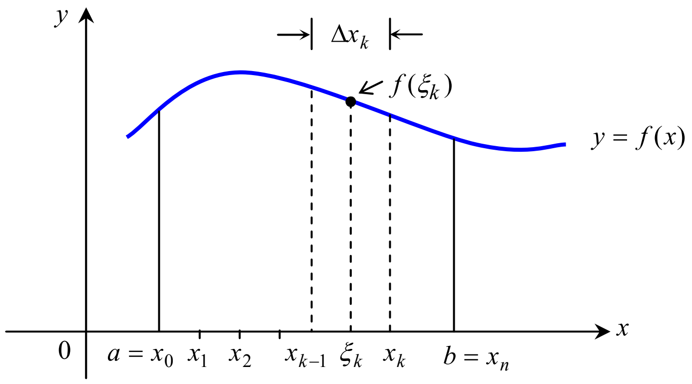
CHAPTER FOUR

Integration in the Complex Plane

§4-1 Contour Integration, Green Theorem and Cauchy's Integral Theorem

1. In the real variable analysis, the definite integral is defined as following：



$$\int_a^b f(x)dx = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} f(\xi_x)\Delta x_k$$

where $x_{k-1} \le \xi_k \le x_k$, $\Delta x_k = x_k - x_{k-1}$, and $\|\Delta\| = \max \Delta x_k$, $k = 1, 2, \cdots, n$.

複變函數之不定積分與實變函數所用之定義相同，因此在 R 區域內若有可解析之函數 $f(z)$ 和 $F(z)$ ，具有下列關係：

$$F'(z) = f(z)$$

則稱 $F(z)$ 為 $f(z)$ 之不定積分（或反導數）寫成

$$F(z) = \int f(z)dz$$

There are some basic properties of definite integral as shown below:

1) $\int_a^b [H(t) \pm G(t)]dt = \int_a^b H(t)dt \pm \int_a^b G(t)dt$

2) $\int_a^b kH(t)dt = k\int_a^b H(t)dt \qquad k = \text{constant}$

3) $\int_a^b H(t)dt = -\int_b^a H(t)dt$

4) $\int_a^b H(t)dt + \int_b^c H(t)dt = \int_a^c H(t)dt \qquad a < b < c$

In elementary calculus, the functions H and G were assumed to be real-value.
But the integral of a complex-valued function of a real variable is defined to be the integral of the real part of the function plus i times the integral of the imaginer part of the function.
That is, if

$$H(t) = H_1(t) + iH_2(t)$$

$$\Rightarrow \quad \int_a^b H(t)dt = \int_a^b H_1(t)dt + i\int_a^b H_2(t)dt$$

2. 求 $\int \sin 3z \cos 3z dz$

i) $\int \sin 3z \cos 3z dz = \frac{1}{3}\int \sin 3z \; d(\sin 3z)$

$$= \frac{1}{3}\sin^2 3z + c_1$$

ii) $\int \sin 3z \cos 3z dz = -\frac{1}{3}\int \cos 3z \, d(\cos 3z)$

$= -\frac{1}{6}\cos^2 3z + c_2$

因 $\sin^2 3z + \cos^2 3z = 1$ ，故 $\sin^2 3z$ 和 $\cos^2 3z$ 實際只差一常數而已。

3. Find $\int z^2 \sin 4z dz$

$\Rightarrow \quad \int z^2 \sin 4z dz$

$= z^2\left(-\frac{1}{4}\cos 4z\right) - (2z)\left(-\frac{1}{16}\sin 4z\right) + (2)\left(\frac{1}{64}\cos 4z\right) + c$

$= -\frac{1}{4}z^2 \cos 4z + \frac{1}{8}z \sin 4z + \frac{1}{32}\cos 4z + c$

寫出之原則：各項中前面括號內之量為自 z^2 開始，然後逐次微分之導數。
而各項後面括號內之量為 $\sin 4z$ 之逐次積分所得之量，並用正、負相間之符號，但常數項恒寫成正項。

4. Show $\int \frac{dz}{z^2 + a^2} = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1 = \frac{1}{2ai}\ln\left[\frac{z - ai}{z + ai}\right] + c_2$

i) Let $z = a\tan u \quad \Rightarrow \quad u = \tan^{-1}\frac{z}{a}$ and $dz = a\sec^2 u du$

$\Rightarrow \quad \int \frac{dz}{z^2 + a^2}$

$= \int \frac{a\sec^2 u du}{a^2(\tan^2 u + 1)} = \frac{1}{a}\int du$

$= \frac{1}{a}u + c_1 = \frac{1}{a}\tan^{-1}\frac{z}{a} + c_1$

ii) Since $\frac{1}{z^2 + a^2} = \frac{1}{(z + ia)(z - ia)}$

$= \frac{1}{2ai}\left[\frac{1}{z - ia} - \frac{1}{z + ia}\right]$

$\Rightarrow \quad \int \frac{dz}{z^2 + a^2} = \frac{1}{2ai}\left[\int \frac{dz}{z - ia} - \int \frac{dz}{z + ia}\right]$

