\pi}{4}$ exists; however,

(b) $\int_1^\infty \frac{1}{1+x^2}dx = \lim_{R\to\infty}\left(\ln R - \ln 1\right)$ fails to exist, as does

(c) $\int_0^\infty \cos x dx = \lim_{R\to\infty} \sin R$.

In case (b), as x increases, the curve $y = 1/x$ does not fall to zero fast enough for the area under the curve to approach a finite limit.

In case (c), a sketch of $y = \cos x$ shows that, along the positive x-axis, the total area under this curve has not meaning.

3\. 1) If the integral

$$\int_{-\infty}^\infty f(x)dx \text{ is convergent,}$$

then we can define that

$$\int_{-\infty}^\infty f(x)dx = \lim_{R\to\infty}\int_{-R}^R f(x)dx$$

2) If the integral

$$\int_{-\infty}^\infty f(x)dx \text{ is divergent, and}$$

$$\lim_{R\to\infty}\int_{-R}^R f(x)dx \quad \text{exists,}$$

then $\lim_{R\to\infty}\int_{-R}^R f(x)dx$ is called the Cauchy's Principal Value of the integral

$$\int_{-\infty}^\infty f(x)dx$$

and is denoted by

$$\text{P.V.} \int_{-\infty}^\infty f(x)dx$$

4\. If

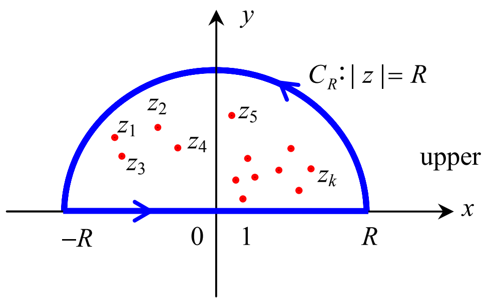
1) the function $f(x)$ is rational and is expressed in terms of the following form:

$$f(x) = \frac{p(x)}{q(x)}$$

where $q(x) \neq 0$, for all real x.

2) the degree of $q(x) \geq$ the degree of $[p(x)+2]$

3) and z_j, $j = 1, 2, 3, \cdots, k$ are poles of $f(z)$ in the upper half plane.



Then, we can obtain that

$$\int_{-\infty}^\infty f(x)dx = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z)$$

<pf.>

Let C_R denote the upper half of the circle and

$$|z| = R$$

described in the positive direction, and take R sufficiently large so that all the poles z_k of $f(z)$ lie in its interior. Hence, let $C = [-R, R] + C_R$. This implies that C is a simple closed curve. And C encloses all poles of $f(z)$ in the upper half plane.

By Cauchy's Residue Theorem, we then have

$$\oint_C f(z)dz = 2\pi i \cdot \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z)$$

$$\Rightarrow \quad \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z)$$

*** When we integrate in the interval of $[-R, R]$, y-component is zero, this means that

$$z = x + i0$$

$$\Rightarrow \quad dz = dx$$

Hence, we have

$$\int_{-R}^{R} f(x)dx = \int_{-R}^{R} f(z)dz$$

Here, we want to prove that

1) $\int_{C_R} f(z)dz \to 0$, when $R \to \infty$, and

2) $\int_{-R}^{R} f(z)dz \Rightarrow \int_{-\infty}^{\infty} f(x)dx$, when $R \to \infty$.

First, suppose that

$$f(z) = \frac{p(z)}{q(z)}$$

$$= \frac{a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0}$$

where $n \geq m + 2$, and $b_n \neq 0$.

$$\Rightarrow \quad f(z) = \frac{z^m \left[a_m + a_{m-1} \cdot \frac{1}{z} + \cdots + a_1 \frac{1}{z^{m-1}} + a_0 \frac{1}{z^m} \right]}{z^n \left[b_n + b_{n-1} \cdot \frac{1}{z} + \cdots + b_1 \cdot \frac{1}{z^{n-1}} + b_0 \frac{1}{z^n} \right]}$$

But, we know that

$$\frac{a_m + a_{m-1} \cdot \frac{1}{z} + \cdots + a_1 \frac{1}{z^{m-1}} + a_0 \frac{1}{z^m}}{b_n + b_{n-1} \cdot \frac{1}{z} + \cdots + b_1 \cdot \frac{1}{z^{n-1}} + b_0 \frac{1}{z^n}} \to \frac{a_m}{b_n}, \quad \text{when } |z| \to \infty$$

Therefore, when $|z| = |R| \to \infty$, we obtain

$$|f(z)| \to \frac{R^m}{R^n} \cdot \left| \frac{a_m}{b_n} \right| \quad \text{-----------} \quad (1)$$

And, the ML inequality tells us that

$$\left| \int_C f(z)dz \right| \leq M\ell$$

$$\Rightarrow \quad |f(z)| \leq M \quad \text{where } \ell \text{ is the arc length.}$$

Thus, equation (1) still satisfies following form:

$$|f(z)| \leq \frac{1}{R^{n-m}} \cdot M$$

Hence

$$\left| \int_{C_R} f(z)dz \right| \leq \frac{1}{R^{n-m}} \cdot M \cdot \pi R = \frac{M\pi}{R^{n-m-1}}$$

Since, $n \geq m + 2$, then

$$n-m-1\geq 1$$

Thus, when $R\to\infty$, we know that

$$\left|\int_{C_R} f(z)dz\right|\to 0$$

This means that

$$\int_{C_R} f(z)dz\to 0\text{, when } R\to\infty$$

Hence, when $R\to\infty$, the following equation

$$\int_{-R}^{R} f(x)dx+\int_{C_R} f(z)dz=2\pi i\sum_{j=1}^{K}\operatorname*{Res}_{z=z_j} f(z)$$

becomes

$$\int_{-R}^{R} f(x)dx+0=2\pi i\sum_{j=1}^{K}\operatorname*{Res}_{z=z_j} f(z)$$

That is,

$$\int_{-R}^{R} f(x)dx=2\pi i\sum_{j=1}^{K}\operatorname*{Res}_{z=z_j} f(z)$$

5. Some Examples

Example 1

Evaluate the integral

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}dx$$

<Sol.> Since $f(x)=\dfrac{x^2}{(x^2+1)^2}$ is an even function

$$\Rightarrow \int_0^\infty \frac{x^2}{(x^2+1)^2}dx=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^2}dx$$

$$=\frac{1}{2}\cdot 2\pi i\operatorname*{Res}_{z=i}\frac{z^2}{(z^2+1)^2}$$

$$=\pi i\left[\frac{d}{dz}\frac{z^2}{(z+i)^2}\right]\Bigg|_{z=i}$$

$$=\pi i\left[\frac{(z+i)^2\cdot 2z-z^2\cdot 2(z+i)}{(z+i)^4}\right]\Bigg|_{z=i}$$

