CHAPTER FIVE

Power Series

§5-1 Introduction and Review of Real Series

1. Examples of power series

1) $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ (1)

2) $\frac{1}{1-x} = \sum_{n=0}^{\infty} \frac{(x+1)^n}{2^{n+1}} = \frac{1}{2} + \frac{(x+1)}{4} + \frac{(x+1)^2}{8} + \cdots$ (2)

2. Power series expansion of real function — Taylor series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \cdots + c_n(x-x_0)^n + \cdots \quad (3)$$

where

$$c_n = \frac{f^{(n)}(x_0)}{n!}. \quad (4)$$

3. Applications of power series

1) Numerical approximation for integral

Example

$$\int_0^{0.2} (e^x - 1)/x dx \xrightarrow{e^x \text{ replaced by its Taylor series}} \int_0^{0.2} (1 + x/2) dx$$

2) Evaluation of the sum of infinite power series

Example

For Eq. (2), substituting $x = -1/2$ gives

$$\frac{2}{3} = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots$$

♣ **Some difficulties**

1) Not all functions of x have a Taylor series expansion.

Example

$x^{1/2} = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow$ Since $x^{1/2}$ does not possess a first- or higher-order derivative at $x = 0$

2) Limitation in convergence of the sum of infinite series

Example

With $x = 2$ on both sides in Eq. (2), we obtain

$-1 = \frac{1}{2} + \frac{3}{4} + \frac{9}{8} + \cdots \quad \Rightarrow$ Clearly, the infinite sum will not yield the numerical value -1.

§5-2 Complex Sequence and Convergence of Complex Series

1. **Definition (convergence and limit of a complex sequence)**

The infinite sequence $p_1(z),\ p_2(z),\ p_3(z),\ \cdots,,\ p_n(z),\ \cdots$ converges and is said to have a limit $P(z)$, for a value of z lying in some region R, if given a constant $\epsilon > 0$ we can find a number N such that

$$|P(z) - p_n(z)| < \epsilon \quad \text{for all } n > N \quad (1)$$

We then write

$$\lim_{n\to\infty} p_n(z) = P(z) \quad (2)$$

Example 1

For the sequence $1+e^{-z}$, $1+e^{-2z}$, $1+e^{-3z}$, …, $1+e^{-nz}$, …, show that the limit is 1 if $x=\mathrm{Re}(z)>0$

<pf.>

With $P(z)=1$, $p_n(z)=1+e^{-nz}$ and with $0<\epsilon<1$, we employ Eq.(1) and obtain the requirement

$$\left|1-e^{-nz}-1\right|=\left|e^{-nz}\right|<\epsilon, \text{ for } n>N.$$

This is equivalent to

$$e^{-nx}<\epsilon \quad \text{or} \quad e^{nx}<\frac{1}{\epsilon}, \text{ for } n>N.$$

$$\Rightarrow \qquad n>\frac{1}{x}\ln\left(\frac{1}{\epsilon}\right)$$

If we take N as the integer that equals or exceeds the positive quantity $\frac{1}{x}\ln\left(\frac{1}{\epsilon}\right)$, then the condition $\left|e^{-nz}\right|<\epsilon$ will be satisfied for all $n>N$.

Note the necessity for our having chosen x as positive as it guarantees that

$$e^{-Nx}>e^{-(N+1)x}>e^{-(N+2)x}$$

i.e. if $\left|e^{-nz}\right|<\epsilon$ is satisfied for $n=N$, then it is satisfied for all $n>N$.

Since we take

$$N\geq\frac{1}{x}\ln\left(\frac{1}{\epsilon}\right)$$

it is clear that N depends on both $x=\mathrm{Re}(z)$ and ϵ, and grows as ϵ shrinks.

$$\Rightarrow \qquad \lim_{n\to\infty}\left(1+e^{-nz}\right)=1$$

2. Limits of complex sequences

If $p_n(z)=v_n(z)+iw_n(z)$ and $P(z)=V(z)+iW(z)$ (v_n, w_n, V, W are real functions), then

1) $\lim_{n\to\infty} p_n(z)=P$ if and only if $\lim_{n\to\infty} v_n(z)=V$ and $\lim_{n\to\infty} w_n(z)=W$

2) If $\lim_{n\to\infty} p_n(z)=P$ and $\lim_{n\to\infty} q_n(z)=Q$, then

a) $\lim_{n\to\infty}\left[p_n(z)+q_n(z)\right]=P+Q$

b) $\lim_{n\to\infty}\left[p_n(z)q_n(z)\right]=PQ$

c) $\lim_{n\to\infty}\left[p_n(z)/q_n(z)\right]=P/Q$, if $Q\neq 0$

3. Some useful limits

1) $\lim_{n\to\infty} r^n=0$, if $|r|<1$.

2) $\lim_{n\to\infty} n^k r^n=0$, if $|r|<1$, k real.

3) $\lim_{n\to\infty}(1+x/n)^n=e^x$, x real.

Example 2

Using the result $\lim_{n\to\infty}\left[1+(1/n)\right]^n=e$, find the limit of the sequence $\left(1+e^{-z}\right)(1+1/1)$, $\left(1+e^{-2z}\right)(1+1/2)^2$, $\left(1+e^{-3z}\right)(1+1/3)^3$, …, $\left(1+e^{-nz}\right)(1+1/n)^n$, … for $\mathrm{Re}(z)>0$.

<Sol.>

Taking $p_n(z)=1+e^{-nz}$, $P=1$, $q_n=(1+1/n)^n$, $Q=e$, we have

$$\lim_{n\to\infty}\left[p_n(z)q_n(z)\right]=PQ=e$$

4. Series and partial sum

1) Series: $u_1(z)+u_2(z)+\cdots=\sum_{j=1}^{\infty}u_j(z)$

2) Partial sum: $S_n(z)=\sum_{j=1}^{n}u_j(z)$

Example: $S_1(z)=u_1(z)$

$S_2(z)=u_1(z)+u_2(z)$

$S_3(z)=u_1(z)+u_2(z)+u_3(z)$

5. **Definition for Ordinary Convergence**

For a convergent series that, given $\epsilon>0$, there exists an integer $N(\epsilon,z)$ such that

$|S_n(z)-S(z)|<\epsilon$, for all $n>N$.

$$\lim_{n\to\infty}S_n(z)=S(z) \quad\Rightarrow\quad S(z)=\sum_{j=1}^{\infty}u_j(z)$$

♣ The set of all values of z for which the series converges is called its ***region of convergence*** **(ROC)**.

Example 3

Show that

$$\sum_{j=1}^{\infty}z^{j-1}=1+z+z^2+\ldots=S(z)=\frac{1}{1-z},\quad |z|<1. \qquad \text{(A)}$$

<pf.>

The n-th partial sum is

$$S_n(z)=1+z+z^2+\ldots+z^{n-1}$$

$$\Rightarrow \qquad S_n(z)-zS_n(z)=\left(1+z+z^2+\ldots+z^{n-1}\right)-\left(z+z^2+\ldots+z^n\right)=1-z^n$$

so that

$$(1-z)S_n(z)=1-z^n$$

Or, for $z\neq 1$

$$S_n(z)=\frac{1-z^n}{1-z}=1+z+z^2+\ldots+z^{n-1}. \qquad \text{(a)}$$

Since the sum in Eq. (A) is $S(z)=1/(1-z)$, we have

$$|S(z)-S_n(z)|=\left|\frac{1-(1-z^n)}{1-z}\right|=\frac{|z|^n}{|1-z|}. \qquad \text{(b)}$$

Referring the above definition, we require for convergence that

$$\frac{|z|^n}{|1-z|}<\epsilon,\ \text{for}\ n>N \qquad \text{(c)}$$

or that

$$\left|\frac{1}{z}\right|^n>\frac{1}{\epsilon|1-z|}.$$

Taking logarithms of the preceding, we obtain

$$n\,\mathrm{Ln}\left|\frac{1}{z}\right|>\mathrm{Ln}\frac{1}{\epsilon|1-z|}.$$

Inside the disc $|z|<1$, we have $|1/z|>1$ and $\mathrm{Ln}|1/z|>0$. The above inequality can be rearranged as

$$n > \frac{\mathrm{Ln}\left(\dfrac{1}{\epsilon\,|1-z|}\right)}{\mathrm{Ln}\left|\dfrac{1}{z}\right|} = \frac{\mathrm{Ln}\left(\epsilon\,\mathrm{Ln}|1-z|\right)}{\mathrm{Ln}|z|} \qquad \text{(d)}$$

If we choose N as appositive integer that equals or exceeds the right side of Eq. (d) and take $n > N$, then Eq. (c) is satisfied. Hence,

$|S_n(z) - S(z)| < \epsilon$, for all $n > N$ ------------ ***Q.E.D. (Quod erat demonstradum!)***

Example 4

Given infinite series $\sum_{n=0}^{\infty} e^{inz} = 1 + e^{iz} + e^{i2z} + \ldots\ldots$, find its region of convergence.

<Sol.>

We know that

$$1 + e^{iz} + e^{i2z} + \ldots\ldots = \frac{1}{1-e^{iz}}, \quad |e^{iz}| < 1.$$

Now

$$|e^{iz}| = |e^{i(x+y)}| = |e^{ix}e^{-y}| = |e^{ix}||e^{-y}| = e^{-y}$$

$|e^{ix}| = 1$ and $e^{-y} > 0$

The requirement for convergence of our given series $|e^{iz}| < 1$ now becomes $e^{-y} < 1$. This means that $y > 0$. Hence, the region of convergence is $\mathrm{Im}\, z > 0$

6. **Theorem**

The convergence of both the real series $\sum_{j=1}^{\infty} R_j(x,y)$ and $\sum_{j=1}^{\infty} I_j(x,y)$ is a necessary and sufficient condition for the convergence of $\sum_{j=1}^{\infty} u_j(z)$, where $u_j(z) = R_j(x,y) + iI_j(x,y)$. If $\sum_{j=1}^{\infty} R_j(x,y)$ and $\sum_{j=1}^{\infty} I_j(x,y)$ converge to the functions $R(x,y)$ and $I(x,y)$, respectively, then $\sum_{j=1}^{\infty} u_j(z)$ converges to $S(z) = R(x,y) + iI(x,y)$. Conversely, if $\sum_{j=1}^{\infty} u_j(z)$ converges to $S(z) = R(x,y) + iI(x,y)$, then $\sum_{j=1}^{\infty} R_j(x,y)$ converges to $R(x,y)$ and $\sum_{j=1}^{\infty} I_j(x,y)$ converges to $I(x,y)$.

Example 5

Given infinite series

$$1 + e^{-y}\cos x + e^{-2y}\cos 2x + \ldots..,$$

which is obtained by taking the real part of each term in the series of **Example 4**. Find the sum of this new series.

<Sol.>

The series of **Example 4** converges to $1/(1-e^{iz})$ in the domain $\mathrm{Im}\, z > 0$. Thus the series of the present example converges to $\mathrm{Re}[1/(1-e^{iz})]$ in this domain. We have

$$\mathrm{Re}\left[\frac{1}{1-e^{iz}}\right] = \mathrm{Re}\left[\frac{e^{-iz/2}}{e^{-iz/2} - e^{iz/2}}\right] = \mathrm{Re}\left[\frac{\cos(z/2) - i\sin(z/2)}{-2i\sin(z/2)}\right] = \mathrm{Re}\left[\frac{i}{2}\cos\left(\frac{z}{2}\right) + \frac{1}{2}\right]$$

Now

$$\cot\left(\frac{z}{2}\right) = \frac{\sin x - i\sinh y}{\cosh y - \cos x}$$

Thus, the sum of our series is

$$\frac{\sinh y}{2\left(\cosh y - \cos x\right)} + \frac{1}{2}$$

7. **Theorem: *n*th Term Test**

The series $\sum_{n=1}^{\infty} u_n(z)$ diverges **if**

$$\lim_{n\to\infty} u_n(z) \neq 0 \qquad (3)$$

or, equivalently, **if**

$$\lim_{n\to\infty} |u_n(z)| \neq 0 \qquad (4)$$

Example 6

Use the above Theorem to show that the series of **Example 3** $\sum_{j=1}^{\infty} z^{j-1}$, diverges for $|z|>1$.

<pf.>

We take $u_n(z) = z^{n-1}$ and $|u_n(z)| = |z^{n-1}| = |z|^{n-1}$. If $|z|=1$, then

$$\lim_{n\to\infty} |u_n(z)| = \lim_{n\to\infty} 1^{n-1} = 1$$

Since this limit is nonzero, the series diverges if $|z|=1$. For $|z|>1$,

$$\lim_{n\to\infty} |z|^{n-1} = \infty$$

which is clearly nonzero. The series again diverges.

Notice that with $|z|<1$ we have

$$\lim_{n\to\infty} |z|^{n-1} = 0$$

However, this is of no use in proving that the series converges for $|z|<1$.

8. ***Some Definitions and Theorems***

1) Definition: Absolute and Conditional Convergence

The series $\sum_{j=1}^{\infty} u_j(z)$ is called ***absolutely convergent*** if $\sum_{j=1}^{\infty} |u_j(z)|$ is convergent.

2) Definition: Conditional Convergence

The series $\sum_{j=1}^{\infty} u_j(z)$ is called ***conditionally convergent*** if it converges but $\sum_{j=1}^{\infty} |u_j(z)|$ diverges.

3) Theorem: An absolutely convergent series is independent in ordinary sense.

4) Theorem: The sum of an absolutely convergent series is independent of the order in which the terms are added.

5) Theorem: Two absolutely convergent series can be multiplied together in the same way as one multiplies two polynomials. The resulting series is absolutely convergent. Its sum, which is independent of how the terms are arranged, is the product of the sums of the two original series. If two absolutely convergent series are

$$\sum_{j=1}^{\infty} u_j(z) = S(z) \quad \text{and} \quad \sum_{j=1}^{\infty} v_j(z) = T(z)$$

$$\Rightarrow \quad (u_1v_1) + (u_1v_2 + u_2v_1) + (u_1v_3 + u_2v_2 + u_3v_1) + \cdots = S(z)T(z) \qquad (5)$$

Define Cauchy Product:

$$c_n(z) = \sum_{j=1}^{n} u_j v_{n-j+1} \qquad (6)$$

Then, Eq. (5) can be rewritten as

$$\sum_{n=1}^{\infty} c_n(z) = S(z)T(z) \qquad (7)$$

6) **Theorem (Ratio Test)**:

For the series $\sum_{j=1}^{\infty} u_j(z)$, consider

$$\Gamma(z) = \lim_{j\to\infty} \left| \frac{u_{j+1}(z)}{u_j(z)} \right| \qquad (8)$$

then

(a) the series converges if $\Gamma(z)<1$, and the convergence is absolute;

(b) the series diverges if $\Gamma(z) > 1$;

(c) Eq. (8) provides no information about the convergence of the series if the indicated limit fails to exist or if $\Gamma(z) = 1$.

Example 7

Use the ratio test and the *n*th term test to investigate the convergence of

$$\sum_{j=1}^{\infty} (-1)^j j 2^{j+1} z^{2j} = -4z^2 + 16z^4 - 48z^6 + \dots.$$

<Sol.>

Let

$$u_j = (-1)^j j 2^{j+1} z^{2j} \quad \text{and} \quad u_{j+1} = (-1)^{j+1}(j+1)2^{j+2} z^{2(j+1)}$$

Thus, we have

$$\left|\frac{u_{j+1}}{u_j}\right| = \left|\frac{(-1)^{j+1}(j+1)2^{j+2} z^{2(j+1)}}{(-1)^j j 2^{j+1} z^{2j}}\right| = \left|\frac{j+1}{j} 2z^2\right|.$$

$$\Rightarrow \quad \Gamma(z) = \lim_{j\to\infty}\left|\frac{u_{j+1}}{u_j}\right| = \lim_{j\to\infty}\left|\frac{j+1}{j}2z^2\right| = 2\left|z^2\right|$$

Now, use part (a) of the above theorem and set $\Gamma < 1$. This requires that

$$2\left|z^2\right| < 1 \quad \text{or} \quad |z| < \frac{1}{\sqrt{2}}.$$

On $|z| = 1/\sqrt{2}$ we have $\Gamma = 1$, which provides no information about convergence. However, observe that on $|z| = 1/\sqrt{2}$, we have

$$\left|u_j(z)\right| = j 2^{j+1}\left(\frac{1}{\sqrt{2}}\right)^{2j} = j\frac{2^{j+1}}{2^j} = 2j$$

Clearly, as $j \to \infty$, we do not have $\left|u_j\right| \to 0$. Thus, according to the above theorem **(*m*th term test)**, the series diverges on $|z| = 1/\sqrt{2}$.