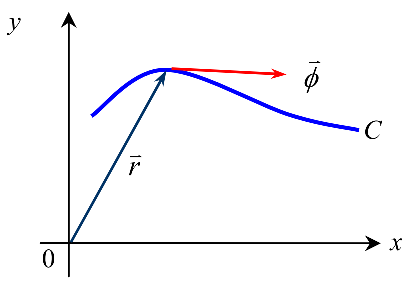
$= \frac{1}{2ai}\left[\ln(z - ia) - \ln(z + ia)\right] + c_2$

$= \frac{1}{2ai}\ln\left[\frac{z - ia}{z + ia}\right] + c_2$

5. Line Integrals (Contour Integral) of the Complex Function
There are two kinds of line integrals:

i) $\int_C f(x, y)dx \quad \Rightarrow$ 使用在 Scalar field

ii) $\int_C P(x, y)dx + Q(x, y)dy \quad \Rightarrow$ 使用在 Vector field



For example, let $\vec{\phi} = P\vec{i} + Q\vec{j}$

$\vec{r} = x\vec{i} + y\vec{j}$

$\Rightarrow \quad d\vec{r} = dx\vec{i} + dy\vec{j}$

$\Rightarrow \quad \vec{\phi} \cdot d\vec{r} = Pdx + Qdy$

In physics, the work is defined as following：

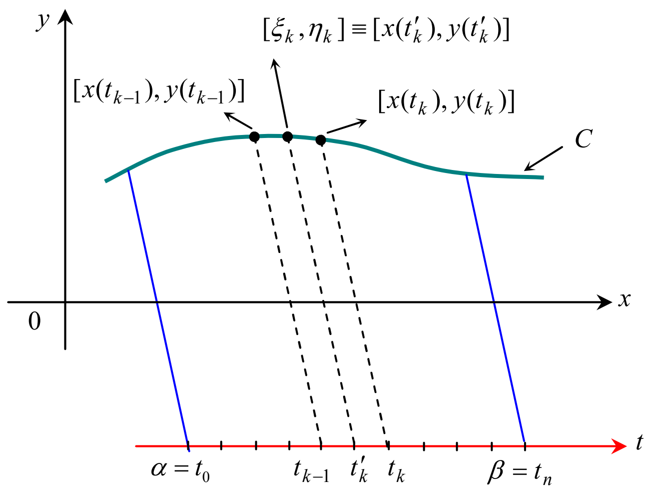
$$w=\int_C \vec{\phi}\cdot d\vec{r}=\int_C Pdx+Qdy$$

以下我們將分別討論此二類型之積分

6. $\int_C f(x,y)ds$ C：curve，s：arc length

參考下列所示之圖，知

若 C: $x=(t)$, $y=y(t)$, and $\alpha\le t\le\beta$



$\Rightarrow$ $\int_C f(x,y)ds$

$$\equiv\lim_{\|\Delta\|\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta s_k \quad ------- \quad (1)$$

Since $\Delta s=\dfrac{\Delta s}{\Delta t}\Delta t$

when $\Delta t\to 0$, we can obtain

$$\frac{\Delta s}{\Delta t}=\frac{ds}{dt}$$

and $\dfrac{ds}{dt}=\sqrt{\left(\dfrac{dx}{dt}\right)^2+\left(\dfrac{dy}{dt}\right)^2}$

So, equation (1) becomes

$$\int_C f(x,y)ds=\int_\alpha^\beta f[(t),y(t)]\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

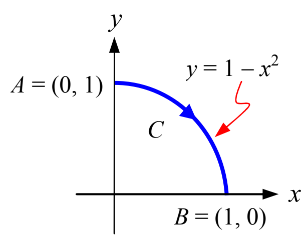
Example 1 Consider the contour shown in the following figure. Let $f(x,y)=xy$. Evaluate (a) $\int_A^B f(x,y)dx$, (b) $\int_A^B f(x,y)dy$, and (c) $\int_A^B f(x,y)ds$. **【本小題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 4.1(類似題), Problem 1, Pearson Education, Inc., 2005.】**