$$=\pi i\cdot\left(\frac{-i}{4}\right)$$

$$=\frac{\pi}{4}$$

Example 2

Show that

$$\int_{-\infty}^{\infty}\frac{x^2+3}{(x^2+1)(x^2+4)}dx=\frac{5}{6}\pi$$

<pf.>

$$\int_{-\infty}^{\infty}\frac{x^2+3}{(x^2+1)(x^2+4)}dx$$

$$=2\pi i\left[\operatorname*{Res}_{z=i}\frac{z^2+3}{(z^2+1)(z^2+4)}+\operatorname*{Res}_{z=2i}\frac{z^2+3}{(z^2+1)(z^2+4)}\right]$$

$$=2\pi i\left[\frac{z^2+3}{(z^2+1)(z^2+4)}\Bigg|_{z=i}+\frac{z^2+3}{(z^2+1)(z^2+4)}\Bigg|_{z=2i}\right]$$

$$= 2\pi i\left[\frac{1}{3i}+\frac{1}{12i}\right]$$

$$= 2\pi\left[\frac{1}{3}+\frac{1}{12}\right] = 2\pi\cdot\frac{5}{12} = \frac{5\pi}{6}$$

Example 3

Evaluate

$$\int_{-\infty}^{\infty}\frac{1}{x^4+1}dx$$

<Sol.> Since $z^4+1=0$

$\Rightarrow$ $z^4=-1=\cos\pi+i\sin\pi$

$\Rightarrow$ $z=\cos\dfrac{\pi+2k\pi}{4}+i\sin\dfrac{\pi+2k\pi}{4}$,

where $k=0,1,2,3$.

Thus, we can obtain that

1) when $k=0$,

$\Rightarrow$ $z=\cos\dfrac{\pi}{4}+i\sin\dfrac{\pi}{4}$

$=e^{i\frac{\pi}{4}}$

2) when $k=1$, $\Rightarrow$ $z=\cos\dfrac{3\pi}{4}+i\sin\dfrac{3\pi}{4}$

$=e^{i\frac{3\pi}{4}}$

3) when $k=2$, $\Rightarrow$ $z=\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}$

$=e^{i\frac{5\pi}{4}}$

4) when $k=3$, $\Rightarrow$ $z=\cos\dfrac{7\pi}{4}+i\sin\dfrac{7\pi}{4}$

$=e^{i\frac{7\pi}{4}}$

Here, only both of $z=e^{i\frac{\pi}{4}}$ and $z=e^{i\frac{3}{4}\pi}$ are the poles of $f(z)$ in the upper half plane, where

$$f(z)=\frac{1}{1+z^4}$$

Hence, we obtain that

$$\int_{-\infty}^{\infty}\frac{1}{1+x^4}dx$$

$$=2\pi i\left[\operatorname*{Res}_{z=e^{i\frac{\pi}{4}}}\frac{1}{z^4+1}+\operatorname*{Res}_{z=e^{i\frac{3}{4}\pi}}\frac{1}{z^4+1}\right]$$

$$=2\pi i\left[\left.\frac{1}{4z^3}\right|_{z=e^{i\frac{\pi}{4}}}+\left.\frac{1}{4z^3}\right|_{z=e^{i\frac{3\pi}{4}}}\right]$$

$$=\frac{\pi i}{2}\left[e^{-\frac{3}{4}\pi i}+e^{-\frac{9}{4}\pi i}\right]$$

$$=\frac{\pi i}{2}\left[-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right]$$

$$=\frac{\pi i}{2}\left[-\sqrt{2}i\right]=\frac{\sqrt{2}}{2}\pi$$

** If $z = z_0$ is a simple pole of $f(z) = \dfrac{p(z)}{q(z)}$

$$\Rightarrow \quad \operatorname*{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$$

Example 4

Find $\int_{-\infty}^{\infty} \dfrac{x^2}{x^4+1} dx$ using residues.

<Sol.>

We first consider $\oint_C z^2/\left(z^4+1\right) dz$ taken around the closed contour C consisting of the line segment $y = 0, -R \le x \le R$, and the semicircle $|z| = R,\ 0 \le \arg(z) \le \pi$. Let us take $R > 1$, which means that C encloses all the poles of $z^2/\left(z^4+1\right)$ in the upper half-plane (abbreviated u.h.p.). Hence,

$$\oint_C \frac{z^2}{z^4+1} dz = 2\pi i \sum \operatorname{Res}\left[\frac{z^2}{z^4+1}\right] \text{ at all poles in u.h.p.}$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx = \frac{2\pi i}{4}\left[e^{-i\pi/4} + e^{-i3\pi/4}\right] = \frac{\pi}{\sqrt{2}}$$

♣ Because $x^2/(x^4+1)$ is an even function, we have, as a bonus,

$$\int_0^{\infty} \frac{x^2}{x^4+1} dx = \frac{1}{2}\frac{\pi}{\sqrt{2}}$$

Example 5

Find $\int_{-\infty}^{\infty} \dfrac{x^2}{x^4+x^2+1} dx$.

<Sol.>

Using the above-mentioned theorem, we have

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+x^2+1} dx = 2\pi i \sum \operatorname{Res}\left(\frac{z^2}{z^4+z^2+1}\right) \text{ at all poles in u.h.p.}$$

Using the quadratic formula, we can solve $z^4 + z^2 + 1 = 0$ for z^2 and obtain

$$z^2 = \frac{-1 \pm i\sqrt{3}}{2} = e^{i2\pi/3},\ e^{-i2\pi/3}$$

Taking square roots yields

$$z = e^{i\pi/3},\ e^{-i2\pi/3},\ e^{-i\pi/3},\ e^{i2\pi/3}$$

Thus $z^2/(z^4+z^2+1)$ has simple poles in the u.h.p at $e^{i\pi/3}$ and $e^{i2\pi/3}$. Evaluating the residues at these two poles in the usual way, we find that the value of the given integral is

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4+x^2+1} dx = 2\pi i \sum \operatorname{Res}\left(\frac{z^2}{z^4+z^2+1}\right) \text{ in u.h.p. } \frac{\pi}{\sqrt{3}}$$

H.W.1 Evaluate the following integrals by means of residue calculus. Use the Cauchy principal value.

(a) $\int_{-\infty}^{\infty} \dfrac{x^3+x^2+x+1}{x^4+1} dx$, (b) $\int_{-\infty}^{\infty} \dfrac{1}{(x+a)^2+b^2} dx$, $a,\ b$ real, $b > 0$

<Ans.> (a) $\pi\sqrt{2}$, (b) π/b

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 22 and 24, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.2 Consider

$$\int_{-\infty}^{\infty} \frac{x^3+x^2}{(x^2+1)(x^2+4)} dx$$

Does the above-mentioned theorem apply directly to this integral? Evaluate this integral by evaluating the sum of the two Cauchy principal values.