§5-3 Uniform Convergence of Series

1. **Definition: Uniform Convergence**

The series $\sum_{j=1}^{\infty} u_j(z)$ whose *n*th partial sum is $S_n(z)$ and is said to converge uniformly to $S(z)$ in a region *R* if, for any $\epsilon > 0$, there exists a number *N* ***independent*** of z so that for all z in *R*

$$\left|S(z) - S_n(z)\right| < \epsilon \quad \text{for all} \quad n > N \qquad (1)$$

2. **Theorem: Weierstrass *M* Test**

Let $\sum_{j=1}^{\infty} M_j$ be a convergent series whose terms $M_1, M_2, \cdots$ are all positive constant. The series $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly in a region *R* if

$$\left|u_j(z)\right| < M_j \quad \text{for all} \quad z \quad \text{in } R \qquad (2)$$

Example 1

Use the *M* test to show that $\sum_{j=1}^{\infty} z^{j-1}$ is uniformly convergent in the disc $|z| \le 3/4$.

<pf.>

From a previous knowledge of real geometric series, if $M_j = (3/4)^{j-1}$, then

$$\sum_{j=1}^{\infty} M_j = 1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + ... = \frac{1}{1-\frac{3}{4}}. \qquad (3)$$

Now with $u_j = z^{j-1}$, we have the given series

$$\sum_{j=1}^{\infty} u_j = \sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + ... \qquad (4)$$

If $|z| \le 3/4$, then the magnitude of each term of the series in Eq. (4) is less than or equal to the corresponding term in Eq. (3), for example, $|z^2| \le (3/4)^2$, $|z^3| \le (3/4)^3$, etc., so that $|u_j(z)| < M_j$ and the ***M* test** is satisfied in the given region.

3. **Some Theorems**

1) Let $\sum_{j=1}^{\infty} u_j(z)$ converge uniformly in a region *R* to $S(z)$. Let $f(z)$ be bounded in *R*, that is $|f(z)| \le k$ (k is constant) throughout *R*. Then in *R*,

$$\sum_{j=1}^{\infty} f(u)u_j(z) = f(z)u_1(z) + f(z)u_2(z) + ... = f(z)S(z)$$

The series converges uniformly to $f(z)S(z)$.

2) Let $\sum_{j=1}^{\infty} u_j(z)$ be a series converging uniformly to $S(z)$ in *R*. If all the functions $u_1(z),\ u_2(z),\ \cdots$ are continuous in *R*, then so is the sum $S(z)$.

3) **Term-by-Term Integration**

Let $\sum_{j=1}^{\infty} u_j(z)$ be a series that is uniformly convergent to $S(z)$ in *R* and let all the terms $u_1(z),\ u_2(z),\ \cdots$ be continuous in *R*. If *C* is a contour in *R*, then

$$\int_C S(z)dz = \sum_{j=1}^{\infty} \int_C u_j(z)dz = \int_C u_1(z)dz + \int_C u_2(z)dz + ...$$

that is, when a uniformly convergent series of continuous functions is integrated term by term the resulting series has a sum that is the integral of the sum of the original series.

Example

Consider

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots, \quad |z| \le r \quad \text{and} \quad r < 1$$

Assume that the contour *C* lies entirely inside the disc $|z| \le r$. The contour is assumed to connect the points $z = 0$ and $z = z'$. Thus, we have

$$\int_0^{z'} \frac{1}{1-z} dz = \int_0^{z'} dz + \int_0^{z'} z dz + \int_0^{z'} z^2 dz + \cdots \qquad (5)$$

In previous chapter, we know that

$$\int_0^{z'} \frac{1}{1-z} dz = -\mathrm{Ln}(1-z)\Big|_0^{z'} = \mathrm{Ln}\left(\frac{1}{1-z'}\right), \qquad (6)$$

We have, finally

$$\mathrm{Ln}\frac{1}{1-z'} = z' + \frac{(z')^2}{2} + \frac{(z')^3}{3} + \cdots = \sum_{j=1}^{\infty} \frac{(z')^j}{j}, \quad |z'| \le r, \quad r < 1$$

The restriction on z' can be written simply $|z'| < 1$.

4) **Theorem: Analyticity of the Sum of a Series**

If $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly to $S(z)$ for all *z* in *R* and if $u_1(z),\ u_2(z),\ \cdots$ are all analytic in R, then $S(z)$ is analytic in R.

5) **Theorem: Term-by-Term Differentiation**

If $\sum_{j=1}^{\infty} u_j(z)$ converges uniformly to $S(z)$ for all z in a region R. If $u_1(z),\ u_2(z),\ \cdots$ are all analytic in R, then at any interior point of this region

$$\frac{dS}{dz} = \sum_{j=1}^{\infty} \frac{du_j(z)}{dz}.$$

Example

Since $1/(1-z) = \sum_{j=1}^{\infty} z^{j-1} = 1 + z + z^2 + \cdots$, where convergence is uniform for $|z \le r|$ (with $r < 1$), we have upon differentiation

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2} = \frac{d}{dz}(1 + z + z^2 + \ldots) = 1 + 2z + 3z^2 + \ldots,$$

or $$\frac{1}{(1-z)^2} = \sum_{j=1}^{\infty} jz^{j-1}, \quad |z| < 1, \quad r < 1.$$

§5-4 Taylor's Series

1. An infinite series

$$\sum_{n=0}^{\infty} c_n (z-c)^n = c_0 + c_1(z-c) + \cdots + c_n(z-c)^n + \cdots$$

is a power series in powers of $(z-c)$, where z is a complex variable and c, $c_n \in C$, $n = 0, 1, 2, \cdots$.

2. 冪級數之收歛與發散之判定

Consider the series

$$\sum_{n=0}^{\infty} a_n = a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

1) Ratio Test $\rightarrow$ Suppose that $a_n \neq 0$, $n = 0, 1, 2, \cdots$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L = \begin{cases} L < 1, & \textbf{absolutely convergence.} \\ L > 1, & \textbf{divergence.} \end{cases}$$

*** If $L = 1$, the test fails.

2) Root Test

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L = \begin{cases} L < 1, & \textbf{absolutely convergence.} \\ L > 1, & \textbf{divergence.} \end{cases}$$

*** If $L = 1$, the test fails.

3. 試判別下列二式是否為收歛?

1) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^n$; 2) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

<Sol.> In calculus, we have two important and useful limits as following：

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \doteqdot 2,71828 \quad \text{----------} \quad (1)$$

$$\lim_{n\to 0} \frac{\sin x}{x} = 1 \quad \text{--------------------------} \quad (2)$$

1) Here, we shall use equation (1).

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$
$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$$
$$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n}$$
$$= \frac{1}{e} \neq 0$$

Since the n-th term of the series is not equal to zero, thus the series is divergent. That is,

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^n \quad \text{is divergent.}$$

2) $$\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\left[\left(\frac{n}{n+1}\right)^{n^2}\right]^{\frac{1}{n}}$$
$$= \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$
$$\Rightarrow \sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2} \quad \text{is convergent.}$$

4. Radius and Circle of Convergence of Power Series

Every power series

$$\sum_{n=0}^{\infty} c_n (z-z_0)^n \qquad \text{-------------} \qquad (1)$$

has a "radius of convergence" R, and can be defined as

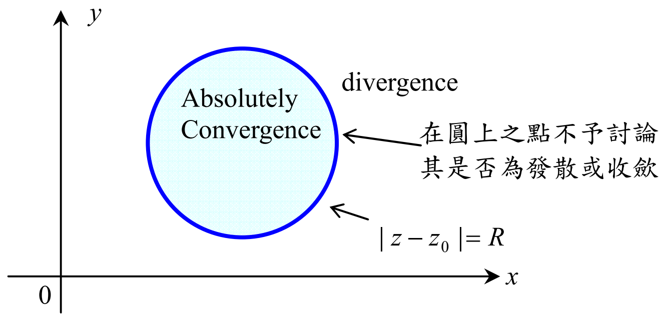
i) $$R = \lim_{n\to\infty}\left|\frac{c_n}{c_{n+1}}\right|$$

ii) $$R = \lim_{n\to\infty}\frac{1}{\sqrt[n]{|C_n|}}$$

such that

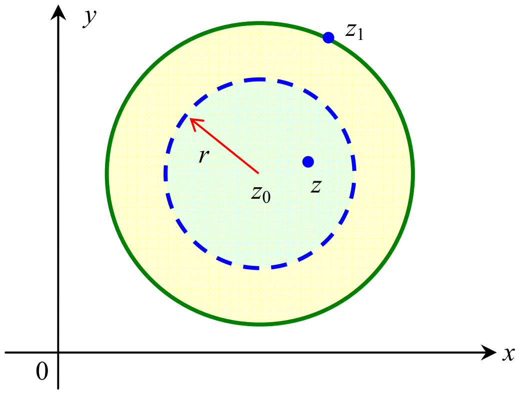
a) $0 < R < \infty$, the series (1) converges absolutely for $|z-z_0| < R$ and diverges for $|z-z_0| > R$

b) $R = 0$, the series converges only at $z = z_0$



5. **Theorem**

If $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ converges when $z = z_1$, then this series converges for all z satisfying $|z-z_0| < |z_1 - z_0|$. The convergence is absolute for these values of z.



6. **Theorem: Uniform Convergence and Analyticity of Power Series**

If $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ converges when $z = z_1$, where $z_1 \neq z_0$, then the series converges uniformly for all z in the disc $|z-z_0| \leq r$, where $r < |z_1 - z_0|$. The sum of the series is an analytic function for $|z-z_0| \leq r$.

<pf.>

We need to use the Weierstrass *M* test to prove this theorem. Consider the convergent series

$$\sum_{n=0}^{\infty} c_n(z_1-z_0)^n = c_0 + c_1(z_1-z_0) + c_2(z_1-z_0)^2 + \ldots, \qquad (2)$$

For the preceding convergent series of constants, we can find a number m that equal or exceeds the magnitude of any of the terms. Thus,

$$\left|c_n(z_1-z_0)^n\right| \leq m, \quad n = 0,\ 1,\ 2,\ \ldots.. \qquad (3)$$

Now consider the original series

$$\sum_{n=0}^{\infty} c_n(z-z_0) = c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \ldots, \qquad (4)$$

where we take $|z-z_0| \leq r$ and $r < |z_1 - z_0|$. Notice that the terms in Eq. (4) can be written

$$c_n(z-z_0)^n = c_n(z_1-z_0)^n \left(\frac{z-z_0}{z_1-z_0}\right)^n$$

Taking magnitudes yields

$$\left|c_n(z-z_0)^n\right| = \left|c_n(z_1-z_0)^n\right| \left|\frac{z-z_0}{z_1-z_0}\right|^n. \qquad (5)$$

Let $p = r/|z_1 - z_0|$, where, by hypothesis, $p < 1$. Since $|z-z_0| \leq r$, we have

$$\left|\frac{z-z_0}{z_1-z_0}\right| \leq p \qquad (6)$$

Simultaneously applying this inequality, as well as Eq.(3), to the right side of Eq. (5), we obtain

$$\left|c_n(z-z_0)^n\right| \leq mp^n \qquad (7)$$

Let $M_n = mp^n$. From Eq.(7), we have

$$\left|c_n(z-z_0)^n\right| \leq M_n. \qquad (8)$$

The summation

$$\sum_{n=0}^{\infty} M_n = \sum_{n=0}^{\infty} mp^n = m\sum_{n=0}^{\infty} p^n, \quad p < 1 \qquad (9)$$

involves a convergent geometric series of real constants.

Eq. (7), Eq. (9), and the theorem of Weirestrass *M* test guarantee the uniform convergence $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ for $|z-z_0| \leq r$. Because the individual items $c_n(z-z_0)^n$ in this series are each analytic function, it follows that the sum of this series is an analytic function in $|z - z_0| \le r$. ***Q.E.D.***

7. In each of the following cases, the radius of convergence is equal to 1.

1) The series $\sum_{n=0}^{\infty} z^n$ does not converge for any point on the circle of convergence.

2) The series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges at every point on the circle of convergence.

3) The series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges for $z = -1$ and diverges for $z = 1$.

8. **Theorem: Taylor's Series**

If the power series is

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n , \quad |z - z_0| < R$$

and suppose that the function f is analytic in the interior of a circle C, with center at z_0 and radius R. Then, we can rewrite the function f in the form of Taylor's series:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \qquad \text{-------------} \quad (1)$$

where $|z - z_0| < R$.

<pf.> Since

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$
$$= c_0 + c_1(z - z_0) + c_2(z - z_0)^2 + \cdots + c_n(z - z_0)^n + \cdots$$

Then, we know that

$$f'(z) = c_1 + 2c_2(z - z_0) + 3c_3(z - z_0)^2 + \cdots + nc_n(z - z_0)^{n-1} + \cdots$$
$$f''(z) = 2c_2 + 3 \cdot 2(z - z_0) + 4 \cdot 3(z - z_0)^2 + \cdots + n(n-1)(z - z_0)^{n-1} + \cdots$$
$$\vdots$$
$$\vdots$$
$$f^{(n)}(z_0) = n!\ c_n$$

$$\Rightarrow \quad f(z_0) = c_0$$
$$f'(z_0) = c_1$$
$$f''(z_0) = 2!\ c_2$$
$$\cdots\cdots\cdots\cdots$$
$$f^{(n)}(z_0) = n!\ c_n$$

Hence, we have

$$f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$$
$$= f(z_0) + f'(c)(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$
$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(z - c)^n$$

where $f^{(0)}(z_0) = f(z_0)$, and $0! = 1$.

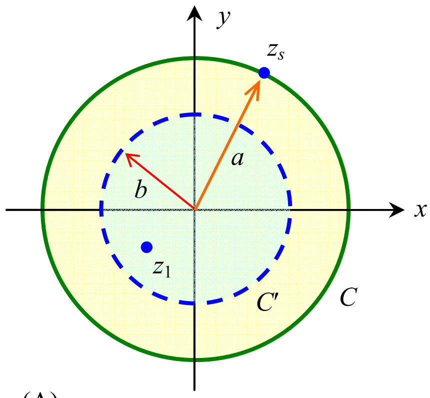
♣ **Maclaurin Series**

If $z_0 = 0$, we call the Taylor series a Maclaurin series, i.e.,

$$f(z) = \sum_{n=0}^{\infty} c_n z^n , \quad |z| < R , \text{ where } c_n = \frac{f^{(n)}(0)}{n!}$$

<pf.>

Assume the function $f(z)$ is analytic at $z=0$. Let z_s be that singularity of $f(z)$ lying closest to $z=0$. Construct a circle C centered at the origin and passes through z_s. The radius of the circle is $a=|z_s|$. Let z_1 lie within this contour. We enclose z_1 by a second circle C' centered at the origin but having a radius less than that of C. Since the radius of C' is b, we have $|z_1|<b<a$. By Cauchy integral formula,



(A)

$$f(z_1)=\frac{1}{2\pi i}\oint_{C'}\frac{f(z)}{(z-z_1)}dz=\frac{1}{2\pi i}\oint_{C'}\frac{f(z)dz}{z\left(1-\frac{z_1}{z}\right)}.$$

Now consider

$$\frac{1}{1-\frac{z_1}{z}}=1+\frac{z_1}{z}+\left(\frac{z_1}{z}\right)^2+\cdots$$

The above series is uniformly convergent when

$$\left|\frac{z_1}{z}\right|\le r,\quad \text{where}\quad r<1$$

If z is confined to the contour C', we observe $|z_1/z|<1$, and we can readily find a value of r satisfying the above equation.

The function $f(z)/z$ is bounded on C'. Thus, we have

$$\frac{f(z)}{z\left(1-\frac{z_1}{z}\right)}=f(z)\frac{1}{z}+f(z)\frac{z_1}{z^2}+f(z)\frac{z_1^{\ 2}}{z^3}+\cdots$$

which is uniformly convergent in some region containing C'. Using a term-by-term integration, we have

$$f(z_1)=\frac{1}{2\pi i}\oint_{C'}\frac{f(z)}{z}dz+\frac{z_1}{2\pi i}\oint_{C'}\frac{f(z)}{z^2}dz+\frac{z_1^{\ 2}}{2\pi i}\oint_{C'}\frac{f(z)}{z^3}dz+\cdots$$

From the extended Cauchy integral formula,

$$\frac{1}{2\pi i}\oint_{C'}\frac{f(z)}{z^{n+1}}dz=\frac{f^{(n)}(0)}{n!},\quad n=0,\ 1,\ 2,\ \cdots$$

Thus, for $|z_1|<b<a$ we have

$$f(z_1)=\sum_{n=0}^{\infty}c_n z_1^{\ n}=c_0+c_1z_1+c_2z_1^{\ 2}+\ldots \qquad \text{(A)}$$

where

$$c_n=\frac{f^{(n)}(0)}{n!}$$

Replacing what is now the dummy variable z_1 in Eq. (A) by z, the correctness of Maclaurin Seires have been demonstrated.