<Sol.>

(a)

$$\int_A^B f(x,y)dx=\int_{(0,1)}^{(1,0)} xydx=\int_{(0,1)}^{(1,0)} x(1-x^2)dx$$

$$=\left[\frac{x^2}{2}-\frac{x^4}{2}\right]_{x=0}^{x=1}=\frac{1}{4}$$



(b)

$$\int_A^B f(x,y)dy = \int_{(0,1)}^{(1,0)} y\sqrt{1-y}dy = -\frac{4}{15}$$

Alternative method for solving the above line integral:

On C, with $f(x,y) = xy$, $y = 1-x^2$, and $dy/dx = -2x$, we obtain

$$\int_A^B f(x,y)dy$$

$$= \int_A^B f(x,y)\frac{dy}{dx}dx = \int_0^1 (xy)(-2x)dx = \int_0^1 \left(-2x^2 + 2x^4\right)dx = -\frac{4}{15}$$

(c) Let C: $x = x(t) = t$, $y = y(t) = 1-t^2$, $1 \ge t \ge 0$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{1+(2t)^2}$$

Hence, we have

$$\int_A^B f(x,y)ds = \int_1^0 xy\left(\frac{ds}{dt}\right)dt = \int_1^0 t\left(1-t^2\right)\sqrt{1+4t^2}\,dt$$

Let $u = 1+4t^2$, we have $du = 8tdt$ and $t^2 = \frac{u-1}{4}$. Substituting these into the above integral, gives

$$\int_1^0 t\left(1-t^2\right)\sqrt{1+4t^2}\,dt$$

$$= \int_5^1 \frac{1}{8}\left(1-\tfrac{u-1}{4}\right)\sqrt{u}du = \frac{1}{32}\left[\frac{2}{5}u^{5/2} - \frac{5\times 2}{3}u^{3/2}\right]_1^5 = \frac{1}{32}\left[\frac{50}{5}\sqrt{5} - \frac{50}{3}\sqrt{5} - \frac{2}{5} + \frac{10}{3}\right]$$

$$= \frac{1}{32}\times\frac{1}{15}\left(150\sqrt{5} - 250\sqrt{5} + 44\right)$$

∗ 在此應注意：我們所取者為微弦長 Δz_k 才對，而非微弧長 Δs_k ——在複變函數中應是如此。

i) 若 $f(z) = u(x,y) + iv(x,y)$ ，其中 u，v 均為實數，則複變函數之線積分可寫成如下式所示：

$$\int_C f(x,y)ds = \int_C (u+iv)(dx+idy)$$

$$= \int_C [udx - vdy] + i\int_C [vdx + udy]$$

ii) 設 $f(z)$ 和 $g(z)$ 為沿著曲線 C 為可積分者，則有

a) $\int_C [f(z) \pm g(z)]dz = \int_C f(z)dz \pm \int_C g(z)dz$

b) $\int_C kf(z)dz = k\int_C f(z)dz$， k 為常數

c) $\int_a^b f(z)dz = -\int_b^a f(z)dz$ $\Rightarrow$ 指線積分之端點為 a 及 b 者，左右項積分路線為同一曲線

d) $\int_a^b f(z)dz = \int_a^c f(z)dz + \int_c^b f(z)dz$ $\Rightarrow$ c 點介於起點 a 與終點 b 之間

Example 2 (a) Compute $\int_{0+i}^{1+2i} \bar{z}^2 dz$ taken along the contour $y = x^2 + 1$ (**Fig. (a)** as shown below).

(b) Perform an integration like that in part (a) using the same integrand and limits, but take as a contour the piecewise smooth curve C shown in **Fig. (b)**.

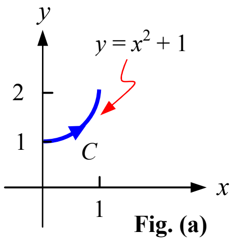


Fig. (a)

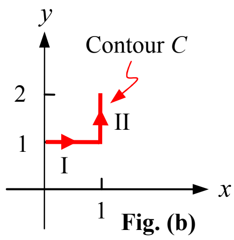


Fig. (b)

<Sol.>

(a) Put $f(z)=\bar{z}^2=(x-iy)^2=x^2-y^2-i2xy=u+iv$. Thus, with $u=x^2-y^2$, $v=-2xy$, we have

$$\int_{0+i}^{1+2i}\bar{z}^2dz=\int_{0+i}^{1+2i}\left(x^2-y^2\right)dx+\int_{0+i}^{1+2i}2xydy+i\int_{0+i}^{1+2i}(-2xy)dx+i\int_{0+i}^{1+2i}\left(x^2-y^2\right)dy$$
$$=\int_{0+i}^{1+2i}\left(x^2-y^2\right)dx+\int_{0+i}^{1+2i}2xydy+i\int_{0+i}^{1+2i}(-2xy)dx+i\int_{0+i}^{1+2i}\left(x^2-y^2\right)dy$$
$$=\int_0^1\left(x^2-\left(x^2+1\right)^2\right)dx+\int_1^2 2\sqrt{y-1}\,ydy$$
$$+i\int_0^1\left(-2x\left(x^2+1\right)\right)dx+i\int_1^2\left(\left(\sqrt{y-1}\right)^2-y^2\right)dy$$
$$=-\frac{23}{15}+\frac{32}{15}-i\frac{3}{2}-i\frac{11}{6}$$
$$=\frac{3}{5}-i\frac{10}{3}$$