<Ans.> $\pi/3$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 26, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.3 (a) When a and b are positive, prove that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2}\frac{1}{(b+a)ab}, \text{ for both } a \neq b \text{ and } a = b.$$

(b) Verify the answer to (a) by performing the integration with Symbolic Math Toolbox in MATLAB. Consider both possible cases.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 27, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.4 Show that for a, b, c real, and $b^2 < 4ac$, the following hold.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{ax^2+bx+c} = \frac{2\pi}{\sqrt{4ac-b^2}}$

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(ax^2+bx+c)^2} = \frac{4\pi a}{\left[\sqrt{4ac-b^2}\right]^3}$

Obtain the result in (b) using residues, but check the answer by differentiating both sides of the equation in (a) with respect to c. You may differentiate under the integral sign.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 28 and 29, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.5 Find $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^{100}+1}$. <Ans.> $(\pi/100)/\sin(\pi/100)$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 30, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.6 Find $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4+x^3+x^2+x+1}$. <Hint> See **H.W. 6**, in page 162, Sec. 6.2.

<Ans.> $(\pi/100)/\sin(\pi/100)$

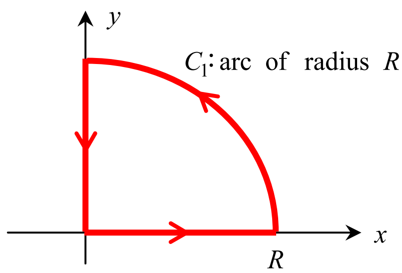
【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 31, Exercise 6.5, Pearson Education, Inc., 2005.】

H.W.7 (a) Explain why $\int_0^{\infty} x/(x^4+1)\,dx$ cannot be evaluated through the use of a closed semicircular in the upper or lower half-plane.

(b) Consider the quarter-circle contour shown in the following figure is the arc of radius $R>1$. Show that

$$\int_0^R \frac{x}{x^4+1}dx - \int_R^0 \frac{y}{y^4+1}dy + \int_{C_1}\frac{z}{z^4+1}dz = 2\pi i\sum \text{Res}\left[\frac{z}{z^4+1}\right] \text{ at all poles in first quadrant}$$

(c) Let $R\to\infty$ and show that $\int_0^{\infty} x/(x^4+1)\,dx = \pi/4$.



【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 36, Exercise 6.5, Pearson Education, Inc., 2005.】

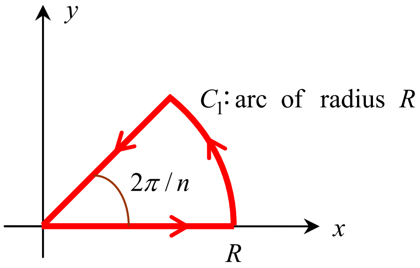
H.W.8 Show that

$$\int_0^{\infty} \frac{x^m}{x^n+1}dx = \frac{\pi}{n\sin[\pi(m+1)/n]}$$

where n and m are nonnegative integers and $n-m\geq 2$.

<Hint> Use the method employed in **H.W. 7** above, but change to the contour of integration shown in below.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 38, Exercise 6.5, Pearson Education, Inc., 2005.】



H.W.9 (a) Show that

$$\int_0^{\infty} \frac{u^{1/l}}{u^k + 1} du = \frac{\pi}{k \sin[\pi(l+1)/(lk)]}$$

where k and l are integers, $l > 0$, which satisfy $l(k-l) \geq 2$. Take $u^{1/l}$ as nonnegative real function in the interval of integration.

<Hint> First work **H.W.8** above. Then, in the present problem, make the change of variable $x = u^{1/l}$ and use the result of **H.W. 8**.

(b) what is $\int_0^{\infty} u^{1/4} /(u^5 + 1) \, du$?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 39, Exercise 6.5, Pearson Education, Inc., 2005.】

§6-6 Improper Integrals From Fourier Analysis

1. **Basic concept**

Residue theory can be useful in evaluating convergent improper integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin ax dx, \quad \int_{-\infty}^{\infty} f(x) \cos ax dx, \text{ or } \int_{-\infty}^{\infty} f(x) e^{iax} dx \qquad \text{(A)}$$

where a denotes a positive constant and $f(x)$ is a rational function of x.

Assume that $f(x) = p(x)/q(x)$, where $p(x)$ and $q(x)$ are polynomials with real coefficients and no factors in common. Also, $q(z)$ has no real zeros. Integrals of type (A) occur in the theory and application of the Fourier integral.

The method described in previous section cannot be applied directly here since

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

and

$$|\cos az|^2 = \cos^2 ax + \sinh^2 ay$$

More precisely, since

$$\sinh ay = \frac{e^{ay} - e^{-ay}}{2}$$

the moduli $|\sin az|$ and $|\cos az|$ increase like e^{ay} as y tends to infinity. The modification illustrated in the example below is suggested by the fact that

$$\int_{-R}^{R} f(x) \cos ax dx + i \int_{-R}^{R} f(x) \sin ax dx = \int_{-R}^{R} f(x) e^{iax} dx$$

together with the fact that the modulus

$$\left|e^{iaz}\right| = \left|e^{ia(x+iy)}\right| = \left|e^{-ay} e^{iax}\right| = e^{-ay}$$

is bounded in the upper half plane $y \geq 0$.

Example 1

To give some insight into the method discussed in this section, we try to evaluate

$$\int_{-\infty}^{+\infty} \frac{\cos(3x)}{(x-1)^2 + 1} dx$$

using the technique of the preceding section.

<Sol.>

From the preceding section, we have

$$\int_{-R}^{+R}\frac{\cos 3x}{(x-1)^2+1}dx+\int_{C_1}\frac{\cos 3z dz}{(z-1)^2+1}=2\pi i\sum \text{Res}\left[\frac{\cos 3z}{(z-1)^2+1}\right] \text{ in u.h.p.}$$

As before, C_1 is an arc of radius R in the upper half-plane. Although the preceding equation is valid for sufficiently large R, it is of no use to us. We would like to show that as $R\to\infty$, the integral over C_1 goes to zero. However,

$$\cos 3z=\frac{e^{3iz}+e^{-3iz}}{2}=\frac{e^{i3x-3y}+e^{-i3x+3y}}{2}$$

As $R\to\infty$, the y-coordinate of points on C_1 become infinite and the term $e^{-i3x+3y}$, whose magnitude is e^{3y}, becomes unbounded. The integral over C_1 thus does not vanish with increasing R.