♣ For Taylor series ($z_0\ne 0$), the integrand is now written as

$$\frac{f(z)}{z-z_1}=\frac{f(z)}{z-z_0\left[1-\frac{(z_1-z_0)}{(z-z_0)}\right]}$$

and a series expansion is made in powers of $(z_1-z_0)/(z-z_0)$.

H.W. 1 Follow the suggestions given above and give a proof valid for any $z_0 \neq 0$.
【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 2, Exercise 5.4, Pearson Education, Inc., 2005.】

♣ If $f(z)$ satisfies the conditions described in the above theorem, then $f(z)$ can be represented within the domain $|z-z_0|<b$ (where $b<a$) by the sum of a power series with a finite number of terms plus a remainder, i.e.,

$$f(z)=\sum_{n=0}^{N-1} c_n(z-z_0)^n + R_N(z)$$

Here c_n is again $f^{(n)}(z_0)/n!$, while $R_N(z)$ is expressed as a contour integration around the circle $|z-z_0|=b$.

$$R_n=\frac{z_1^n}{2\pi i}\oint_{C'}\frac{f(z)}{z^n(z-z_1)}dz$$

H.W. 2 (a) Refer to the proof of the above theorem and to the above figure. Use Eq. (A) (in previous page) to show that

$$f(z_1)=\frac{1}{2\pi i}\oint_{C'} f(z)\left[\frac{1}{z}+\frac{z_1}{z^2}+\cdots+\frac{z_1^{n-1}}{z^n}+\left(\frac{z_1}{z}\right)^n\frac{1}{z-z_1}\right]dz$$

<Hint> Refer to the following equation

$$\sum_{j=1}^{\infty} z^{j-1}=1+z+z^2+\ldots=S(z)=\frac{1}{1-z},\quad |z|<1.$$

which implies that

$$\frac{1}{1-z}=1+z+z^2+\cdots+z^{n-1}+\frac{z^n}{1-z}$$

and replace z by z_1/z

(b) Use the expression for $f(z_1)$ given in part (a) to show, after integration, that

$$f(z_1)=f(0)+f'(0)z_1+\frac{f''(0)}{2!}z_1^2+\cdots+\frac{f^{(n-1)}(0)}{(n-1)!}z_1^{n-1}+R_n, \quad (2)$$

where

$$R_n=\frac{z_1^n}{2\pi i}\oint_{C'}\frac{f(z)}{z^n(z-z_1)}dz \quad (3)$$

We see that R_n, the remainder, represents the difference between $f(z_1)$ and the first n terms of its Maclaurin expansion.

(c) We can replace an upper bound on the remainder in the above equation. Assume $|f(z)\le m|$ everywhere on $|z|=b$ (the contour C'). Use the *ML* inequality to show that

$$|R_n|\le\left|\frac{z_1}{b}\right|^n\frac{mb}{b-|z_1|} \quad (4)$$

<Hint> Note that for z lying on the contour C'

$$\left|\frac{1}{z-z_1}\right|\le\frac{1}{b-|z_1|}$$

Why?

In passing, we notice that since $|z_1/b|<1$, the remainder R_n in eq. (4) tends to zero as $n\to\infty$. Using this limit, we find the right side of Eq. (2) becomes the Maclaurin series of $f(z_1)$. This constitutes a derivation of the Maclaurin expansion shown in the last equation (in page 114) not requiring the use of uniform convergence. A similar derivation applies for the Taylor Series.

(d) Suppose we wish to determine the approximate value of $\cosh i$ by the finite series $i^0 + i^2/2! + \cdots + i^{10}/10!$. Taking the contour C' in Eq. (3) as $|z| = 2$, show by using Eq. (4) that the error made cannot exceed $(\cosh 2)/2^{10} \doteq 3.67 \times 10^{-3}$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 32, Exercise 5.4, Pearson Education, Inc., 2005.】

9. Some Examples

Example 1

Let us show that

$$\frac{1}{z^2} = \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$$

where $|z-2| < 2$.

<pf.> Let function *f* be defined by $f(z) = \dfrac{1}{z^2}$, $\forall\ z \neq 0$.

Clearly, *f* is analytic for all *z* interior to the circle $c = |z-2| = 2$.

Differentiating *f* with respect to *z* by *n* times, we obtain

$$f^{(n)}(z) = \frac{(-1)^n(n+1)!}{z^{n+2}} \quad , \quad n = 0, 1, 2, \cdots$$

Thus,

$$f^{(n)}(2) = \frac{(-1)^n(n+1)!}{2^{n+2}} \quad , \quad n = 0, 1, 2, \cdots$$

Utilizing Eq. (1), with $c = z$, we have

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)!}{2^{n+2}\cdot n!}(z-2)^n$$

$$= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n(n+1)\left(\frac{z-2}{2}\right)^n$$

where $n = 0, 1, 2, \cdots$.

Example 2

Expand e^z in (a) a Maclaurin series and (b) a Taylor series about $z = i$.

<Sol.>

(a) $e^z = c_0 + c_1 z + c_2 z^2 + \cdots$

In the coefficient formula of Taylor series, with $z_0 = 0$,

$$c_n = \frac{\dfrac{d^n}{dz^n}e^z\,|_{z=0}}{n!} = \frac{e^z}{n!}\,|_{z=0} = \frac{1}{n!}$$

Thus

$$e^z = \sum_{n=0}^{\infty}\frac{1}{n!}z^n$$

(b) $e^z = c_0 + c_1(z-i) + c_2(z-i)^2 + \cdots$

In the coefficient formula of Taylor series, with $z_0 = i$,

$$c_n = \frac{\left.\dfrac{d^n}{dz^n}e^z\right|_{z=i}}{n!} = \left.\frac{e^z}{n!}\right|_{z=i} = \frac{e^i}{n!}$$

thus

$$e^z = \sum_{n=0}^{\infty} \frac{e^i}{n!}(z-i)^n$$

Example 3

Expand

$$f(z) = \frac{1}{1-z}$$

in Taylor series $\sum_{n=0}^{\infty} c_n (z+1)^n$. For what values of z must the series converge to $f(z)$?

<Sol.>

We can find $c_0 = 1/2$, $c_1 = 1/4$, and in general, $c_n = 1/2^{n+1}$. Thus,

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}(z+1)^n$$

** 茲節錄一些有用級數展開式：

1) $\frac{1}{1-z} = 1 + z + z^2 + \cdots + z^n + \cdots$

$$= \sum_{n=0}^{\infty} z^n , \quad |z| < 1$$

2) $e^n = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!} , \quad \text{for all } z.$$

3) $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} , \quad \text{for all } z.$$

4) $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots$

$$= \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} , \quad \text{for all } z.$$

5) $\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} , \quad |z| < \infty .$

6) $\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} , \quad |z| < \infty .$

7) $\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1} , \quad |z| < 1 .$

8) $(1+z)^p = 1 + pz + \frac{p(p-1)}{2!} z^2 + \cdots + \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} z^n + \cdots , \quad |z| < 1 .$

9) $\ln(1+z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{n+1}}{n+1} , \quad |z| < 1 .$

10) $\ln\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} \frac{2z^{n+1}}{2n+1} , \quad |z| < 1 .$

10. **Some Theorems**

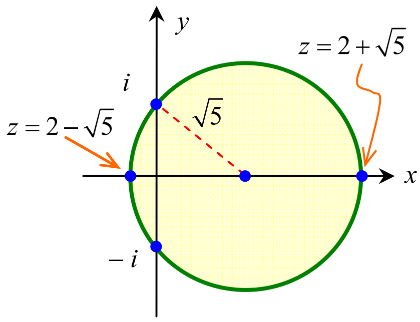
1) The Taylor series expansion about z_0 of the analytic function $f(z)$ is the only power series using powers of $(z - z_0)$ that will converge to $f(z)$ everywhere in a circular domain centered at z_0 .
2) Let $f(z)$ be expanded in a Taylor series about z_0 . The largest circle within which this series

converges to $f(z)$ at each point is $|z-z_0|=a$, where a is distance from z_0 to the nearest singular point of $f(z)$.

Example 4

Without actually obtaining the Taylor series give the largest circle throughout which the indicated expansion is valid:

$$f(z)=\frac{1}{z^2+1}=\sum_{n=0}^{\infty}c_n(z-2)^n$$



<Sol.>

The singularities of $f(z)$ lie at $\pm i$. The nearest singularity to $z=2$ is, in this case, either $+i$ or $-i$. The distance from $z=2$ to these points is $\sqrt{5}$. Thus, the Taylor series converges to $f(z)$ throughout the circular domain $|z-2|<\sqrt{5}$.

Example 5

Consider the real Taylor series expansion

$$\frac{1}{x^2+1}=\sum_{n=0}^{\infty}c_n(x-2)^n$$

Determine the largest interval along the x-axis inside which the series converges to $1/(x^2+1)$.

<Sol.>

By requiring z to be a real variable ($z=x$) in the previous example, we require $2-\sqrt{5}<x<2+\sqrt{5}$ for convergence.

♣ Remarks on Analyticity

A function $f(z)$ is analytic in a domain *D* if

(a) $f'(z)$ exists throughout *D*;

(b) $f(z)$ has derivatives of all orders throughout *D*;

(c) $f(z)$ has a Taylor series expansion valid in a neighborhood of each point in *D*;

(d) $f(z)$ is the sum of a convergent power series in a neighborhood of each point in *D*.

§5-5 Techniques for Obtaining Taylor Series Expansions

1. Substitution Method

Example 1

Given

$$\frac{1}{1-w}=1+w+w^2+\cdots,\quad |w|<1.$$

$$\Rightarrow \quad \frac{1}{z}=1+(1-z)+(1-z)^2+(1-z)^3+\cdots$$
$$=1-(z-1)+(z-1)^2-(z-1)^3+\cdots,\quad |z-1|<1 \qquad (1)$$

H.W. 1 Show that, for $N\ge 0$,

$$\frac{1}{(1-w)^N}=\sum_{n=0}^{\infty}c_n z^n,\quad \text{where } c_n=\frac{(N-1+n)!}{n!(N-1)!},\quad |z|<1$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 7, Exercise 5.5, Pearson Education, Inc., 2005.】

2. **Term-by-Term Differentiation and Integration**

Example 2

Use term-by-term differentiation and the result in Eq. (1) to obtain the expansion of $1/z^2$ about $z=1$.

<Sol.>

Differentiating both sides of Eq. (1) with respect to z and multiplying by (-1), we obtain

$$\frac{1}{z^2}=1-2(z-1)+3(z-1)^2+\cdots=\sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n \qquad (2)$$

valid for $|z-1|<1$.

H.W. 2 Differentiate the series of Eq. (2) to show that

$$\frac{1}{z^3}=1-\frac{3\cdot 2}{2}(z-1)+\frac{4\cdot 3}{2}(z-1)^2-\frac{5\cdot 4}{2}(z-1)^3+\cdots,\quad |z-1|<1.$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 5, Exercise 5.5, Pearson Education, Inc., 2005.】

Example 3

Obtain the Maclaurin expansion of

$$\mathrm{Si}(z)=\int_0^z f(z')dz', \qquad (3)$$

where

$$f(z')=\frac{\sin z'}{z'},\quad z'\neq 0, \qquad (4a)$$

$$f(0)=1,\quad z'=0. \qquad (4b)$$

The function $\mathrm{Si}(z)$ is called the sine integral and cannot be evaluated in terms of elementary functions. It appears often in problems involving electromagnetic radiation.

<Sol.>

From the examples in page 118, we have

$$\sin z'=z'-\frac{(z')^3}{3!}+\frac{(z')^5}{5!}+\cdots,$$

$$\Rightarrow \qquad \frac{\sin z'}{z'}=1-\frac{(z')^2}{3!}+\frac{(z')^4}{5!}+\cdots.$$

We now integrate as follows:

$$\int_0^z \frac{\sin z'}{z'}dz'=\int_0^z dz'+\int_0^z\frac{-(z')^2}{3!}dz'+\int_0^z\frac{(z')^4}{5!}dz'+\cdots=z-\frac{z^3}{3\cdot 3!}+\frac{z^5}{5\cdot 5!}+\cdots.$$

Thus,

$$\mathrm{Si}(z)=\sum_{n=0}^{\infty}c_n z^{2n+1} \qquad (5)$$

where

$$c_n=\frac{(-1)^n}{(2n+1)!(2n+1)}.$$

The expansion is valid throughout the z-plane.

H.W. 3 (a) Explain how the following series is derive:

$$\frac{1}{1+z^2}=1-z^2+z^4-\cdots,\quad |z|<1$$

(b) Integrate the series in part (a) along a contour connecting the origin to an arbitrary point z, where $|z|<1$, to show that

$$\tan^{-1} z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{2n+1}, \quad |z|<1. \qquad \text{(A)}$$

(c) We might put $z=1$ in the preceding expansion to obtain $\tan^{-1} 1 = \pi/4 = 1 - 1/3 + 1/5 - \cdots$. This expansion, which could be used to obtain $\pi/4$, is valid, although not justified by our method, which assumed $|z|<1$.

This series converges slowly and is not useful for computing π. A more useful series is obtained in the following.

Prove that $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$ and with aid of (b) derive the more rapidly converging series:

$$\frac{\pi}{4} = \left(\frac{1}{2}+\frac{1}{3}\right) - \frac{\left(\frac{1}{8}+\frac{1}{27}\right)}{3} + \frac{\left(\frac{1}{32}+\frac{1}{243}\right)}{5} - \frac{\left(\frac{1}{128}+\frac{1}{2187}\right)}{7} + \cdots$$

(c) Compare the two series for $\pi/4$ given in (b) by using the first 10 terms in each and seeing how well $\pi/4$ is approximated. MATLAB is recommended here.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 2 and 8, Exercise 5.5, Pearson Education, Inc., 2005.】

<Ans.>

(c) MATLAB Command:

```
% Two ten term series approximations to pi/4 for problem (c) in H.W. 2
    format long
    s=0;
    for n=0:9
        s=(-1)^n*1/(2*n+1)+s;
    end
    s1=s
    s=0;
    for n=0:9
        s=(-1)^n*(.5^(2*n+1) +(1/3)^(2*n+1)) /(2*n+1)+s;
    end
    s2=s
    exact=pi/4
```

MATLAB Output

```
s1 =
    0.76045990473235
s2 =
    0.78539814490159
exact =
    0.78539816339745
```

3. **Series Expansions of Branches of Multivalued Functions**

Example 4

Find the Maclaurin expansion of $f(z) = (z+1)^{1/2}$, where the principal branch of the function is used. Where is the expansion valid?

<Sol.>

Recall that the branch in question is identical to $e^{1/2\,\mathrm{Ln}(z+1)}$ and that its derivative is given by

$$e^{(1/2)\mathrm{Ln}(z+1)} \frac{1}{2(z+1)} = \frac{(z+1)^{1/2}}{2(z+1)}.$$

We may of course differentiate indefinitely and thus have

$$f^{(1)}(z) = \frac{1}{2}(z+1)^{1/2-1}, \quad f^{(2)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)(z+1)^{1/2-2},$$

$$f^{(3)}(z) = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)(z+1)^{1/2-3}, \quad \text{etc.}$$

In general,

$$f^{(n)}(z)=\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)(z+1)^{1/2-n} \qquad (6)$$

Note that $(z+1)^{1/2-n}$ must be interpreted as

$$\frac{(z+1)^{1/2}}{(z+1)^n}=\frac{e^{(1/2)\mathrm{Ln}(z+1)}}{(z+1)^n}.$$

When $z=0$, this function equals $e^{1/2\,\mathrm{Ln}(z+1)}/1^n=1$. With this result and Eq. (6) and the coefficient formula of Taylor series ($c_n=f^{(n)}(0)/n!$), we finally have

$$(1+z)^{(1/2)}=\sum_{n=0}^{\infty}c_n z^n \qquad (7a)$$

where

$$c_0=1,$$

$$c_n=\frac{1}{n!}\left[\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\cdots\left(\frac{1}{2}-(n-1)\right)\right],\quad n\ge 1 \qquad (7b)$$

The singularity of $(z+1)^{1/2}$ nearest the origin is the branch point $z=-1$. Thus, Eq.(7) is valid in the domain $|z|<1$.