The correct approach in solving the given problem is to begin by finding

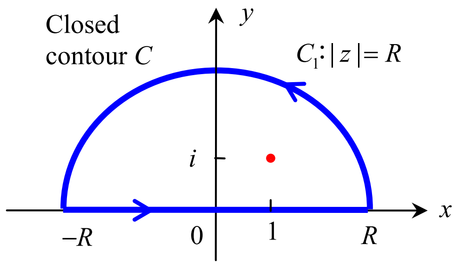
$$\int_{-\infty}^{+\infty}\frac{e^{i3x}}{(x-1)^2+1}dx$$

Its value can be determined if we use the technique of the previous section, that is, we integrate

$$\int_{-\infty}^{+\infty}\frac{e^{i3z}}{(z-1)^2+1}dz$$

around the contour of the figure shown below and obtain

$$\int_{-R}^{+R}\frac{e^{3ix}}{(x-1)^2+1}dx+\int_{C_1}\frac{e^{3iz}}{(z-1)^2+1}dz=2\pi i\sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right] \text{ in u.h.p.} \quad (1)$$



Assuming we can argue that the integral over arc C_1 vanishes as $R\to\infty$ (the troublesome e^{-i3z} no longer appears), we have, in this limit,

$$\int_{-\infty}^{+\infty}\frac{e^{3ix}}{(x-1)^2+1}dx=2\pi i\sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right] \text{ in u.h.p.}$$

Putting $e^{i3x}=\cos 3x+i\sin 3x$ and rewriting the integral on the left as two separate expressions, we have

$$\int_{-\infty}^{+\infty}\frac{\cos 3x}{(x-1)^2+1}dx+i\int_{-\infty}^{+\infty}\frac{\sin 3x}{(z-1)^2+1}dx=2\pi i\sum_{res}\frac{e^{3iz}}{(z-1)^2+1} \text{ in u.h.p.}$$

When we equate corresponding parts (reals and imaginaries) in this equation, the values of two real integrals are obtained:

$$\int_{-\infty}^{+\infty}\frac{\cos 3x}{(x-1)^2+1}dx=\text{Re}\left[2\pi i\sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right]\right] \text{ at all poles in u.h.p.} \quad (2)$$

$$\int_{-\infty}^{+\infty}\frac{\sin 3x}{(x-1)^2+1}dx=\text{Im}\left[2\pi i\sum \text{Res}\left[\frac{e^{3iz}}{(z-1)^2+1}\right]\right] \text{ at all poles in u.h.p.} \quad (3)$$

Equation (2) contains the result being sought, while the integral in Eq. (3) has been evaluated as a bonus.

Solving the equation $(z-1)^2=-1$ and finding that $z=1\pm i$, we see that on the right sides of Eqs. (2) and (3), we must evaluate a residue at the simple pole $z=1+i$. From the previous section, we obtain

$$2\pi i\,\text{Res}\left(\frac{e^{3iz}}{(z-1)^2+1},1+i\right)=2\pi i\lim_{z\to(1+i)}\frac{e^{3iz}}{2(z-1)}$$
$$=\pi e^{-3+3i}=\pi e^{-3}\left[\cos 3+i\sin 3\right]$$

Using the result in Eqs. (2) and (3), we have, finally,

$$\int_{-\infty}^{+\infty} \frac{\cos 3x}{(x-1)^2+1} dx = \pi e^{-3} \cos 3 \quad \text{and} \quad \int_{-\infty}^{+\infty} \frac{\sin 3x}{(x-1)^2+1} dx = \pi e^{-3} \sin 3$$

2. **Theorem**

Let $f(z)$ have the following property in the half-plane $\operatorname{Im} z \geq 0$. There exist constants, $k > 0$, R_0, and μ such that

$$|f(z)| \leq \frac{\mu}{|z|^k} \quad \text{for all } |z| \geq R_0$$

Then if C_1, is the semicircular arc $Re^{i\theta}$, $0 \leq \theta \leq \pi$, and $R > R_0$, we have

$$\lim_{R\to\infty} \int_{C_1} f(z) e^{ivz} dz = 0 \quad \text{when } v > 0 \qquad (4)$$

When $v < 0$, there is a corresponding theorem that applies in the lower half-plane.

Notice that when the factor e^{ivz} is not present, we require $k > 1$, whereas the validity of Eq. (4) requires the less-restrictive condition $k > 0$.

To prove Eq. (4), we rewrite the integral on the left, which we call I, in terms of polar coordinates; taking $z = Re^{i\theta}$, $dz = Re^{i\theta} i d\theta$, we have

$$I = \int_{C_1} f(z) e^{ivz} dz = \int_0^{\pi} f(R\mathrm{e}^{i\theta}) e^{ivR\mathrm{e}^{i\theta}} iR\mathrm{e}^{i\theta} d\theta \qquad (5)$$

Recall now the inequality

$$\left| \int_a^b g(\theta) d\theta \right| \leq \int_a^b |g(\theta)| d\theta$$

Applying this to Eq. (5) and resulting that $\left| e^{i\theta} \right| = 1$, we have

$$|I| \leq R \int_0^{\pi} \left| f(R\mathrm{e}^{i\theta}) \right| \left| e^{ivR\mathrm{e}^{i\theta}} \right| d\theta \qquad (6)$$

We see that

$$\left| e^{ivR\mathrm{e}^{i\theta}} \right| = \left| e^{ivR(\cos\theta + i\sin\theta)} \right| = \left| e^{-vR\sin\theta} \right| \left| e^{ivR\cos\theta} \right|$$

Now

$$\left| e^{ivR\cos\theta} \right| = 1$$

and since $e^{-vR\sin\theta} > 0$, we find that

$$\left| e^{ivR\mathrm{e}^{i\theta}} \right| = e^{-vR\sin\theta}$$

Rewriting Eq. (6) with the aid of the previous equation, we have

$$|I| \leq R \int_0^{\pi} \left| f(R\mathrm{e}^{i\theta}) \right| e^{-vR\sin\theta} d\theta$$

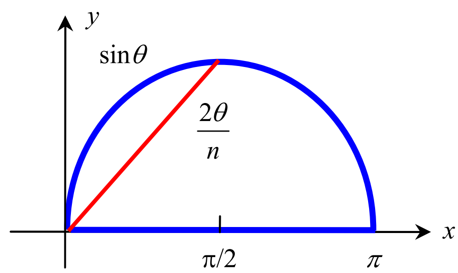
With the assumption $|f(z)| = \left| f(Re^{i\theta}) \right| \leq \mu / R^k$, it should be clear that

$$|I| \leq R \int_0^{\pi} \frac{\mu}{R^k} e^{-vR\sin\theta} d\theta = \frac{\mu}{R^{k-1}} \int_0^{\pi} e^{-vR\sin\theta} d\theta \qquad (7)$$

Since $\sin\theta$ is symmetric about $\theta = \pi/2$ (see the figure shown below), we can perform the integration on the right in Eq. (7) from 0 to $\pi/2$ and the double the result.

Hence

$$|I| \leq \frac{2\mu}{R^{k-1}} \int_0^{\pi/2} e^{-vR\sin\theta} d\theta \qquad (8)$$



The above figure also shows that over the interval $0 \leq \theta \leq \pi/2$, we have

$$|I| \leq \frac{\mu}{R^{k-1}} \int_0^{\pi/2} e^{-vR\theta 2/\pi} d\theta = \frac{\pi\mu}{vR^k} \left[1 - e^{-vR} \right]$$

With $R \to \infty$, the right-hand side of his equation becomes zero, which implies $I \to 0$ in the same limit. Thus

$$\lim_{R\to\infty} \int_{C_1} f(z) e^{ivz} dz = 0$$

♣ Jordan's Lemma

$$\lim_{R\to\infty} \int_{C_1} \frac{P(z)}{Q(z)} e^{ivz} dz = 0 \quad \text{if} \quad v > 0, \ \text{degree } Q - \text{degree } P \geq 1 \qquad (9)$$

3. Improper Integrals From Fourier Analysis

Evaluation for $\int e^{ivz} P(z)/Q(z) dz$:

All zeros of $Q(z)$ in u.h.p. are assumed enclosed by the contour, and we also assume $Q(x) \neq 0$ for all real values of x. Therefore,

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{ivx} dx + \int_{C_1} \frac{P(z)}{Q(z)} e^{ivz} dz = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right] \quad \text{at all poles in u.h.p.} \qquad (10)$$

Now, provided the degrees of Q and P are as described in Eq. (9), we put $R \to \infty$ in Eq. (10) and discard the integral over C_1 in this equation by invoking Jordan's lemma. We obtain the following:

$$\int_{-\infty}^{+\infty} \frac{P(x)}{Q(x)} e^{ivx} dx = 2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right] \qquad (11)$$

The derivation of Eq. (11) requires that $v > 0$, $Q(x) \neq 0$ for $-\infty < x < \infty$, and the degree of Q exceed the degree of P by at least 1.

We now apply Euler's identity on the left in eq. (11) and obtain

$$\int_{-\infty}^{+\infty} (\cos vx + i \sin vx) \frac{P(x)}{Q(x)} dx = 2\pi i \sum \text{Res}\left[e^{ivz} \frac{P(z)}{Q(z)}\right] \quad \text{in u.h.p.}$$

We can then equate corresponding parts (reals and imaginaries) on each side of the preceding equation with the result that

$$\int_{-\infty}^{+\infty} \cos vx \frac{P(x)}{Q(x)} dx = \text{Re}\left[2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right]\right] \quad \text{in u.h.p.} \qquad (12a)$$

$$\int_{-\infty}^{+\infty} \sin vx \frac{P(x)}{Q(x)} dx = \text{Im}\left[2\pi i \sum \text{Res}\left[\frac{P(z)}{Q(z)} e^{ivz}\right]\right] \quad \text{in u.h.p.} \qquad (12b)$$

where degree Q − degree $P \geq 1$, $Q(x) \neq 0$ for $-\infty < x < \infty$, and $v > 0$.

4. If

1) function f is rational, and is in the following form as

$$f(x) = \frac{p(x)}{q(x)}$$

where $q(x) \neq 0$, for all real x.

2) the degree of $q(x) \geq$ the degree of $[p(x)+1]$

3) z_j, $j = 1, 2, 3, \cdots, k$ are the poles of $f(z)e^{imz}$ located in the upper half plane

4) and $m > 0$,

then we obtain that

$$\int_{-\infty}^{\infty} f(x) e^{imz} dx = 2\pi i \sum_{j=1}^{K} \operatorname*{Res}_{z=z_j} f(z) e^{imz}$$

5. Some Examples

Example 2

Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+4)(x^2+9)} dx$$

<Sol.>
***

Because the function *f* can be given by

$$f(z)=\frac{1}{(z^2+4)(z^2+9)}=\frac{p(z)}{q(z)}$$

which satisfies the conditions mentioned above.

Thus, in the upper half plane the poles of $f(z)e^{imz}$ are given by

$$q(z)=(z^2+4)(z^2+9)=0$$

namely $z=2i$ and $z=3i$. Taking $m=1$, we can obtain

$$\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+4)(x^2+9)}dx$$

$$=\mathrm{Re}\left\{2\pi i\left[\operatorname*{Res}_{z=2i}\frac{e^{iz}}{(z^2+4)(z^2+9)}+\operatorname*{Res}_{z=3i}\frac{e^{iz}}{(z^2+4)(z^2+9)}\right]\right\}$$

$$=\mathrm{Re}\left\{2\pi i\left[\frac{e^{iz}}{(z+2i)(z^2+9)}\bigg|_{z=2i}+\frac{e^{iz}}{(z^2+4)(z+3i)}\bigg|_{z=3i}\right]\right\}$$

$$=\mathrm{Re}\left\{2\pi i\left[\frac{e^{-2}}{20i}-\frac{e^{-3}}{30i}\right]\right\}$$

$$=\pi\left(\frac{e^{-2}}{10}-\frac{e^{-3}}{15}\right)$$

Example 3

Evaluate

$$\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx$$

<Sol.>

$$\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx$$

$$=\mathrm{Im}\left\{2\pi i\left[\operatorname*{Res}_{z=2i}\frac{e^{iz}}{(z^2+4)(z^2+9)}+\operatorname*{Res}_{z=3i}\frac{e^{iz}}{(z^2+4)(z^2+9)}\right]\right\}$$

$$=\mathrm{Im}\left\{2\pi i\left[\frac{e^{-2}}{20i}-\frac{e^{-3}}{30i}\right]\right\}$$

$$=\mathrm{Im}\left[\pi\left(\frac{e^{-2}}{10}-\frac{e^{-3}}{15}\right)\right]$$

$$=0$$

Also, we can solve the problem in the following way:

Since

$$f(x)=\frac{\sin x}{(x^2+4)(x^2+9)}$$

is an odd function, then

$$\Rightarrow\int_{-\infty}^{\infty}\frac{\sin x}{(x^2+4)(x^2+9)}dx=0$$

Example 4

Evaluate

$$\int_{0}^{\infty}\frac{x\sin x}{(x^2+1)(x^2+4)}dx$$

<Sol.> Since $f(x)=\dfrac{x\sin x}{(x^2+1)(x^2+4)}$

is an even function, we can

$$\int_0^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+1)(x^2+4)}dx$$

$$= \frac{1}{2}\,\mathrm{Im}\left\{2\pi i\left[\operatorname*{Res}_{z=i}\frac{ze^{iz}}{(z^2+1)(z^2+4)} + \operatorname*{Res}_{z=2i}\frac{ze^{iz}}{(z^2+1)(z^2+4)}\right]\right\}$$

$$= \mathrm{Im}\left\{\pi i\left[\left.\frac{ze^{iz}}{(z^2+1)(z^2+4)}\right|_{z=i} + \left.\frac{ze^{iz}}{(z^2+1)(z^2+4)}\right|_{z=2i}\right]\right\}$$

$$= \mathrm{Im}\left[\pi i\left(\frac{e^{-1}}{6}-\frac{e^{-2}}{6}\right)\right]$$

$$= \frac{\pi}{6}\left(\frac{1}{e}-\frac{1}{e^2}\right)$$

Example 5

Evaluate

$$\int_{-\infty}^{+\infty}\frac{xe^{i\omega x}}{x^4+1}dx,\quad \int_{-\infty}^{+\infty}\frac{x\cos\omega x}{x^4+1}dx, \text{ and } \int_{-\infty}^{+\infty}\frac{x\sin\omega x}{x^4+1}dx \quad \text{for } \omega > 0.$$