H.W. 4 (a) Let α be any complex number except zero or a positive integer. Using the branch of $(1+z)^{\alpha}$ defined by $e^{\alpha\,\mathrm{Ln}(1+z)}$ (principal branch), show that for $|z|<1$,

$$(1+z)^{\alpha}=1+\alpha z+\frac{\alpha(\alpha-1)z^2}{2!}+\frac{\alpha(\alpha-1)(\alpha-3)z^3}{3!}+\cdots=1+\sum_{n=1}^{\infty}c_n z^n$$

where $c_n=\left(\frac{1}{n!}\right)\left[\alpha(\alpha-1)(\alpha-3)\cdots(\alpha-(n-1))\right]$. Follow the method of the above Example **(Example 3, Section 5.5, in textbook)**.

(b) Show that if α is a positive integer, then $(1+z)^n=1+\sum_{n=1}^{\alpha}c_n z^n$, where c_n is given in Part (a). where is this expansion valid?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 29, Exercise 5.5, Pearson Education, Inc., 2005.】

H.W. 5 (a) Use the result derived in **H.W. 3 (a)** and a change of variable to show that

$$\frac{1}{(1-z)^{1/2}}=1+\frac{z}{2}+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{z^2}{2!}++\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{z^3}{3!}+\cdots,\quad |z|<1.$$

Use the first four terms of this series to evaluate approximately $\sqrt{2}$. Compare this with the value obtained from your calculator.

(b) Show that

$$\frac{1}{(1-z^2)^{1/2}}=1+\frac{1}{2}z^2+\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\frac{z^4}{2!}++\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)\frac{z^6}{3!}+\cdots,\quad |z|<1.$$

(c) Use the preceding result and a term-by-term integration to show that

$$\sin^{-1}z=z+\frac{z^3}{2\cdot 3\cdot 1!}+\frac{1\cdot 3z^5}{2^2\cdot 5\cdot 2!}+\frac{1\cdot 3\cdot 5z^7}{2^3\cdot 7\cdot 3!}+\cdots,\quad |z|<1.$$

where this branch of $\sin^{-1}z$ is analytic inside the unit circle, and $\sin^{-1}(0)=0$. Note that

$\cos^{-1}z=(\pi/2)-(\text{the seriers on the above right})$, provided $|z|<1$.

(d) Use the series for $\sin^{-1}z$ to obtain a numerical series for $\pi/6$. Use the first four terms of your result to evaluate approximately $\pi/6$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 30, Exercise 5.5,

Pearson Education, Inc., 2005.】

4. **Multiplication and Division of Series**

Example 5

(a) Using series multiplication, obtain the Maclaurin expansion of $f(z)=e^z/(1-z)$.

(b) Use your result to obtain the value of the 1oth derivative of $f(z)$ at $z=0$.

<Sol.>

(a)

With $e^z=\sum_{n=0}^{\infty} z^n/n!$ (valid for all z) and $1/(1-z)=\sum_{n=0}^{\infty} z^n$ (for $|z|<1$), we have

$$f(z)=\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots\right)(1+z+z^2+\cdots)$$

$$=1+(1+1)z+\left(1+1+\frac{1}{2!}\right)z^2+\left(1+1+\frac{1}{2!}+\frac{1}{3!}\right)z^3+\cdots,$$

or, equivalently

$$\frac{e^z}{1+z}=\sum_{n=0}^{\infty} c_n z^n, \qquad (8a)$$

where

$$c_n=\sum_{j=0}^{n}\frac{1}{j!}. \qquad (8b)$$

Eq. (8a) is valid only for $|z|<1$.

(b)

It is a little tedious to obtain the 10th derivative of $f(z)$ by differentiating this function 10 times.

Note, however, that in the Maclaurin expansion $f(z)=\sum_{n=0}^{\infty} c_n z^n$, we have $c_n=f^{(n)}(0)/n!$.

Thus, using the result of part (a) and taking $n=10$, we find

$$f^{10}(0)=10!\sum_{j=0}^{\infty}\frac{1}{j!}=10!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{10!}\right).$$

♣ **The Quotient and Product of Two Analytic Functions**

1) Suppose $f(z)$ and $g(z)$ are both analytic at z_0. If $g(z_0)\neq 0$, the quotient

$$h(z)=\frac{f(z)}{g(z)} \qquad (9)$$

is analytic at z_0 and can be expanded in Taylor series about this point.

2) Let the series $f(z)$, $g(z)$, and $h(z)$ are

$$h(z)=\sum_{n=0}^{\infty} c_n(z-z_0)^n,\quad f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n,\quad g(z)=\sum_{n=0}^{\infty} b_n(z-z_0)^n$$

where a_n and b_n are presumed known and the coefficients c_n are unknown.

3) According to the Cauchy product, we have

$$\sum_{n=0}^{\infty} c_n(z-z_0)^n\sum_{n=0}^{\infty} b_n(z-z_0)^n=\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

and

$$c_0b_0+(c_0b_1+c_1b_0)(z-z_0)^1+(c_0b_2+c_1b_1+c_2b_0)(z-z_0)^2+\cdots$$

$$=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$$

Equating coefficients of corresponding powers of $(z-z_0)$, we have

$$c_0b_0=a_0, \qquad (10a)$$

$$c_0b_1+c_1b_0=a_1, \qquad (10b)$$

$$c_0 b_2 + c_1 b_1 + c_2 b_0 = a_2, \qquad (10c)$$

……

From the first equation, we have

$$c_0 = \frac{a_0}{b_0} \qquad (11a)$$

and then

$$c_1 = \frac{a_1}{b_0} - \frac{a_0 b_1}{b_0^2} \qquad (11b)$$

$$c_2 = \frac{a_2}{b_0} - \frac{a_1 b_1 + a_0 b_2}{b_0^2} + \frac{a_0 b_1^2}{b_0^3} \qquad (11c)$$

……

The process can be repeated to yield any coefficient c_n, where n is as large as we wish.

Example 6

Obtain the Maclaurin expansion of $(e^z - 1)/\cos z$ from the Naclaurin series for $e^z - 1$ and $\cos z$.

<Sol.>

From previous paragraph, we have

$$e^z - 1 = z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \qquad (12) \qquad \text{and} \qquad \cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots \qquad (13)$$

We divide these series as follows:

$$
\begin{array}{r|l}
 & z + \frac{z^2}{2!} + z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + \ldots \\
\hline
1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots & z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots \\
 & z \qquad - \frac{z^3}{2!} \qquad + \\
\cline{2-2}
 & \frac{z^2}{2!} + z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + \frac{z^4}{4!} + \ldots \\
 & \frac{z^2}{2!} \qquad - \frac{z^4}{(2!)^2} + \ldots \\
\cline{2-2}
 & z^3\left(\frac{1}{3!} + \frac{1}{2!}\right) + z^4\left(\frac{1}{4!} + \frac{1}{(2!)^2}\right) \\
 & + \ldots
\end{array}
$$

Recalling that $\cos z = 0$ for $z = \pm\pi/2$, we have

$$\frac{e^z - 1}{\cos z} = \sum_{n=0}^{\infty} c_n z^n$$

valid for $|z| < \pi/2$. Our division shows that

$$c_0 = 0, \quad c_1 = 1, \quad c_2 = 1/2!, \quad c_3 = (1/3! + 1/2!) = 2/3$$

H.W. 6 (a) The Bernoulli numbers $B_0, B_1, B_2, \ldots$ are defined by

$$B_n = n! c_n$$

where

$$f(z) = \left\{ \begin{array}{ll} \frac{z}{e^z - 1}, & z \neq 0 \\ 1, & z = 0 \end{array} \right\} = \sum_{n=0}^{\infty} c_n z^n .$$

Note that $f(z)$ is analytic at $z=0$ since, for all z,

$$\frac{z}{e^z-1}=\frac{z}{z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots}=\frac{1}{1+\frac{z}{2!}+\frac{z^2}{3!}+\cdots}$$

Perform long division on the right-hand quotient to show that $B_0=1$, $B_1=-1/2$, $B_2=1/6$.

(b) Show that the coefficients of odd order beyond 1, i.e., B_3, B_5, B_7, … are all zero.

<Hint> $f(z)+z/2=(z/2)\cosh(z/2)/\sinh(z/2)$ is an even function of z. **See Problem 30, Section 5.4, Textbook.**

(c) Where is the series expansion of **Part (a)** valid?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 27, Exercise 5.5, Pearson Education, Inc., 2005.】

5. **The Method of Partial Fractions**

Consider a rational algebraic function

$$f(z)=\frac{P(z)}{Q(z)}$$

where P and Q are polynomials in z. If $Q(z_0)\neq 0$, then $f(z)$ has a Taylor expansion about z_0.

Rule I (Nonrepeated factors)

Let $P(z)/Q(z)$ be a rational function, where the polynomial $P(z)$ is of lower degree than the polynomial $Q(z)$. If $Q(z)$ can be factored into the form

$$Q(z)=C(z-a_1)(z-a_2)\cdots(z-a_n) \qquad (14)$$

where $a_1,\ a_2,\ \cdots$ are all different constants and C is a constant, then

$$\frac{P(z)}{Q(z)}=\frac{A_1}{z-a_1}+\frac{A_2}{z-a_2}+\cdots+\frac{A_n}{z-a_n} \qquad (15)$$

where $A_1,\ A_2,\ \cdots$ are constants. Eq. (15), called the partial fraction expansion of $P(z)/Q(z)$, is valid for all $z\neq a_j$ ($j=1,\ 2,\ \cdots,\ n$).

Rule II (Repeated Factors)

Let $Q(z)$ be factored as in Eq. (14), except that $(z-a_1)$ appears raised to the m_1 power, $(z-a_2)$ appears raised to the m_2 power, etc. Then $P(z)/Q(z)$ can be decomposed as in Eq. (15), except that for each factor of $Q(z)$ of the form $(z-a_j)^{m_j}$, where $m_j\geq 2$, we replace $A_j/(z-a_j)$ in Eq. (15) by

$$\frac{A_{j1}}{(z-a_j)}+\frac{A_{j2}}{(z-a_j)^2}+\cdots+\frac{A_{jm_j}}{(z-a_j)^{m_j}}$$

Example

Rule I tell us that $$\frac{z}{(z-1)(z+1)}=\frac{A_1}{z-1}+\frac{A_2}{z+1}$$

Rule II tell us that $$\frac{z}{(z-1)^2(z+1)}=\frac{A_{11}}{z-1}+\frac{A_{12}}{(z-1)^2}+\frac{A_2}{z+1}$$

♣ **Four useful Maclaurin expansion**

(1) $$\frac{1}{1-w}=1+w+w^2+\cdots,\quad |w|<1; \qquad (16a)$$

(2) $$\frac{1}{1+w}=1-w+w^2-w^3+\cdots,\quad |w|<1; \qquad (16b)$$

(3) $$\frac{1}{(1-w)^2}=1+2w+3w^2+\cdots,\quad |w|<1; \qquad (16c)$$

(4) $$\frac{1}{(1+w)^2} = 1 - 2w + 3w^2 + \cdots, \quad |w| < 1; \qquad (16d)$$

Example 7

Expand

$$f(z) = \frac{z}{z^2 - z - 2} = \frac{z}{(z+1)(z-2)}$$

in a Taylor series about the point $z = 1$.

<Sol.>

From **Rule I**, we have

$$\frac{z}{(z+1)(z-2)} = \frac{a}{z+1} + \frac{b}{z-2}. \qquad (17)$$

Clearing the fractions in Eq. (17) yields

$$z = a(z-2) + b(z+1)$$

We can find a and b by letting z in the above equation equal -1 and 2. For another approach, we rearrange the previous equation as

$$z = (a+b)z + (-2a+b)$$

Thus, we have

z^1 coefficient: $1 = a + b$

z^0 coefficient: $0 = -2a + b$

whose solution is

$$a = 1/3, \quad b = 2/3$$

Hence, from Eq. (17)

$$\frac{z}{(z+1)(z-2)} = \frac{1/3}{z+1} + \frac{2/3}{z-2}. \qquad (18)$$

To expand $z/[(z+1)(z-2)]$ in powers of $(z-1)$, we expand each fraction on the right in Eq. (18) in these powers. Thus,

$$\frac{1/3}{z+1} = \frac{1/3}{(z-1)+2} = \frac{1/6}{1+\dfrac{(z-1)}{2}} = \frac{1}{6}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right], \quad \text{for } |z-1| < 2 \qquad (19)$$

The preceding series is obtained with the substitution $w = (z-1)/2$ in Eq. (16b). The requirement $|z-1| < 2$ is identical to the constraint $|w| < 1$.

Similarly,

$$\frac{2/3}{z-2} = \frac{2/3}{(z-1)-1} = \frac{-2/3}{1-(z-1)} = -\frac{2}{3}[1 + (z-1) + (z-1)^2 + \cdots], \quad \text{for } |z-1| < 1 \qquad (20)$$

where we have used Eq. (16a) and taken $w = z - 1$. The series in Eqs.(19) and (20) are now substituted in the right side of Eq. (18). Thus,

$$\frac{z}{(z+1)(z-2)} = \underbrace{\frac{1}{6}\left[1 - \frac{(z-1)}{2} + \frac{(z-1)^2}{4} - \cdots\right]}_{|z-1|<2} - \underbrace{\frac{2}{3}[1 + (z-1) + (z-1)^2 + \cdots]}_{|z-1|<1}.$$

In the domain $|z-1| < 1$, ***both*** series converge and their terms can be combined

$$\frac{z}{(z+1)(z-2)} = \left(\frac{1}{6} - \frac{2}{3}\right) + \left(-\frac{1}{12} - \frac{2}{3}\right)(z-1) + \left(\frac{1}{24} - \frac{2}{3}\right)(z-1)^2 + \cdots.$$

or

$$\frac{z}{(z+1)(z-2)} = \sum_{n=0}^{\infty} c_n (z-1)^n. \quad |z-1| < 1 \qquad (21)$$

where

$$c_n = \frac{1}{6}\left(-\frac{1}{2}\right)^n - \frac{2}{3}.$$

Example 8

Expand

$$f(z) = \frac{z}{(z+1)^2(z-2)}$$

in MAclaurin series.

<Sol.>

From **Rule II**, we have

$$\frac{z}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}. \qquad (22)$$

Clearing fractions, we obtain

$$z = A(z+1)(z-2) + B(z-2) + C(z+1)^2 \qquad (23)$$

or

$$z = (A+C)z^2 + (-A+B+2C)z + (-2A-2B+C). \qquad (24)$$

By putting $z=2$ and then $z=-1$ in Eq. (23), we discover that $C = 2/9$ and $B = 1/3$. Note that z^2 does not appear on the left in Eq. (24), which means z^2 must not appear on the right; hence $A = -C = -2/9$. Thus from Eq. (22)

$$\frac{z}{(z+1)^2(z-2)} = \frac{-2/9}{z+1} + \frac{1/3}{(z+1)^2} + \frac{2/9}{z-2} \qquad (25)$$

We now expand each fraction in powers of z. Taking $w = z$, we have, from Eq. (16b),

$$\frac{-2/9}{1+z} = -\frac{2}{9}[1 - z + z^2 - \cdots], \quad |z| < 1,$$

and, from Eq. (16a)

$$\frac{1/3}{(1+z)^2} = \frac{1}{3}[1 - 2z + 3z^2 - 4z^3 + \cdots], \quad |z| < 1,$$

With $w = z/2$ in Eq. (16a), we obtain

$$\frac{2/9}{z-2} = \frac{-1/9}{1 - z/2} = -\frac{1}{9}[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots], \quad |z| < 2.$$

The substitution of the three preceding series on the right in Eq. (25) yields

$$\frac{z}{(z+1)^2(z-2)} = -\frac{2}{9}\underbrace{[1 - z + z^2 - \cdots]}_{|z|<1} + \frac{1}{3}\underbrace{[1 - 2z + 3z^2 - 4z^3 + \cdots]}_{|z|<1}$$

$$-\frac{1}{9}\underbrace{\left[1 + \frac{z}{2} + \frac{z^2}{4} + \cdots\right]}_{|z|<2}$$

Inside $|z| = 1$, we can add the three series together and obtain

$$\frac{z}{(z+1)^2(z-2)} = \sum_{n=0}^{\infty} c_n z^n, \quad |z| < 1. \qquad (26)$$

where

$$c_n = (-1)^{n+1}\frac{2}{9} + \frac{(-1)^n}{3}(n+1) - \frac{1}{9}\left(\frac{1}{2}\right)^n$$

H.W. 7 Obtain the following Taylor expansions. Give a general formula for the nth coefficients, and state the circle within which your expansion is valid.