<Sol.>

Using Eqs.(11) and (12) for the evaluations, we see all involve our finding

$$2\pi i\sum \mathrm{Res}\left[\frac{z}{z^4+1}e^{i\omega z}\right] \text{ at the poles of } \frac{z}{z^4+1} \text{ in the upper half-plane}$$

The poles of $f(z) = z/(z^4+1)$ in u.h.p are $\pm\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}$. Thus, we have

$$2\pi i\sum \mathrm{Res}\left[\frac{z}{z^4+1}e^{i\omega z}\right] \text{ in u.h.p.} = \frac{\pi i}{2}\left[\frac{e^{i\omega(1+i)/\sqrt{2}}}{i}+\frac{e^{i\omega(-1+i)/\sqrt{2}}}{-i}\right]$$

$$\Rightarrow \qquad 2\pi i\sum \mathrm{Res}\left[\frac{z}{z^4+1}e^{i\omega z}\right] \text{ in u.h.p.} = i\pi e^{-\omega/\sqrt{2}}\sin(\omega/\sqrt{2})$$

Applying Eqs.(11) and (12) to this last result, we evaluate three integrals:

$$\int_{-\infty}^{+\infty}\frac{xe^{i\omega x}}{x^4+1}dx = i\pi e^{-\omega/\sqrt{2}}\sin(\omega/\sqrt{2})$$

$$\int_{-\infty}^{+\infty}\frac{x\cos(\omega x)}{x^4+1}dx = \mathrm{Re}\left[i\pi e^{-\omega/\sqrt{2}}\sin(\omega/\sqrt{2})\right] = 0$$

$$\int_{-\infty}^{+\infty}\frac{x\sin(\omega x)}{x^4+1}dx = \mathrm{Im}\left[i\pi e^{-\omega/\sqrt{2}}\sin(\omega/\sqrt{2})\right] = \pi e^{-w/\sqrt{2}}\sin(\omega/\sqrt{2})$$

6. Jordan's Inequality

The $\cos\theta$ is a strictly decreasing function for $0 \le \theta \le \pi/2$.

Therefore, the mean (or average) coordinate of the graph $y = \cos x$ over the range $0 \le x \le \theta$ also decreasing steadily.

This means that the coordinate is given by

$$\frac{1}{\theta}\int_0^{\theta}\cos x\,dx = \frac{\sin\theta}{\theta}$$

Hence, for $0 \le \theta \le \pi/2$, we have

$$\frac{2}{\pi} \le \frac{\sin\theta}{\theta} \le 1$$

Thus, we have

$$\frac{2\theta}{\pi} \le \sin\theta \le \theta, \text{ where } 0 \le \theta \le \frac{\pi}{2} \qquad \text{(A)}$$

and

$$\int_0^{\pi} e^{-m\sin\theta}d\theta < \frac{\pi}{m}, \qquad \text{for } m > 0$$

which is known as Jordan's Inequality.

Observe that

$$\int_0^{\pi} e^{-m\sin\theta}d\theta = \int_0^{\frac{\pi}{2}} e^{-m\sin\theta}d\theta + \int_{\frac{\pi}{2}}^{\pi} e^{-m\sin\theta}d\theta$$

where m is a real number.
Let $\theta = \pi - \phi$, substitute it into the last integral above.
We can obtain that

$$\int_{\frac{\pi}{2}}^{\pi} e^{-m\sin\theta}d\theta = -\int_{\frac{\pi}{2}}^{0} e^{-m\sin(\pi-\phi)}d\theta$$

$$= \int_0^{\frac{\pi}{2}} e^{-m\sin\theta}d\theta$$

Thus, we have

$$\int_0^{\pi} e^{-m\sin\theta}d\theta = 2\int_0^{\frac{\pi}{2}} e^{-m\sin\theta}d\theta$$

From Eq. (a), if $m > 0$

$$e^{-m\sin\theta} \le e^{-2m\theta/\pi} \quad \text{when} \quad 0 \le \theta \le \pi/2$$

and so

$$\int_0^{\frac{\pi}{2}} e^{-m\sin\theta}d\theta \le \int_0^{\frac{\pi}{2}} e^{-2m\theta/\pi}d\theta = \frac{\pi}{2m}\left(1-e^{-m}\right)$$

Hence,

$$\int_0^{\frac{\pi}{2}} e^{-m\sin\theta}d\theta \le \frac{\pi}{2m}, \quad m > 0 \qquad \text{(B)}$$

Combining Eqs.(A) and (B) yields

$$\int_0^{\pi} e^{-m\sin\theta}d\theta < \frac{\pi}{m}, \qquad \text{for} \quad m > 0$$

H.W. 1 Evaluate the following integrals by residue calculus. Use Cauchy principal values and take advantage of even and odd symmetries where appropriate.

(a) $\displaystyle\int_{-\infty}^{\infty} \frac{(x^3+x^2)\cos(\sqrt{2}x)}{x^4+1}dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{xe^{ix/3}}{(x-i)^2+4}dx$

(c) $\displaystyle\int_{-\infty}^{\infty} \frac{(x^3+x^2+x)\sin(x/2)}{(x^2+1)(x^2+4)}dx$ (d) $\displaystyle\int_0^{\infty} \frac{\cos x}{(x^2+4)^2}dx$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 7, 8, 12, and 13, Exercise 6.6, Pearson Education, Inc., 2005.】

H.W.2 Show that

$$\int_0^{\infty} \frac{\sin mx \sin nx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma}\sinh na, \text{ for } m \ge n \ge 0. \text{ Assume } a \text{ is positive.}$$

<Hint> Express $\sin mx \sin nx$ as a sum involving $\cos(m+n)x$ and $\cos(m-n)x$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 14, Exercise 6.6, Pearson Education, Inc., 2005.】

H.W.3 The following is an extension of the work used in deriving Eq.(11) and deals with integrands having removable singularities. In parts (b) and (c), take $m,\ n > 0$.

(a) Show that $\dfrac{e^{imz}-e^{inz}}{z}$ has a removable singularity at $z = 0$.

(b) Use the above to prove that $\displaystyle\int_0^{\infty} \frac{\sin mx - \sin nx}{x}dx = 0$.

(c) Show that $\displaystyle\int_{-\infty}^{\infty} \frac{\sin mx - \sin nx}{x(x^2+a^2)}dx = \frac{\pi}{a^2}\left[e^{-na}-e^{-ma}\right]$, where $a > 0$.

(d) Show that $\int_{-\infty}^{\infty} \frac{\cos^2(\pi x/2)}{x^4-1}dx = -\frac{\pi}{4}\left(1+e^{-\pi}\right)$.