(a) $\dfrac{z+1}{(z-1)^2(z+2)}$ expanded about $z=2$;

(b) $\dfrac{1}{(z-1)^2(z+1)^2}$ expanded about $z=2$;

(c) $\dfrac{e^z}{(z-1)(z+1)}$ expanded about $z=0$; and

(d) $\dfrac{z^3+2z^2+z-1}{z^2-4}$ expanded about $z=1$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 19-21 and 23, Exercise 5.5, Pearson Education, Inc., 2005.】

§5-6 Laurent Series

♣ **Basic Concept: Limitation for Taylor Series**

(1) Negative exponents never appear in Taylor series.
(2) A Taylor series expansion is only valid in a disc-shaped domain.

Example 1

Consider a series expansion

$$\frac{1}{1-\frac{1}{z}}=\frac{z}{z-1}=1+\frac{1}{z}+\frac{1}{z^2}+\ldots,\ \text{for}\ \left|\frac{1}{z}\right|<1,\ \text{ or equivalently, } |z|>1$$

$$\Rightarrow \qquad \frac{z}{z-1}=1+z^{-1}+z^{-2}+\cdots=\cdots+z^{-2}+z^{-1}+1,\ \text{for}\ |z|>1. \qquad (1)$$

This series is not a Taylor series as the above-mentioned reasons.

Example 2

Consider the Taylor series

$$\frac{1}{1-(z/2)}=1+\frac{z}{2}+\frac{z^2}{4}+\cdots,\quad |z|<2.$$

or

$$\frac{2}{2-z}=1+\frac{z}{2}+\frac{z^2}{4}+\cdots,\quad |z|<2.$$

If we add together Eq. (1) and the preceding, we have the series expansion

$$\frac{z}{z-1}+\frac{2}{2-z}==\cdots+z^{-2}+z^{-1}+2+\frac{z}{2}+\frac{z^2}{4}+\cdots,\quad 1<|z|<2$$

This is a special case of series called **Laurent series**. The ring-shaped domain $1<|z|<2$ is the intersection of the sets of points where the two series used in the calculation are valid.

1. Definition (Laurent series)

The Laurent series expansion of a function $f(z)$ is an expansion of the form

$$f(z)=\sum_{n=-\infty}^{\infty}c_n(z-z_0)^n=\cdots+c_{-2}(z-z_0)^{-2}+c_{-1}(z-z_0)^{-1}+c_0+c_1(z-z_0)+\cdots \qquad (3)$$

where the series converges to $f(z)$ in a region or domain.

♣ **Examples of Laurent series are often obtained from some simple manipulations on Taylor series.**

Example 3

Given a series

$$e^u = 1 + u + \frac{u^2}{2!} + \cdots, \text{ all finite } u$$

Putting $u = (z-1)^{-1}$ in the preceding equation, we have

$$e^{1/(z-1)} = 1 + (z-1)^{-1} + \frac{(z-1)^{-2}}{2!} + \frac{(z-1)^{-3}}{3!} + \ldots, \quad z \neq 1$$

$$\Rightarrow \quad e^{1/(z-1)} = \cdots + \frac{(z-1)^{-3}}{3!} + \frac{(z-1)^{-2}}{2!} + \frac{(z-1)^{-1}}{1!} + 1, \quad z \neq 1 \qquad (4)$$

This is a Laurent series with no positive powers of $(z-1)$.

Multiplying both sides of Eq. (4) by $(z-1)^2$, we have

$$(z-1)^2 e^{1/(z-1)} = \cdots + \frac{(z-1)^{-1}}{3!} + \frac{1}{2!} + (z-1) + (z-1)^2, \quad z \neq 1 \qquad (5)$$

This is a Laurent series with both negative and positive powers of $(z-1)$.

♣ **Applications of Laurent series:**

1) An understanding of the calculus residue
2) Basis of the *z*-transformation.

2. Theorem (Laurent's Theorem)

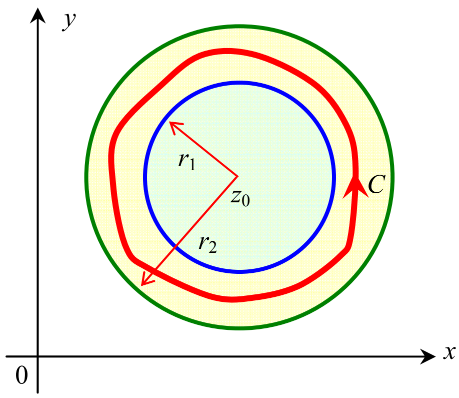
Let $f(z)$ be analytic in *D*, an annular domain $r_1 < |z - z_0| < r_2$. If z lies in *D*, $f(z)$ can be represented by the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \cdots + c_{-2}(z-z_0)^{-2} + c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + c_2(z-z_0)^2 + \cdots \qquad (6)$$

The coefficients are given by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \qquad (7)$$

where *C* is any simple closed contour lying in *D* and enclosing the inner boundary $|z - z_0| = r_1$. The series is uniformly convergent in any region centered at z_0 and lying in *D*.



<pf.>

For simplicity, we consider a proof in which z_0 is zero; that is, we seek an expansion in an annulus centered at the origin.

Annulus: $r_1 < |z - 0| < r_2$.

Contour *C'* lies in the annulus. Observe that *C'* encloses the point z_1 and that $f(z)$ is analytic on and inside the contour.

Cauchy integral formula:

$$f(z_1) = \frac{1}{2\pi i} \oint_{C'} \frac{f(z)}{z - z_1} dz \qquad (8)$$

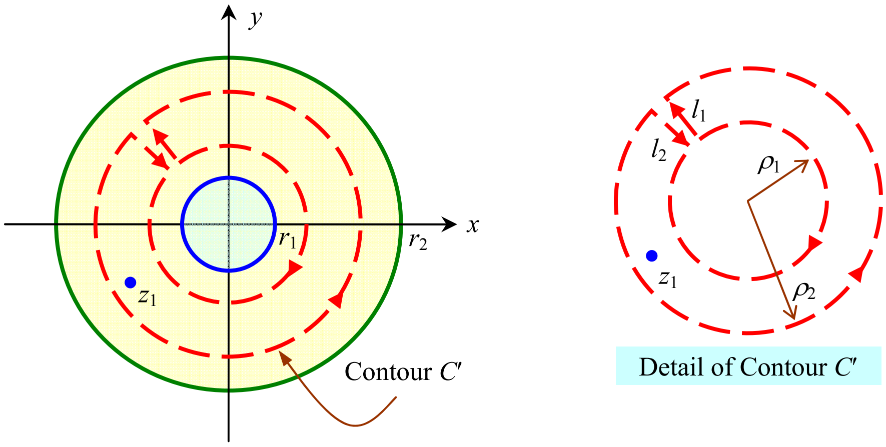
The portions of the preceding integral taken along the contiguous lines l_1 and l_2 cancel because of the opposite directions of integration. Thus, Eq.(8) becomes

$$f(z_1) = I_A + I_B \qquad (9)$$

where

$$I_A = \frac{1}{2\pi i} \oint_{|z|=\rho_2} \frac{f(z)}{z - z_1} dz \qquad (10)$$

and



$$I_B = \frac{1}{2\pi i}\oint_{|z|=\rho 1}\frac{f(z)}{z-z_1}dz \qquad (11)$$

Follow the derivation of the Taylor series in the previous section, we have

$$I_A = \frac{1}{2\pi i}\oint_{|z|=\rho_2}\frac{f(z)}{z\left(1-\frac{z_1}{z}\right)}dz = \frac{1}{2\pi i}\oint_{|z|=\rho_2}\frac{f(z)}{z}\left(1+\frac{z_1}{z}+\left(\frac{z_1}{z}\right)^2+\cdots\right)dz$$
$$= \sum_{n=0}^{\infty} c_n z_1{}^n \qquad (12)$$

where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=\rho_2}\frac{f(z)}{z^{n+1}}dz, \quad n = 0,\ 1,\ 2,\ \cdots \qquad (13)$$

In Eq. (12), we require that $|z_1/z|<1$ or $|z_1|<\rho_2$ (since $|z|<\rho_2$).

In the integral I_B, we reverse the direction of integration and **compensate with a minus sign** in the integrand. Thus,

$$I_B = \frac{1}{2\pi i}\oint_{|z|=\rho_1}\frac{f(z)}{z_1-z}dz = \frac{1}{2\pi i}\oint_{|z|=\rho_1}\frac{f(z)}{z_1\left(1-\frac{z}{z_1}\right)}dz \qquad (14)$$

Now

$$\frac{1}{1-\frac{z}{z_1}} = 1+\frac{z}{z_1}+\left(\frac{z}{z_1}\right)^2+\cdots \quad \text{if} \quad \left|\frac{z}{z_1}\right|<1 \quad \text{or} \quad |z|<|z_1|$$

This series converges uniformly in a region containing the circle $|z|=\rho_1$ (since $|z|=\rho_1<|z_1|$). Using this series in Eq. (14), and integrating, we obtain

$$I_B = \frac{1}{2\pi i}\oint_{|z|=\rho_1}\frac{f(z)}{z_1}\left(1+\frac{z}{z_1}+\left(\frac{z}{z_1}\right)^2+\cdots\right)dz$$
$$= \frac{z_1{}^{-1}}{2\pi i}\oint_{|z|=\rho_1} f(z)dz + \frac{z_1{}^{-2}}{2\pi i}\oint_{|z|=\rho_1} z f(z)dz + \frac{z_1{}^{-3}}{2\pi i}\oint_{|z|=\rho_1} z^2 f(z)dz + \cdots \qquad (15)$$

We have the constant z_1 outside the integral signs. We may rewrite Eq. (15) more succinctly as

$$I_B = \sum_{n=-\infty}^{-1} c_n z_1{}^n, \quad |z_1| > \rho_1, \qquad (16)$$

where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=\rho_1} z^{-n-1} f(z)dz = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z^{n+1}}dz, \quad n = \cdots, -2, -1. \qquad (17)$$

Combining Eqs. (16) and (12) into the right of Eq. (9), we have

$$f(z_1) = \underbrace{\sum_{n=0}^{+\infty} c_n z_1^n}_{|z_1|<\rho_2} + \underbrace{\sum_{n=-\infty}^{-1} c_n z_1^n}_{|z_1|<\rho_1} \qquad (18)$$

where

$$c_n = \frac{1}{2\pi i}\oint_{|z|=\rho_1} \frac{f(z)}{z^{n+1}}dz, \quad n = 0, \pm 1, \pm 2, \ldots. \qquad (19)$$

We can rewrite Eq. (18) as a single summation,

$$f(z_1) = \sum_{n=-\infty}^{+\infty} c_n z_1^n \qquad (20)$$

that is valid when z_1 satisfies $\rho_1 < |z_1| < \rho_2$. This restriction can be relaxed to $r_1 < |z_1| < r_2$.

Replacing z_1 by z in Eq. (20), we find that we have derived Eq. (6) for the special case $z_0 = 0$.

♣ We may conclude that the coefficients for our Laurint series with $z_0 = 0$ are given by

$$c_n = \frac{f^{(n)}(0)}{n!}, \quad \text{for } n \geq 0 \qquad (21)$$

1) This maneuver is not permitted here!!!

2) The Cauchy integral formula and its extension apply only when $f(z)$ in Eq. (19) is analytic not only on *C* but throughout its interior.

⇒ We have made no assumption concerning $f(z)$.

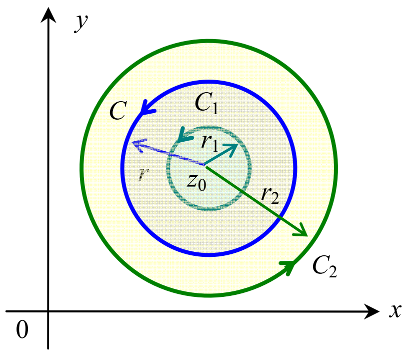
3. Definition (Isolated Singular Point)

The point z_p is an isolated singular point of $f(z)$ if $f(z)$ is not analytic at z_p but is analytic in a deleted neighborhood of z_p.

Example 4

$1/[(z-1)(z-2)^3]$ has isolated singular points at $z = 1$ and $z = 2$ since we can find a disc, centered at each of these points, in which this function is everywhere analytic except for the center.

4. Another Description of Laurent's Series



As the figure shown, we have

$C_1: |z - z_0| = r_1$

$C_2: |z - z_0| = r_2$

$C: |z - z_0| = r$

where $r_1 < r < r_2$.

If the function $f(z)$ is analytic on c_1 and c_2, and $f(z)$ is analytic in the annulus:

$D: r_1 < |z - z_0| < r_2$

Then, for every *z* in *D*, we have

$$\Rightarrow \quad f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} \quad \text{-------------} \quad \text{(A)}$$

where

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

$$b_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{-n+1}}dz$$

and the $\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}$ is called the principle part of the Laurent Series.

** 因為 z_0 為圓心，而 z 則為 C 之圓周，二者不可能相等，因此 $z-z_0$ 不可能為 0。
所以，當 $n=1$ 時

$$\Rightarrow \quad b_1=\frac{1}{2\pi i}\oint_C f(z)dz$$

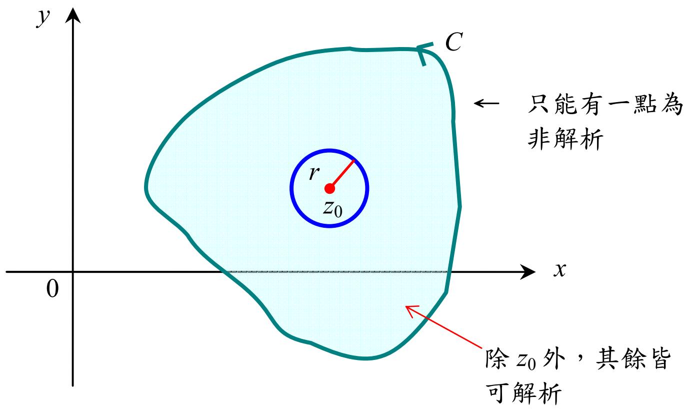
$$\Rightarrow \quad \oint_C f(z)dz=2\pi i\, b_1$$

where b_1 is called the residue of $f(z)$ at $z=z_0$, thus

$$\operatorname*{Res}_{z=z_0} f(z)=b_1$$

** If $f(z)$ is analytic on a simple closed curve C and at every points interior of C except at $z=z_0$,

$$\Rightarrow \quad \oint_C f(z)dz=2\pi i \operatorname*{Res}_{z=z_0} f(z)$$



Example 5

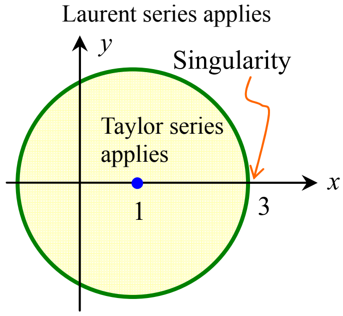
Expand

$$f(z)=\frac{1}{z-3}$$

in a Laurent series in powers of $(z-1)$. State the domain in which the series converges to $f(z)$.

<Sol.>

The only singularity of $f(z)$ is at $z=3$. A Taylor series representation of $f(z)$ is valid in the domain $|z-1|<2$. But, with $z_0=1$, we can represent $f(z)$ in a Laurent series in the domain $|z-1|>2$. Recall that



$$\frac{1}{1-w}=1+w+w^2+\cdots,\quad |w|<1. \qquad (22)$$

Now

$$\frac{1}{z-3}=\frac{1}{(z-1)-2}=\frac{1/(z-1)}{1-2/(z-1)} \qquad (23)$$

Comparing Eq. (22) and (23) and taking $w=2/(z-1)$, we obtain our Laurent series. Thus,

$$\frac{1}{z-3}=\frac{1}{z-1}\left[1+\frac{2}{z-1}+\frac{4}{(z-1)^2}+\cdots\right]$$

$$=(z-1)^{-1}+2(z-1)^{-2}+4(z-1)^{-3}+\cdots$$

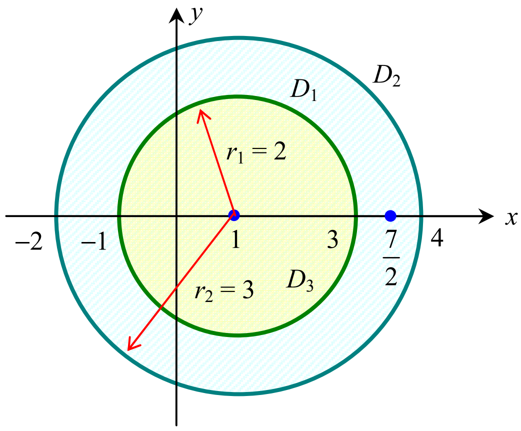
The condition $|w|<1$ in Eq. (22) becomes $|2/(z-1)|<1$ or $|z-1|>2$.

Example 6

Expand

$$f(z)=\frac{1}{(z+1)(z+2)}$$

in a Laurent series in powers of $(z-1)$ valid in an annular domain containing the point $z=7/2$. State the domain in which the series converges to $f(z)$. Consider also other series representation of $f(z)$ involving powers of $(z-1)$ and state where they are valid.



<Sol.>

Refer to the shown figure.

Since $f(z)$ has singularities at -2 and -1, we see that one such domain is D_1 defined by $2<|z-1|<3$, while another is D_2 given by $|z-1|>3$. A Taylor series representation is also available in the domain D_3 described by $|z-1|<2$.

Since $z=7/2$ lies in D_1, it is the Laurent expansion valid in this domain that we seek.

We break $f(z)$ into partial fractions. Thus,

$$\frac{1}{(z+1)(z+2)}=\frac{1}{(z+1)}-\frac{1}{(z+2)} \qquad (24)$$

Rewrite the first fraction as

$$\frac{1}{(z+1)}=\frac{1}{(z-1)+2}=\frac{1/2}{1+(z-1)/2} \qquad (25)$$

or, alternatively, as

$$\frac{1}{(z+1)}=\frac{1}{(z-1)+2}=\frac{1/(z-1)}{1+2/(z-1)} \qquad (26)$$

Recall that

$$\frac{1}{1+w}=1-w+w^2-w^3+\cdots, \quad |w|<1.$$

With $w=(z-1)/2$, we expand eq. (25) to obain

$$\frac{1}{z+1}=\frac{1}{2}\left[1-\frac{(z-1)}{2}+\frac{(z-1)^2}{4}+\cdots\right], \quad \text{if } \left|\frac{z-1}{2}\right|<1 \text{ or } |z-1|<2 \qquad (27)$$

Taking $w=2/(z-1)$, we expand Eq. (26) as follows:

$$\frac{1}{z+1}=\frac{1}{(z-1)}\left[1-\frac{2}{(z-1)}+\frac{4}{(z-1)^2}+\cdots\right] \quad \text{if } \left|\frac{2}{z-1}\right|<1 \text{ or } |z-1|>2 \qquad (28)$$

$$=(z-1)^{-1}-2(z-1)^{-2}+4(z-1)^{-3}-\ldots$$

We have expressed $1/(z+1)$ as a Taylor series and a Laurent series, both in powers of $(z-1)$.

Similarly, for the fraction in Eq.(24), with $w=(z-1)/3$, we have

$$-\frac{1}{z+2}=\frac{-1}{(z-1)+3}=\frac{-1/3}{1+\left(\dfrac{z-1}{3}\right)}$$

$$=-\frac{1}{3}\left[1-\frac{(z-1)}{3}+\frac{(z-1)^2}{9}-\cdots\right], \quad |z-1|<3 \qquad (29)$$

and, with $w=3/(z-1)$

$$-\frac{1}{z+2}=\frac{-1}{(z-1)+3}=\frac{-1/(z-1)}{1+3/(z-1)}$$

$$=-\frac{1}{z-1}\left[1-\frac{3}{(z-1)}+\frac{9}{(z-1)^2}-\cdots\right] \qquad (30)$$

$$=-(z-1)^{-1}+3(z-1)^{-2}-9(z-1)^{-3}+\cdots, \quad |z-1|>3$$

In the domain D_1, the series in Eqs.(28) and (29) converge to their respective functions (but, the series in Eqs. (27) and (30) are of no use). Using these equations, we replace each fraction on the right in Eq. (24) by a series and obtain

$$\frac{1}{(z+1)(z+2)}=\underbrace{(z-1)^{-1}-2(z-1)^{-2}+4(z-1)^{-3}-\cdots}_{|z-1|>2}$$
$$\underbrace{-\frac{1}{3}+\frac{1}{9}(z-1)-\frac{1}{27}(z-1)^2+\cdots,}_{|z-1|<3} \qquad (31)$$

which, when written in more succinctly, reads

$$\frac{1}{(z+1)(z+2)}=\sum_{n=-\infty}^{+\infty}c_n(z-1)^n, \quad 2<|z-1|<3 \qquad (32)$$

where

$$c_n=(-\frac{1}{3})^{n+1}, \quad n\geq 0 \qquad (33a)$$

and

$$c_n=(-1)^{n+1}2^{-n-1}, \quad n\leq -1 \qquad (33b)$$

♣ A Laurent series expansion of $f(z)$ in the domain $|z-1|>3$, that is, D_2, is possible. We represent the partial fractions in Eq. (24) by the series shown in (28) and (30). Both are valid in D_2. Adding these series, we have

$$\frac{1}{(z+1)(z+2)}=\sum_{n=-\infty}^{-2}c_n(z-1)^n, \quad |z-1|>3,$$

where

$$c_n=(-1)^n\left[3^{-n-1}-2^{-n-1}\right], \quad n=\cdots,-3,-2.$$

Example 7

Expand

$$f(z)=\frac{1}{z(z-1)}$$

in a Laurent series that is valid in a deleted neighborhood of $z=1$. State the domain throughout which the series is valid.

<Sol.>

Observe that $f(z)$ has singularities at $z=1$ and $z=0$. The annulus $0<|z-1|<1$ is the largest deleted neighborhood of $z=1$ that excludes both singularities of $f(z)$.

Decomposing $f(z)$ into fractions, we obtain

$$\frac{1}{z(z-1)}=-\frac{1}{z}+\frac{1}{z-1} \qquad (34)$$

This equality breaks down at $z=0$ and $z=1$. The second fraction, $(z-1)^{-1}$, is already expanded in powers of $(z-1)$. It is a one term Laurent series.

For the fraction $-1/z$, we have the choice of two series containing powers of $(z-1)$. Thus

$$-\frac{1}{z}=\frac{-1}{1+(z-1)}=-\left(1-(z-1)+(z-1)^2-\cdots\right),\quad |z-1|<1 \tag{35}$$

and

$$-\frac{1}{z}=\frac{-1/(z-1)}{1+1/(z-1)}=-(z-1)^{-1}\left(1-\frac{1}{(z-1)}+\frac{1}{(z-1)^2}-\cdots\right) \tag{36}$$
$$=-(z-1)^{-1}+(z-1)^{-2}-(z-1)^{-3}+\cdots\quad |z-1|>1$$

Using Eq. (35) on the right in Eq. (34) to represent $-1/z$, we get

$$\frac{1}{z(z-1)}=\underbrace{-1+(z-1)-(z-1)^2+\cdots}_{|z-1|<1}+\underbrace{(z-1)^{-1}}_{z\neq 1}$$

or, more neatly,

$$\frac{1}{z(z-1)}=\sum_{n=-1}^{\infty}(-1)^{n+1}(z-1)^n,\quad 0<|z-1|<1.$$

♣ Had we used Eq. (36) instead of Eq. (35) to represent $-1/z$ on the right in Eq. (34), we would have obtained the Laurent expansion

$$\frac{1}{z(z-1)}=(z-1)^{-2}-(z-1)^{-3}+(z-1)^{-4}-\cdots$$

This expansion is valid in the same annulus as the series in Eq. (36), that is, $|z-1|>1$, which is not the required deleted neighborhood of $z=1$.

♣ Laurent series for transcendental functions are sometimes obtained either by division of Taylor series or by a recursive procedure equivalent to series division.

Example 8

Expand $1/\sin z$ in a Laurent series valid in a deleted neighborhood of the origin. Where in the complex plane will your series converge to this function?

<Sol.>

Recall that

$$\sin z=0\quad\text{when}\quad z=0,\ \pm\pi,\ \pm 2\pi,\ \cdots$$

Thus, $z=0,\ -\pi,\ \pi$ are isolated singular points of $1/\sin z$. A Laurent expansion of this function, employing powers of z, is thus possible in the punctured disc $0<|z|<\pi$.

We seek a series expansion of the form $1/\sin z=\sum_{n=-\infty}^{\infty}c_n z^n$. Note that

$$\frac{z}{\sin z}=z\sum_{n=-\infty}^{\infty}c_n z^n=\ldots+c_{-3}z^{-2}+c_{-2}z^{-1}+c_{-1}+c_0 z+c_1 z^2+\ldots \tag{37a}$$

Now from L'Hôspital's rule, we

$$\lim_{z\to 0}\frac{z}{\sin z}=\lim_{z\to 0}\frac{1}{\cos z}=1$$

If the series on the right in Eq. (37a) is to produce this same limit, we require that

$$c_{-2}=c_{-3}=c_{-4}=\cdots=0$$

Otherwise, the terms $c_{-2}z^{-1},\ c_{-3}z^{-2},\ c_{-4}z^{-3}$, etc., on the right would become infinite as $z\to 0$. Having eliminated all c_n for $n\le -2$, we have

$$\frac{1}{\sin z}=c_{-1}z^{-1}+c_0+c_1 z+c_2 z^2+\ldots,\quad (0<|z|<\pi)$$

Multiplying both sides of the preceding equation by $\sin z$ and using the expansion

$$\sin z=z-\frac{z^3}{3!}+\frac{z^5}{5!}-+\cdots,$$

we have

$$1 = (z - \frac{z^3}{3!} + \frac{z^5}{5!} + ...)(c_{-1}z^{-1} + c_0 + c_1 z + c_2 z^2 + ...)$$

Now multiplying the series on the above right and equating the coefficients of the various powers of z to the corresponding coefficients on the left, we find

z^0 term: $1 = c_{-1}$,

z^1 term: $0 = c_0$,

z^2 term: $0 = c_1 - \frac{c_{-1}}{3!}$,

z^3 term: $0 = c_2 - \frac{c_0}{3!}$,

z^4 term: $0 = c_3 - \frac{c_1}{3!} + \frac{c_{-1}}{5!}$,

$\cdots$, etc.

Then, we find the coefficients of all even powers,

$$c_0 = c_2 = c_4 = \cdots = 0$$

and

$$c_{-1} = 1,\ c_1 = 1/6,\ c_3 = -1/5! + (1/3!)/3! = 7/360,\ \cdots$$

and when n is odd, the general form of c_n is shown as below

$$c_n = \left[\frac{c_{n-2}}{3!} - \frac{c_{n-4}}{5!} + \frac{c_{n-6}}{7!} + \cdots \pm \frac{c_{-1}}{(n+2)!}\right]$$

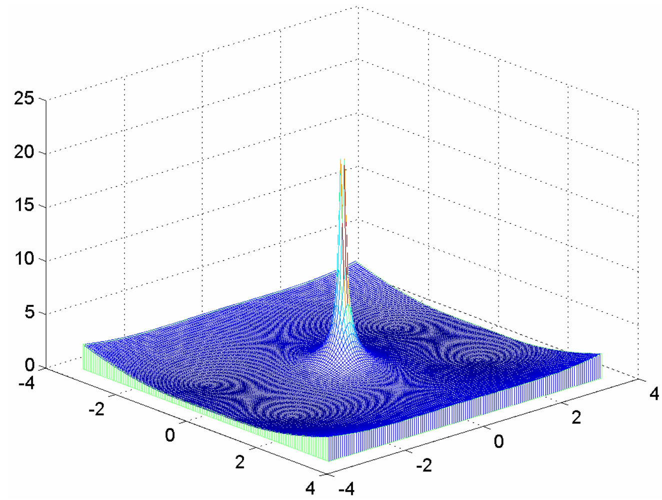
Thus, we have

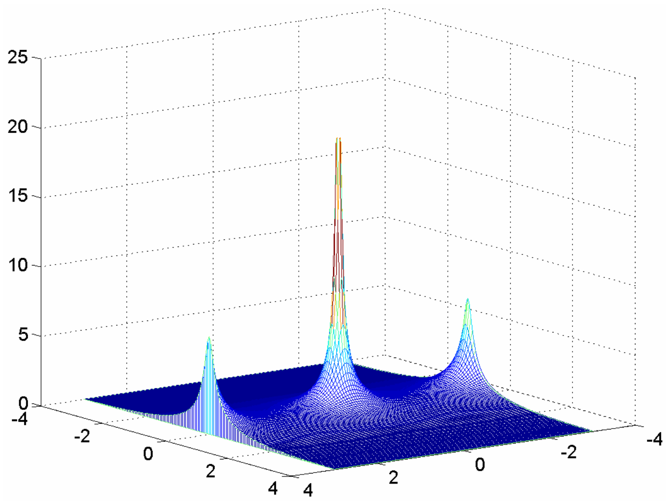
$$\frac{1}{\sin z} = \frac{1}{z} + \frac{z}{6} + \frac{7z^3}{360} + \cdots, \quad 0 < |z| < \pi \qquad (37b)$$

In the following figures, we have plotted an approximation to $|1/\sin z|$ obtained by our using the first five terms in the Laurent expansion of $1/\sin z$; i.e., we have graphed $\left|\frac{c_{-1}}{z} + c_0 + c_1 z + c_2 z^2 + c_3 z^3\right|$ for the domain $0 < |z| < \pi$. For comparison, we have plotted in the shown figure the function $|1/\sin z|$.

```
% section 5.6, approximate plot of | 1/sin z |
x=[-3.5:0.05:3.5];
y=[-3.5:0.05:3.5] ;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./Z+(1/6).*Z+(7/360).*Z.^3;
wm=abs(w);
meshz(X,Y,wm);view(100,70)
```

```
% section 5.6, exact plot of | 1/sin z |
x=[-3:0.05:3];
y=[-4:0.05:4] ;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=sin(Z);
wm=1./abs(w);
meshz(X,Y,wm);view(150,70)
```

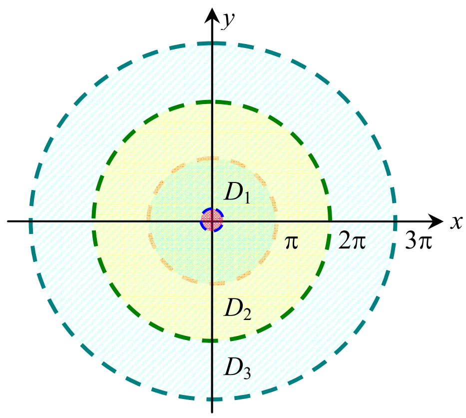




♣ From the location of the singularities of $1/\sin z$, we see that it should be possible to obtain another Laurent series, in powers of z, valid in D_2 of the following figure, i.e.,

$$\frac{1}{\sin z} = \sum_{n=-\infty}^{\infty} d_n z^n, \quad \pi < |z| < 2\pi$$

Similarly, there is a third Laurent series valid in the domain D_3 described by $2\pi < |z| < 3\pi$.



H.W. 1 The exponential integral $E_1(a)$ is defined by the improper integral

$$E_1(a) = \int_a^{\infty} \frac{e^{-x}}{x} dx, \quad a > 0$$

Thus,

$$E_1(a) - E_1(b) = \int_a^b \frac{e^{-x}}{x} dx,$$

Use a Laurent expansion for e^{-z}/z and a term-by-term integration to show that

$$E_1(a) - E_1(b) = \text{Ln}\frac{b}{a} - (b-a) + \frac{b^2 - a^2}{(2!)(2)} - \frac{b^3 - a^3}{(3!)(3)} + \cdots$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 18, Exercise 5.6,

Pearson Education, Inc., 2005.】

H.W.2 (a) Extend the work of the previous **Example 8** to show that in the expansion

$$\frac{1}{\sin z}=\sum_{n=-1}^{\infty} c_n z^n\ ,\quad 0<|z|<\pi$$

we can get c_n from the recursion formula

$$c_n=\left[\frac{c_{n-2}}{3!}-\frac{c_{n-4}}{5!}+\frac{c_{n-6}}{7!}+\cdots\pm\frac{c_{-1}}{(n+2)!}\right]$$

when n is odd. Recall that $c_n=0$ if n is even and that $c_{-1}=1$.

(b) Find c_5 for the series.

(c) Consider the Laurent expansion $1/\sinh z=\sum_{n=-1}^{\infty} a_n z^n$ for $0<|z|<\pi$. Find, by means of a change of variable, the simple relationship between coefficients a_n and c_n of part (a).

(d) Derive a recursion formula like that given in part (a) for the a_n coefficients. Proceed as we did in **Example 8**.