<Hint> Study $\frac{1+e^{i\pi z}}{z^4-1}$, noting the removable singularities.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 14, Exercise 6.6, Pearson Education, Inc., 2005.】

H.W.4 To establish the well-known result

$$\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2},$$

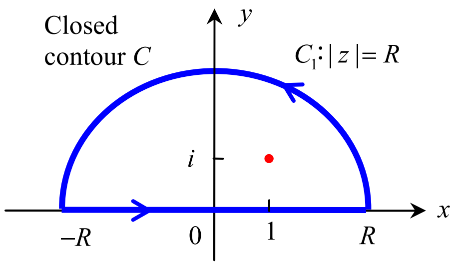
we proceed as follows:

(a) Show that

$$f(z) = \frac{e^{iz}-1}{z}$$

Has a removable singularity at $z=0$. How should $f(0)$ be defined to remove the singularity?

(b) Using the contour shown below, prove that



$$\int_{-R}^{R} \frac{e^{ix}-1}{x}dx + \int_{C_1} \frac{e^{iz}-1}{z}dz = 0$$

and also

$$\int_{-R}^{R} \frac{\cos x-1}{x}dx + i\int_{-R}^{R} \frac{\sin x}{x}dx = \int_{C_1} \frac{1}{z}dz - \int_{C_1} \frac{e^{iz}}{z}dz$$

(c) Evaluate the first integral on the above right by using the polar representation of C_1: $z = Re^{i\theta}$, $0 \le \theta \le \pi$. Pass to the limit $R \to \infty$ and explain why the second integral on the right goes to zero. Thus prove that

$$\int_{-\infty}^{\infty} \frac{\cos x-1}{x}dx = 0$$

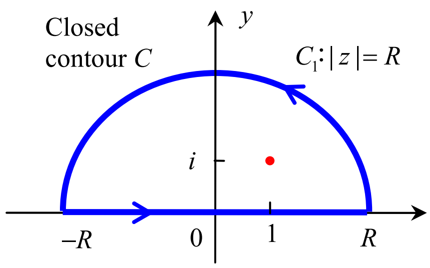
and

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}dx = \pi$$

and finally that

$$\int_0^{\infty} \frac{\sin x}{x}dx = \frac{\pi}{2}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 24, Exercise 6.6, Pearson Education, Inc., 2005.】



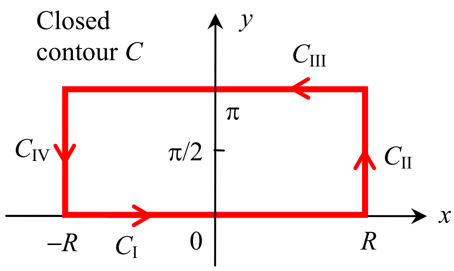
H.W.5 Consider the problem of evaluating

$$I = \int_{-\infty}^{\infty} \frac{\cos x}{\cosh x}dx = \mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{ix}}{\cosh x}dx$$

If we try to evaluate the preceding integral by employing the contour shown above and the methods leading to Eq. (12) we get into difficult because the function $e^{iz}/\cosh z$ has an infinite number of poles in the upper half-plane, i.e., at $z = i(n\pi + \pi/2)$, $n = 0, 1, 2, \ldots$, which are the zeros of $\cosh z$. Thus the Theorem in Point 2, in page 182 is inapplicable. However, we can determine I with the aid of the contour C shown below.

(a) Using residues show that

$$\oint_C \frac{e^{iz}}{\cosh z} dz = 2\pi e^{-\pi/2}$$



(b) Using the appropriate values of z integrals along the top and bottom portions of the rectangle show that

$$\int_{-R}^{R} \frac{e^{ix}}{\cosh x} dx + \int_{-R}^{R} \frac{e^{i(x+i\pi)}}{\cosh(x+i\pi)} dx + \int_{C_{II}} \frac{e^{iz}}{\cosh z} dz + \int_{C_{IV}} \frac{e^{iz}}{\cosh z} dz = 2\pi e^{-\pi/2}$$

(c) Combine the first two integrals on the left in the preceding equation to show that

$$\int_{-R}^{R} \frac{e^{ix}}{\cosh x} dx (1+e^{-\pi}) + \int_{C_{II}} \frac{e^{iz}}{\cosh z} dz + \int_{C_{IV}} \frac{e^{iz}}{\cosh z} dz = 2\pi e^{-\pi/2}$$

(d) Let $R \to \infty$ in the preceding. Using the ***ML*** inequality argue that the integrals along C_{II} and C_{IV} are zero in this limit.

<Hint> Recall that $|\cosh z| = \sqrt{\sinh^2 x + \cos^2 y}$ (see chapter 3).

Thus on C_{II} and C_{IV} we have

$$\left| \frac{e^{iz}}{\cosh z} \right| \le \frac{1}{\sinh R}$$

Prove finally that

$$\int_{-\infty}^{\infty} \frac{\cos x}{\cosh x} dx = \frac{2\pi e^{-\pi/2}}{1+e^{-\pi}} = \frac{\pi}{\cosh(\pi/2)}$$

and explain why

$$\int_{0}^{\infty} \frac{\cos x}{\cosh x} dx = \frac{\pi}{2\cosh(\pi/2)}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 26, Exercise 6.6, Pearson Education, Inc., 2005.】

H.W.6 Using a technique similar to that employed in **H.W.5** prove the following:

(a) $\displaystyle\int_{0}^{\infty} \frac{\cosh x}{\cosh ax} dx = \frac{\pi}{a\cos(\pi/2a)}$ for $a > 1$

<Hint> Use a rectangle like that in the above figure but having height π/a. Take $\cosh z/\cosh az$ as the integrand, and prove that the integrations along segments C_{II} and C_{IV} each go to zero as $R \to \infty$.

(b) $\displaystyle\int_{-\infty}^{\infty} \frac{e^x}{1+e^{ax}} dx = \frac{\pi}{a\sin(\pi/a)}$ for $a > 1$

<Hint> Use a rectangle like that in the above figure but having height $2\pi/a$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 27 and 28, Exercise 6.6, Pearson Education, Inc., 2005.】

H.W.7 The result

$$\int_{-\infty}^{\infty} e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{a}, \quad a > 0$$

is derived in many standard texts on real calculus. Use this identity to show that

$$\int_{-\infty}^{\infty} e^{-m^2x^2} \cos bx \, dx = \frac{\sqrt{\pi}}{m} e^{-b^2/(4m^2)}$$

where b is a real number and $m > 0$.