(e) Using MATLAB, obtain figures like those in Figures (shown in the previous page) so that one can compare $|1/\sinh z|$ with a five-term Laurent expansion approximating this function. Use the domain $0<|z|<\pi$ as in the previous figures and a five-term Laurent series.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 23, Exercise 5.6, Pearson Education, Inc., 2005.】

<Ans.>

♣ MATLAB commands for plotting $|1/\sin z|$

% section 5.6, plot of $|1/\sinh z|$

```
clear
nm=5;
d(1)=1;
for k=2:nm
for j=1:k-1
    u(j)=gamma(2*k-2*j+2);
end
d(k)=sum(d./u);
d;
end
nr=25;
r=linspace(0.05,pi-0.05,nr);
nth=91;
theta=linspace(0.2*pi,nth);
[T,R]=meshgrid(theta,r);
[X,Y]=pol2cart(T,R) ;
z=X+i*Y ;
mm=length(d) ;
ff=0 ;
for p=1:mm
    ff=d(p)*z.^(2*p-3)+ff;
end
% ff=1./sinh(z);
%use for figure (a)
meshz(X,Y,abs(ff)); view(135,30)
```

% section 5.6, exact plot of $|1/\sin z|$

```
x=[-3:0.05:3];
y=[-4:0.05:4] ;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=sin(Z);
```

```
wm=1./abs(w);
meshz(X,Y,wm);view(150,70)
```

H.W. 3 One way of defining the Bessel functions of the first kind is by means of an integral:

$$J_n(w) = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\cos(n\theta - w\sin\theta)d\theta$$

where n is an integer. The number n is called the order of the Bessel function. There is a connection between this integral and the coefficients of z in a Laurent expansion of $e^{w(z-1/z)/2}$.
Let

$$e^{w(z-1/z)/2} = \sum_{n=-\infty}^{\infty} c_n z^n, \quad |z| > 0 \qquad (38)$$

Show using Laurent's theorem that

$$c_n = J_n(w) \qquad (39)$$

<Hint> Refer to eq. (7). Take as a contour $|z| = 1$. Make a change of variables to polar coordinate ($z = e^{i\theta}$). Then, use Euler's identity and symmetry to argue that a portion of your result is zero.
The expression $e^{(w/2)(z-z^{-1})}$ is called a ***generating function*** for these Bessel functions.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 26, Exercise 5.6, Pearson Education, Inc., 2005.】

H.W. 4 (a) Refer to Eqs.(38) and (39). Show that

$$J_n(w) = \sum_{k=0}^{\infty}\frac{(-1)^k (w/2)^{n+2k}}{k!(n+k)!}, \quad n = 0,\ 1,\ 2,\ \cdots$$

<Hint> The left side of Eq. (38) is $e^{(wz/2)}e^{-w/(2z)}$. Multiply the Maclaurin series for the foirst term by a Laurent series for the second term.

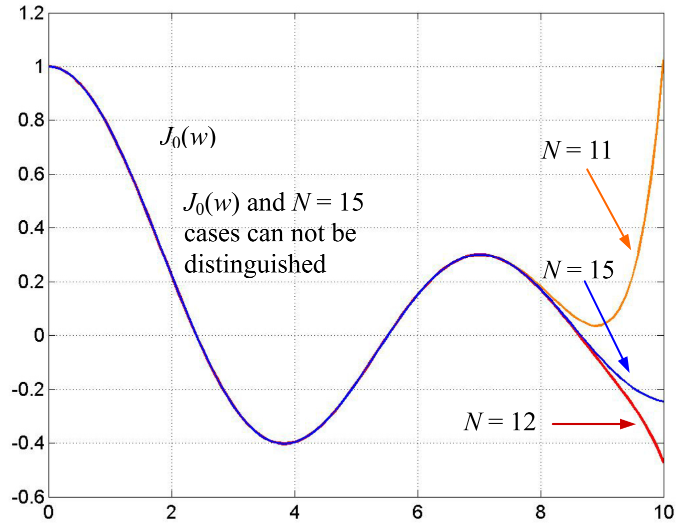
(b) Let w be a real variable in the preceding. Consider the Bessel function $J_0(w)$, which we will try to approximate using three different *N*th partial sums in the series derived above. Using MATLAB, plot on one set of axes these sums for the cases $N = 11,\ 12,$ and 15 for the interval $0 \le w \le 10$. (You may wish to re-index the sum.) Notice that rather significant differences. Bessel functions are built into MATLAB and there is no usually a need to use series approximations. Plot on the same axes $J_0(w)$ for $0 \le w \le 10$ using the MATLAb supplied function, and compare it with the three partial-sum approximation.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 27, Exercise 5.6, Pearson Education, Inc., 2005.】

<Ans.>

♣ MATLAB commands:

```
% for H.W. 4 (b)
clear
x=linspace(0,10,100);
nm=[11 12 15]
for jj=1:3
    y=x.*0;
for k=1:nm(jj)
    k=k-1;
    y=(-1)^(k)*(x/2).^(2*k)/(gamma(k+1))^2+y;
end
plot(x,y); hold on
end
y2=besselj(0,x);
plot(x,y2);grid
```



§5-7 The z Transformations

1. **Definition of *z* Transform**

The z transform of the function $f(t)$, that is, $\mathbb{Z}[f(t)]$, is given by

$$\mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} f(nT)z^{-n} = f(0) + f(T)z^{-1} + f(2T)z^{-2} + \cdots \qquad (1)$$

where $T > 0$. The function so obtained is called $F(z)$. We say that $\mathbb{Z}[f(t)] = F(z)$.

1) Eq. (1) is a Laurent series with ***no positive exponent*** in any term
2) $f(t)$ is defined only for $t = nT$, $n = 0, 1, 2, \cdots$.
3) The transformation is the conversion of a sequence of numbers $c_n = f(nT)$ $(n = 0, 1, 2, \cdots)$ to a function of z by means of $\sum_{n=0}^{\infty} c_n z^{-n}$.
4) In some treatment of z-transform, it is convenient to take $T = 1$. In this case, we have $f(n) \leftrightarrow F(z)$.
5) Let us set $w = 1/z$ in Eq.(1), we have

$$F(z) = \mathbb{Z}[f(t)] = \sum_{n=0}^{\infty} c_n w^n \text{, for } |w| \le r \text{, where } r < \rho \text{ and } \rho > 0$$

Thus, $\sum_{n=0}^{\infty} c_n (1/z)^n = F(z)$ is an analytic function of z for $|1/z| \le r$ or $|z| \ge 1/r$. This means that the z-transform $F(z)$, defined by a Laurent series, that is analytic in the z-plane in an annular domain whose outer radius is **infinite**.

6) **Inversion of *z*-transform:**

$$\mathbb{Z}^{-1}[f(z)] = f(nT)$$

2. ***z* Transform Inversion Formula**

$$f(nT) = \frac{1}{2\pi i} \oint_C F(z) z^{n-1} dz, \quad n = 0, 1, 2, \cdots \qquad (2)$$

Here, C is any circle centered at the origin with radius greater than R.

♣ **Unit step function**

$$u(t) = 1, \quad t \ge 0 \qquad (3a)$$
$$u(t) = 0, \quad t < 0 \qquad (3b)$$

Example 1

Find $\mathbb{Z}[u(t)]$, the transform of the unit step function.

<Sol.>

It is obvious that

$$u(nT)=1 \quad \text{for} \quad n=0,\ 1,\ 2,\ \cdots$$

Thus, from Eq. (1), we have

$$\mathbb{Z}[u(t)]=\mathbb{Z}[1]=\sum_{n=0}^{\infty} z^{-n}=1+\frac{1}{z}+\frac{1}{z^2}+\cdots$$

Recalling that

$$\frac{1}{1-w}=1+w+w^2+\cdots, \text{ for } |w|<1$$

With $w=1/z$, we have

$$1+\frac{1}{z}+\frac{1}{z^2}+\cdots=\frac{1}{1-1/z}=\frac{z}{z-1}$$

which is valid for $|1/z|<1$ or $|z|>1$. Thus,

$$\mathbb{Z}[u(t)]=\mathbb{Z}[1]=\frac{z}{z-1}, \quad |z|>1$$

Example 2

Find the *z* transform of $f(t)=tu(t)$.

<Sol.>

Here, $f(nT)=nT$ for $n=0,\ 1,\ 2,\ \cdots$. Thus,

$$\mathbb{Z}[tu(t)]=\sum_{n=0}^{\infty}(nT)z^{-n}=T\left[\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\ldots\right] \qquad (4)$$

Recall that

$$\frac{\omega}{(1-\omega)^2}=\omega+2\omega^2+3\omega^3+\ldots., \quad |\omega|<1.$$

If we replace w with $1/z$ in the preceding and multiply both sides by T, we have

$$\frac{T(1/z)}{(1-1/z)^2}=T\left[\frac{1}{z}+\frac{2}{z^2}+\frac{3}{z^3}+\ldots\right], \quad |z|>1.$$

Comparing this equation with Eq. (4), we obtain

$$\mathbb{Z}[tu(t)]=\frac{T(1/z)}{(1-1/z)^2}=\frac{Tz}{(z-1)^2}$$

3. **Linearity of the z transformation**

If

$$f(t) \leftrightarrow F(z)$$
$$g(t) \leftrightarrow G(z)$$

then we have

1) $\mathbb{Z}[cf(t)]=cF(z)$, where *c* is constant.

2) $\mathbb{Z}[f(t)+g(t)]=\mathbb{Z}[f(t)]+\mathbb{Z}[g(t)]=F(z)+G(z)$

3) $\mathbb{Z}^{-1}[F(z)+G(z)]=\mathbb{Z}^{-1}[F(z)]+\mathbb{Z}^{-1}[G(z)]=f(t)+g(t)$

Example 3

$$\mathbb{Z}[(1+t)u(t)]=\frac{z}{z-1}+\frac{Tz}{(z-1)^2}=\frac{z^2-z+Tz}{(z-1)^2}$$

Example 4

If $F(z)=(z+1)/z^2$, find $\mathbb{Z}^{-1}\left[F(z)\right]$.

<Sol.>

Rewriting $F(z)$ as a two-term Laurent series, we have

$$F(z)=(1/z)+(1/z^2)$$

A glance at Eq. (1) shows that

$$f(0)=0\,,\quad f(T)=f(2T)=1\text{, and}\quad f(nT)=0,\quad n\geq 3$$

Example 5

If $F(z)=(z+1)/(z-1)$, find $\mathbb{Z}^{-1}\left[F(z)\right]$.

<Sol.>

$F(z)$ can be expanded in a Laurent series valid for $|z|>1$. We have

$$F(z)=\frac{z+1}{z-1}=\frac{(z-1)+2}{z-1}=1+\frac{2}{z-1}$$

Now,

$$\frac{2}{z-1}=\frac{2}{z}\frac{1}{1-(1/z)}=\frac{2}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\cdots\right],\quad |z|>1$$

Thus,

$$F(z)=1+\frac{2}{z}+\frac{2}{z^2}+\cdots,\quad |z|>1$$

Studying the coefficients and using Eq. (1), we conclude that

$$f(0)=1\text{, and}\quad f(nT)=2,\quad n\geq 1$$

♣ A given $F(z)$ does not necessarily have an inverse z transform.

1) If $F(z)$ has no Laurent series of the form $\sum_{n=0}^{\infty}c_n z^{-n}$, no inverse transform is possible.

2) $\lim\limits_{z\to\infty}F(z)=c_0 \quad\Rightarrow\quad$ It means that $F(z)$ may have inverse transform.

4. **Translation properties of z transform**

1) **First translation formula:**

If $\mathbb{Z}\left[f(t)\right]=F(z)=\sum\limits_{n=0}^{\infty}f(nT)z^{-n}$, then

$$\mathbb{Z}\left[f\left(t-kT\right)\right]=\sum_{n=k}^{\infty}f\left(nT-kT\right)z^{-n}=\sum_{n=k}^{\infty}f\left(\left(n-k\right)T\right)z^{-n}$$

where recalling that $f(t)=0,\quad t<0$, we see that $f(nT-kT)=0$, when $n<k$ and $k\geq 0$.

We now re-index this summation using $m=n-k$. Thus,

$$\mathbb{Z}\left[f\left(t-kT\right)\right]=\sum_{m=0}^{\infty}f\left(mT\right)z^{-(m+k)}=z^{-k}\sum_{m=0}^{\infty}f\left(mT\right)z^{-m}$$

$$\Rightarrow\qquad \mathbb{Z}\left[f\left(t-kT\right)\right]=z^{-k}F\left(z\right)\qquad(6)$$

2) **Second translation formula**

Consider $\mathbb{Z}\left[f\left(t+kT\right)\right]$ when $k=1$. We have

$$\mathbb{Z}\left[f\left(t+T\right)\right]=\sum_{n=0}^{\infty}f\left(nT+T\right)z^{-n}=\sum_{n=0}^{\infty}f\left(\left(n+1\right)T\right)z^{-n}$$

$$=f\left(T\right)+f\left(2T\right)z^{-1}+f\left(3T\right)z^{-2}+\ldots.$$

Adding and subtracting $f(0)z$ in this last series, we obtain

$$\mathbb{Z}\left[f(t+T)\right]=\underbrace{\left[f(0)z+f(T)z^{0}+f(2T)z^{-1}+\cdots\right]}_{zF(z)}-f(0)z.$$

Thus,

$$\mathbb{Z}\left[f(t+T)\right]=zF(z)-zf(0) \qquad (7)$$

When $k=2$, we have

$$\mathbb{Z}\left[f(t+2T)\right]=\sum_{n=0}^{\infty}f(nT+2T)z^{-n}=f(2T)+f(3T)z^{-1}+f(4T)z^{-2}+\cdots$$

$$=\underbrace{\left[f(0)z^{2}+f(T)z+f(2T)+f(3T)z^{-1}+f(4T)z^{-2}+\ldots\right]}_{z^{2}F(z)}$$

$$-z^{2}f(0)-zf(T).$$

Thus,

$$\mathbb{Z}\left[f(t+2T)\right]=z^{2}F(z)-z^{2}f(0)-zf(T). \qquad (8)$$

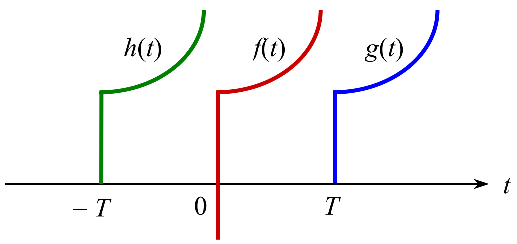
General case for $k\geq 0$:

$$\mathbb{Z}\left[f(t+kT)\right]=z^{k}F(z)-z^{k}f(0)-z^{k-1}f(T)-z^{k-2}f(2T)$$

$$-\cdots-zf((k-1)T).$$

Example 6

If $f(t)=e^{at}u(t)$, then $F(z)=z/(z-e^{aT})$ for $|z|>|e^{aT}|$. Use this result to find $\mathbb{Z}\left[g(t)\right]$, where $g(t)=e^{a(t-T)}u(t-T)$. Also, find $\mathbb{Z}\left[h(t)\right]$, where $h(t)=e^{a(t+T)}u(t+T)$. Assume that $a>0$.



<Sol.>

Since $g(t)=f(t-T)$, we use Eq. (6) with $k=1$ to get $G(z)$. Thus,

$$G(z)=z^{-1}\frac{z}{z-e^{aT}}=\frac{1}{z-e^{aT}}, \quad \text{for } |z|>e^{aT}$$

Since $h(t)=f(t+T)$, we use Eq. (7) to get $H(z)$. Noting that $f(0)=1$, we have

$$\mathbb{Z}\left[h(t)\right]=\frac{z^{2}}{z-e^{aT}}-z=\frac{ze^{aT}}{z-e^{aT}}$$

5. **z Transforms of Products of Functions**

Let $\mathbb{Z}\left[f(t)\right]=\sum_{n=0}^{\infty}c_{n}z^{-n}=F(z)$ and $\mathbb{Z}\left[g(t)\right]=\sum_{n=0}^{\infty}d_{n}z^{-n}=G(z)$, where $c_{n}=f(nT)$ and $d_{n}=g(nT)$. By definition $\mathbb{Z}\left[f(t)g(t)\right]=\sum_{n=0}^{\infty}f(nT)g(nT)z^{-n}$. Thus,

$$\mathbb{Z}\left[f(t)g(t)\right]=\sum_{n=0}^{\infty}c_{n}d_{n}z^{-n} \qquad (9)$$

Let $F(z)$ and $G(z)$ both be analytic in the domain $|z|>R$. From Laurent expansion we have

$$F(w)=\sum_{m=0}^{\infty}c_{m}w^{-m}, \quad |w|>R$$

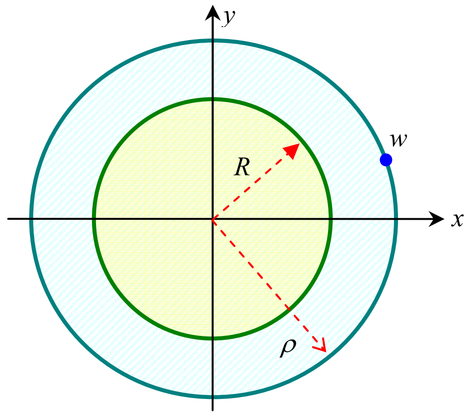
and

$$G(z/w)=\sum_{m=0}^{\infty}d_{n}(z/w)^{-n}=\sum_{m=0}^{\infty}d_{n}w^{n}z^{-n}, \quad |z/w|>R\text{, or } |z|>R|w|$$

Multiplying our series, we have

$$F(w)G(z/w)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_m d_n w^{n-m}z^{-n}, \qquad (10)$$

where we choose $|w|>R$ and $|z|>R|w|$.