<Hint> Assume $b>0$. Integrate $e^{-m^2z^2}$ around a contour similar to that in the above figure. Take $b/(2m^2)$ as the height of the rectangle. Argue that the integrals along C_{II} and C_{IV} vanish as $R\to\infty$. Why does your result apply to the case $b\le 0$?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3[rd] ed., Problem 29, Exercise 6.6, Pearson Education, Inc., 2005.】

§6-7 Integrals Involving Indented Contour

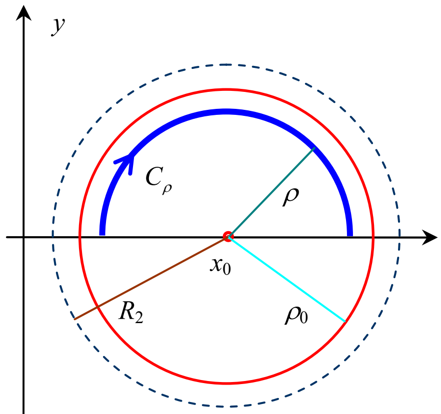


Fig. A

1. **Theorem**

Suppose that

(1) a function $f(z)$ has a simple pole at a point $z=x_0$ on the real axis, with a Laurent series representation in a punctured disk $0<|z-x_0|<R_2$ (see **Fig.A**) and with residue B_0;

(2) C_ρ denotes the upper half of a circle $|z-x_0|=\rho$, where $\rho<R_2$ and the clockwise direction is taken.

Then

$$\lim_{\rho\to 0}\int_{C_\rho} f(z)dz=-B_0\pi i \qquad (1)$$

<pf.>

Assuming that the conditions (1) and (2) are satisfied, we start the proof of the theorem by writing the Laurent series in part (1) as

$$f(z)=g(z)+\frac{B_0}{z-x_0} \qquad \text{for } 0<|z-x_0|<R_2$$

where

$$g(z)=\sum_{n=0}^{\infty}a_n(z-x_0)^n, \qquad (|z-x_0|<R_2)$$

Thus

$$\int_{C_\rho} f(z)dz=\int_{C_\rho} g(z)dz+B_0\int_{C_\rho}\frac{1}{z-x_0}dz \qquad (2)$$

Now the function $g(z)$ is continuous when $|z-x_0|<R_2$, according to **the convergence theorem of power series**. Hence if we choose a number ρ_0 such that $\rho<\rho_0<R_2$ (see **Fig. A**), it must be bounded on the *closed disc* $|z-x_0|<\rho_0$, according to the section of discussing **continuity**. That is, there is a nonnegative constant M such that

$$|g(z)|\le M \qquad \text{where} \quad |z-x_0|<\rho_0;$$

and, since the length L of the path C_ρ is $L=\pi\rho$, it follows that

$$\left|\int_{C_\rho} g(z)dz\right|\le ML=M\pi\rho$$

Consequently,

$$\lim_{\rho\to 0}\int_{C_\rho} g(z)dz=0 \qquad (3)$$

Inasmuch as the semicircle — $-C_\rho$ has parametric representation

$$z=x_0+\rho e^{i\theta} \qquad (0\le\theta\le\pi)$$

the second integral on the right in Eq. (2) has the value

$$\int_{C_\rho}\frac{1}{z-x_0}dz=-\int_{-C_\rho}\frac{1}{z-x_0}dz=-\int_0^{\pi}\frac{1}{\rho e^{i\theta}}\rho i e^{i\theta}d\theta=-i\int_0^{\pi}d\theta=-i\pi$$

Thus

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{1}{z - x_0} dz = -\pi i \qquad (4)$$

Limit (1) now follows by letting ρ tend to zero on each side of equation (2) and referring to limits (3) and (4).

♣ **Concept review**

1) Let the function $f(z)$ possesses a simple pole at the point z_0, and let C be a circle of radius r centered at z_0.
2) Suppose c_{-1} is the residue of $f(z)$ at z_0.
3) Assuming that $f(z)$ has no other singularities on and inside C, we know that

$$\oint_C f(z)dz = 2\pi i c_{-1} = 2\pi i \operatorname*{Res}_{z_0} f(z) \qquad (1)$$

2. **Theorem**

Let the function $f(z)$ have a simple pole at the point z_0. An arc C_0 of radius r is constructed using z_0 as its center. The arc subtends an angle α at z_0 (see **Fig. B**). Then

$$\lim_{r \to 0} \int_{C_0} f(z)dz = 2\pi i \left[\frac{\alpha}{2\pi} \operatorname{Res}\left[f(z), z_0 \right] \right] \qquad (2)$$

where the integration is done in the counterclockwise direction. (For clockwise integration, a factor of -1 is placed on the right of Eq. (2).)

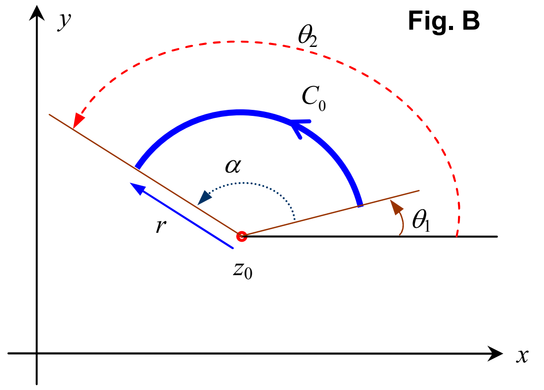


Fig. B

♦ If C_0 is a circle, then $\alpha = 2\pi$.

<pf.>

We first expand $f(z)$ in a Laurent series about z_0.

Because of the simple pole at z_0, the series assumes the form

$$f(z) = \frac{c_{-1}}{(z - z_0)} + \sum_{n=0}^{\infty} c_n (z - z_0)^n = \frac{c_{-1}}{(z - z_0)} + g(z)$$

where

$$g(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n \equiv \text{Taylor series is analytic at } z_0$$

and
$$c_{-1} = \operatorname*{Res}_{z_0} f(z)$$

We now integrate the series expansion of $f(z)$ along C_0 in **Fig.B**. Thus

$$\int_{C_0} f(z)dz = \int_{C_0} \frac{c_{-1}}{(z - z_0)} dz + \int_{C_0} g(z)dz \qquad (3)$$

Applying ***ML* inequality** to the second integral on the right in Eq.(3), we have

$$\left| \int_{C_0} g(z)dz \right| \le Mr\alpha \qquad (4)$$

where

$$|g(z)| \le M \equiv \text{upper bound of } g(z) \text{ in a neighborhood of } z_0$$

$$r\alpha \equiv \text{ the length of contour } C_0$$

From eq. (4), we see that

$$\lim_{r \to 0} \int_{C_0} g(z)dz = 0 \qquad (5)$$

With $z = z_0 + re^{i\theta}$, $dz = ire^{i\theta} d\theta$, and the limits on θ indicated in **Fig.B**, we have

$$\int_{C_0} \frac{c_{-1}}{(z-z_0)} dz = \int_{\theta_1}^{\theta_1+\alpha} \frac{c_{-1} i r e^{i\theta} d\theta}{r e^{i\theta}} = c_{-1} \alpha i = 2\pi i \frac{\alpha}{2\pi} c_{-1} \qquad (6)$$

$$\int_{-1}^{+1} \frac{1}{x} dx, \quad \int_{1}^{3} \frac{1}{(x-2)} dx, \quad \text{and} \quad \int_{-\pi/2}^{\pi/2} \frac{1}{\sin x} dx$$

13. 課本§22-4 利用 Residue Theorem 求級數和的方法.