Refer to the figure shown on the left.

We take $\rho>R$, and we place our variable w on this circle so that $|w|=\rho$. In Eq. (10), we will require that $|z|>R\rho$. Hence, the Laurent expansion in Eq. (10) is uniformly convergent in a domain in the w-plane containing the circle $|w|=\rho$. The following Laurent expansion is also uniformly convergent in this domain:

$$\frac{1}{2\pi i w}F(w)G(z/w)=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}c_m d_n\frac{z^{-n}w^{n-m}}{w}$$

We can thus integrate this series term by term around $|w|=\rho$, so that

$$\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{f(w)G(z/w)}{w}dw=\frac{1}{2\pi i}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\oint_{|w|=\rho}c_m d_n z^{-n}\frac{w^{n-m}}{w}dw. \qquad (11)$$

Recalling that

$$\oint_{|w|=\rho}w^k dw=\begin{cases}0, & k\neq -1\\ 2\pi i, & k=-1\end{cases}$$

We notice that the integrands on the right in Eq. (11) are zero except when $n=m$. Then,

$$\oint_{|w|=\rho}z^{-n}w^{n-m}/w\,dw=2\pi i z^{-n}\text{, for } n=m$$

Thus, Eq. (11) becomes

$$\frac{1}{2\pi i}\oint_{|\omega|=\rho}\frac{F(\omega)G(z/\omega)}{\omega}d\omega=\sum_{n=0}^{\infty}c_n d_n z^{-n}. \qquad (12)$$

Comparing the above with Eq. (9), we have our desired result:

$$\mathbb{Z}\left[f(t)g(t)\right]=\frac{1}{2\pi i}\oint_{|\omega|=\rho}\frac{F(\omega)G(z/\omega)}{\omega}d\omega. \qquad (13)$$

In this integral, we require that $|z|>R\rho$, where $\rho>R$. Recall that R is such that $F(w)$ and $G(w)$ are analytic for $|w|>R$.

Example 7

Find $\mathbb{Z}\left[te^{at}u(t)\right]$ from Eq. (13).

<Sol.>

Let $f(t)=tu(t)$ and $g(t)=e^{at}u(t)$. Thus, we have

$$\mathbb{Z}[f(t)]=\frac{Tz}{(z-1)^2}=F(z)$$

and

$$\mathbb{Z}[g(t)]=\frac{z}{(z-e^{aT})}=G(z)$$

Notice that $F(z)$ is analytic except at $z=1$, while $G(z)$ is analytic except at $z=e^{aT}$. Substituting in Eq. (13), we find

$$\mathbb{Z}\left[te^{at}u(t)\right]=\frac{1}{2\pi i}\oint_{|w|=\rho}\frac{Tw}{w(w-1)^2}\frac{z/w}{(z/w-e^{aT})}dw$$
$$=\frac{zT}{2\pi i}\oint_{|w|=\rho}\frac{1}{(w-1)^2}\frac{1}{(z-we^{aT})}dw \qquad (14)$$

We require $\rho > R$, where R is the radius of a circle in the w-plane outside which $F(w)$ and $G(w)$ are analytic. Thus, $R>1$ and $R>\left|e^{aT}\right|$.

Recall that Eq. (13) is valid for $|z|>R\rho$. We have, for w lying on or inside the contour $|w|=\rho$,

$$\left|\omega e^{aT}\right|=|\omega|\left|e^{aT}\right|\le\rho\left|e^{aT}\right|<\rho R<|z|.$$

The preceding, $\left|we^{aT}\right|<|z|$, tell us that $z-we^{aT}=0$ can not be satisfied on and inside our contour of integration.

Using the extended Cauchy integral formula, Eq. (14) can be evaluated as

$$\mathbb{Z}\left[te^{aT}u(t)\right]=zT\frac{d}{d\omega}\left[\frac{1}{z-\omega e^{aT}}\right]_{\omega=1}=\frac{zTe^{aT}}{\left(z-e^{aT}\right)^2} \qquad (15)$$

6. Inverse z Transform of a Product of Two Functions

Definition of Convolution

$$f(t)*g(t)=\sum_{k=0}^{\infty}f(kT)g\left((n-k)T\right),\quad n=0,\ 1,\ 2,\ \cdots \qquad (16)$$

♣ **Commutative property:**

$$g(t)*f(t)=\sum_{k=0}^{\infty}g(kT)f\left((n-k)T\right)=f(t)*g(t),$$

when $f(t)$ and $g(t)$ are zero for $t<0$

♣ The sum in eq. (16) need to be carried only from $k=0$ to $k=n$.

Let $h(t)=f(t)*g(t)=\sum_{k=0}^{\infty}f(kT)g\left((n-k)T\right)$, which defines $h(t)$ for $t=nT$. Now

$$\mathbb{Z}\left[h(t)\right]=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{\infty}f(kT)g\left((n-k)T\right)\right]z^{-n}$$

The inner sum needs to be carried out only as far as n. Taking $f(kT)=a_k$ and $g(jT)=b_j$, we have

$$\mathbb{Z}\left[h(t)\right]=\sum_{n-0}^{\infty}\sum_{k-0}^{n}a_k b_{n-k}z^{-n}. \qquad (17)$$

Now $\mathbb{Z}\left[f(t)\right]=\sum_{k=0}^{\infty}a_k z^{-k}=F(z)$ and $\mathbb{Z}\left[g(t)\right]=\sum_{j=0}^{\infty}b_j z^{-j}=G(z)$. Thus,

$$F(z)G(z)=\sum_{k=0}^{\infty}a_k z^{-k}\sum_{j=0}^{\infty}bjz^{-j}$$
$$=\left(a_0+a_1/z+a_2/z^2+\cdots\right)\left(b_0+b_1/z+b_2/z^2+\cdots\right)$$
$$=a_0b_0+\left(a_0b_1+a_1b_0\right)z^{-1}+\left(a_0b_2+a_1b_1+a_2b_0\right)z^{-2}+\cdots$$

Hence, we see that

$$F(z)G(z)=\sum_{n=0}^{\infty}c_n z^{-n}$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Comparing this series with Eq. (17), we have, finally,

$$\mathbb{Z}\left[h(t)\right] = \mathbb{Z}\left[f(t) * g(t)\right] = F(z)G(z). \qquad (18)$$

Thus,

the z transform of the convolution of two functions is the product of the z transform of each function,

and, conversely

the inverse z transform of the product of two functions is the convolution of the inverse transform of each function.

Example 8

Using the concept of convolution, find

$$\mathbb{Z}^{-1}\left[\frac{z^2}{\left(z-e^{aT}\right)\left(z-1\right)}\right] = h(nT).$$

<Sol.>

Rewriting the expression in the brackets and using the inverse of Eq. (18), we have

$$\mathbb{Z}^{-1}\left[\frac{z}{z-e^{aT}}\frac{z}{(z-1)}\right] = f(t) * g(t),$$

where

$$f(nT) = \mathbb{Z}^{-1}\left[\frac{z}{z-e^{aT}}\right] \quad \text{and} \quad g(nT) = \mathbb{Z}^{-1}\left[\frac{z}{z-1}\right].$$

Recalling that

$$\mathbb{Z}^{-1}\left[\frac{z}{(z-1)} f(t)\right] = u(t)$$

and

$$\mathbb{Z}^{-1}\left[\frac{z}{(z-e^{aT})}\right] = e^{at}u(t)$$

where $t = nT$ in both cases.

Taking $f(nT) = e^{anT}u(nT)$ and $g(nT) = u(nT)$ and performing their convolution, we get

$$h(nT) = \sum_{k=0}^{\infty} e^{akT} u\left((n-k)T\right).$$

Now $u\left((n-k)T\right) = 0$ for $k > n$ and $u\left((n-k)T\right) = 1$ for $n \geq k$. We can thus rewrite the preceding as

$$h(nT) = \sum_{k=0}^{n} e^{akT} = \sum_{k=0}^{n} \left(e^{aT}\right)^k.$$

Recalling that $\sum_{k=0}^{n} p^k = (1-p^{k+1})/(1-p)$, and taking $p = e^{aT}$, we have

$$h(nT) = \frac{1-e^{a(n+1)T}}{1-e^{aT}} = \mathbb{Z}^{-1}\left[\frac{z^2}{\left(z-e^{aT}\right)\left(z-1\right)}\right]$$

7. Difference Equation and the z Transform

Let $f(nT)$ be a function defined for $n = 0, 1, 2, \cdots$, and assume $T > 0$.

Find the closed-form of the solution of the equation

$$f\left((n+1)T\right) - 2f(nT) = 0.$$

given that $f(0) = 1$.

1) **Method I:**

Put $n=0$, $f(0)=1$ and obtain $f(T)=2$.

Then putting $n=1$, $f(T)=2$, we get $f(2T)=4$.

Continuing in this way, we find

$$f(nT)=2^n,\quad n=0,\ 1,\ 2,\ \cdots$$

2) **Method II: *z*-transform**

We perform a z transform on both sides of the given equation taking $t=nT$, $\mathbb{Z}[0]=0$, $2\mathbb{Z}[f(nT)]=2F(z)$. With $f(0)=1$, from the translation formula, we have

$$\mathbb{Z}\left[f\left((n+1)T\right)\right]=zF(z)-z$$

Thus, the transformed equation,

$$\mathbb{Z}\left[f\left((n+1)T\right)\right]-2\mathbb{Z}\left[f\left(nT\right)\right]=\mathbb{Z}[0]$$

becomes

$$zF(z)-z-2F(z)=0$$

Hence, we obtain

$$F(z)=\frac{z}{z-2}$$

To obtain $f(nT)$, we have

$$F(z)=\frac{1}{1-2/z}=1+\frac{2}{z}+\frac{4}{z^2}+\frac{8}{z^3}+\ldots=\sum_{n=0}^{\infty}f(nT)z^{-n}.$$

Thus,

$$f(nT)=2^n.$$

♣ **General form of the linear difference equation:**

$$a_0f(t+NT)+a_1f\left(t+(N-1)T\right)+a_2f\left(t+(N-2)T\right)+\cdots+a_Nf(t)=g(t).\qquad(19)$$

Here, $t=nT$, $n=0,\ 1,\ 2,\ \cdots$, and $g(t)$ must be defined for these values of *t*. *N* is an integer $\geq N$.

Example 9

The Fibonacci sequence of numbers was first described in the early thirteenth century by the Italian mathematician Leonardo Fabonacci (1170-1250). The sequence is 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, …. Each element of the sequence is the sum of the two preceding elements. Fabonacci described these numbers in the solution of a problem in the growth of a rabbit population. The numbers arise also in plant growth, puzzles, and in aesthetics. For $n\geq 0$, the *n*th element of the sequence, $f(n)$, satisfies the difference equation $f(n+2)=f(n+1)+f(n)$, or

$$f(n+2)-f(n+1)-f(n)=0\qquad(20)$$

The preceding is of the form shown in Eq. (19) if we take $T=1$, $N=2$, $a_0=1$, $a_1=-1$, $a_2=-1$. Note that $f(0)=0$, $f(1)=1$, $f(2)=1$, etc. Our problem is to find a closed-form solution of Eq. (20) by using *z* transform.

<Sol.>

Taking the *z* transform of Eq. (20), we have

$$\mathbb{Z}[f(n+2)]-\mathbb{Z}[f(n+1)]-\mathbb{Z}[f(n)]=0$$

With $T=1$, $f(0)=0$, $f(1)=1$, we obtain

$$\mathbb{Z}[f(n+1)]=zF(z)$$

and

$$\mathbb{Z}[f(n+2)]=z^2F(z)-z$$

Substituting these into our transformed equation, we have

$$z^2F(z)-z-zF(z)-F(z)=0$$

from which we obtain

$$F(z)=\frac{z}{z^2-z-1}$$

We expand the preceding in a Laurent series containing z to only nonpositive powers. Partial fraction are handy here. Thus,

$$F(z)=\frac{z}{z^2-z-1}=\frac{1}{\sqrt{5}}\left[\frac{(1+\sqrt{5})/2}{z-(1+\sqrt{5})/2}-\frac{(1-\sqrt{5})/2}{z-(1-\sqrt{5})/2}\right]$$

Each fraction can be expanded in negative powers of z, and we obtain

$$F(z)=\sum_{n=0}^{\infty}c_n z^{-n},\quad |z|>(1+\sqrt{5})/2$$

where

$$c_n=\frac{1}{\sqrt{5}2^n}\left[(1+\sqrt{5})^n-(1-\sqrt{5})^n\right]$$

Since $c_n=f(n)$, the problem is solved.

For example, the 20th Fabonacci number ($n=20$) is 6765.

8. **MATLAB and z Transform**
   1) Symbolic Mathematics Toolbox in MATLAB
   2) MATLAB functions: **ztrans** and **iztrans**

H.W. 1 Show that $\mathrm{Ln}\left(z/(z-1)\right)$ is analytic in a cut plane defined by the branch cut $y=0,\ \ 0\le x\le 1$. Expand this function in a Laurent series valid for $|z|>1$, and use your result to show that

$$\mathbb{Z}\left[\frac{T}{t}u(t-T)\right]=\mathrm{Ln}\left(z/(z-1)\right)$$

We define $u(t-T)/t=0$ when $t=0$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 12, Exercise 5.8, Pearson Education, Inc., 2005.】

H.W. 2 Show that

$$\mathbb{Z}\left[\sin(\alpha t)\right]=\frac{z\sin(\alpha T)}{z^2-2z\cos(\alpha T)+1},\quad |z|>1,\quad \alpha \text{ is real.}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 3, Exercise 5.8, Pearson Education, Inc., 2005.】

H.W. 3 (a) If $\mathbb{Z}\left[f(t)\right]=F(z)$, show that

$$\mathbb{Z}\left[e^{\beta t}f(t)\right]=F(ze^{-\beta T})$$

(b) Use the preceding result and the result of **H.W. 2** to show that

$$\mathbb{Z}\left[e^{\beta t}\sin(\alpha t)\right]=\frac{ze^{\beta T}\sin(\alpha T)}{z^2-2ze^{\beta T}\cos(\alpha T)+e^{\beta T}},\quad |z|>e^{\beta T},\quad \text{and } \alpha,\ \beta \text{ real}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 16, Exercise 5.8, Pearson Education, Inc., 2005.】

H.W. 4 If $\mathbb{Z}\left[f(t)\right]=F(z)$, where $F(z)$ is analytic for $|z|>R$, show that

$$f(nT)=\frac{1}{2\pi i}\oint_C F(z)z^{n-1}dz$$

where C is the circle $|z|=R_0,\quad R_0>R$. C can also be any closed contour into which $|z|=R_0$ can be deformed, by the principle of deformation of contours.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 17, Exercise 5.8, Pearson Education, Inc., 2005.】

H.W. 5 The gamma function, written $\Gamma(z)$, is an important analytic function of a complex variable and is treated at some length in next chapter. Here, as a prelude, we see its connection to the z transform.

(a) The gamma function is defined as $\Gamma(z) = \lim_{L\to\infty}\int_0^L u^{z-1}e^{-u}du$, commonly written $\int_0^\infty u^{z-1}e^{-u}du$.

Here u is a real variable, z a complex variable, and $u^{z-1} = e^{-(z-1)\mathrm{Ln}\,u}$. In the next chapter, we learn that $\Gamma(z)$ is analytic for $\mathrm{Re}\,z > 0$. Do an integration by parts to show that

$$\Gamma(z+1) = z\Gamma(z)$$

(b) Show that $\Gamma(1)=1$, $\Gamma(2)=1$, $\Gamma(3)=2$. Taking $n \ge 0$ as an integer, show by induction that

$$\Gamma(n+1) = n!$$

(c) Show that

$$\mathbb{Z}\left[1/\Gamma(t/T+1)\right] = e^{1/z}, \quad |z|>0.$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 19, Exercise 5.8, Pearson Education, Inc., 2005.】

H.W. 6 (a) Use the result derived in part (c) of **H.W. 5**, the transform shown in the previous lecture note, and Eq.(13) to show that

$$\mathbb{Z}\left[\frac{e^{at}}{\Gamma(t/T+1)}\right] = e^{e^{aT}/z}.$$

(b) Derive this same formula by using the results of part (a) in **H.W. 3** and part (c) in **H.W. 5**.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 20, Exercise 5.8, Pearson Education, Inc., 2005.】