(b) Along **path I**, we have $y=1$, so that $f(z)=\bar{z}^2=(x-i)^2=x^2-1-i2x=u+iv$. Thus, $u=x^2-1$, $v=-2x$ .Since $y=1$, $dy=0$. The limits of integration along **path I** are (0, 1) and (1, 1). Hence, we have

$$\int_I f(z)dz=\int_0^1\left(x^2-1\right)dx+i\int_0^1(-2x)dx=-\frac{2}{3}-i$$

Along **path II**, $x=1$, $dx=0$, so that $f(z)=\bar{z}^2=(1-iy)^2=1-y^2-i2y=u+iv$. Thus, $u=1-y^2$, $v=-2y$. The limits of integration along **path II** are (1, 1) and (1, 2). Hence, we have

$$\int_{II} f(z)dz=\int_1^2 2ydy+i\int_1^2\left(1-y^2\right)dy=3-i\frac{4}{3}$$

The value of the integral along C is obtained by summing the contributions from **I and II**. Thus,

$$\int_C \bar{z}^2dz=-\frac{2}{3}-i+3-\frac{4}{3}i=\frac{7}{3}-\frac{7}{3}i$$

H.W. 1 Evaluate $\int_C z^2dz$, where C is the parabolic arc $y=x^2$, $1\le x\le 2$. The direction of integration is from (1, 1) to (2, 4). **【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Example 3, Section 4.2, Pearson Education, Inc., 2005.】**

<Ans.> $\int_C z^2dz=-\frac{86}{3}-6i$

H.W. 2 Perform the following integrations:

(a) $\int_1^{-1}\frac{1}{z}dz$ along $|z|=1$, upper half plane

(b) $\int_1^{-1}\frac{1}{z}dz$ along $|z|=1$, lower half plane

(c) $\int_1^{i}\bar{z}^4\,dz$ along $|z|=1$, first quadrant

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 8-10, Exercise 4.2, Pearson Education, Inc., 2005.】

<Ans.> (a) $\int_1^{-1}\frac{1}{z}dz=\pi i$; (b) $\int_1^{-1}\frac{1}{z}dz=-\pi i$; (c) $\int_1^{i}\bar{z}^4\,dz=\frac{1-i}{3}$

H.W. 3 (a) Find a parametric representation of the shorter of the two arcs lying along $(x-1)^2+(y-1)^2=1$ that connects $z=1$ and $z=i$.

(b) Find $\int_1^i \bar{z}\,dz$ along the arc of (a), using the parametrization you have found.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 13, Exercise 4.2, Pearson Education, Inc., 2005.】

<Ans.> (a) $z = 1 + i + e^{it}$, $-\pi/2 \le t \le -\pi$; (b) $\int_1^i \bar{z}\,dz = i\left(2 - \frac{\pi}{2}\right)$

7. **ML Inequality:**

$\left|\int_C f(z)dz\right| \le ML$ ，其中 $|f(z)| \le M$ ，即 M 為曲線 C 上 $|f(z)|$ 之上界值，而 L 為曲線 C 之長度。

<pf.> 因 $\left|\sum_{K=1}^{n} z_K\right| \le \sum_{K=1}^{n} |z_K|$

故
$$\left|\sum_{K=1}^{n} f(\xi_K)\Delta z_K\right| \le \sum_{K=1}^{n} |f(\xi_K)\Delta z_K| = \sum_{K=1}^{n} |f(\xi_K)||\Delta z_K|$$

故有
$$\left|\int_C f(z)dz\right| \le \int_C |f(z)dz| = \int_C |f(z)||dz| = \int_C \sqrt{u^2+v^2}\sqrt{dx^2+dy^2} = \int_C \underbrace{\sqrt{u^2+v^2}}_{|f(z)|}\,ds$$

因此，若 $f(z)$ 在曲線 C 上為有界函數，亦即 $|f(z)| \le M$ ，而且曲線 C 的長度為 L 時，則有

$\left|\int_C f(z)dz\right| \le M\int_C ds = ML$ ，故得證。

1) 若 $f(z) \equiv k = \text{constant}$，並令 C 為連接 z_0 及 z_n 之任意曲線。利用線積分之原始定義，首先寫出

$$s_n = \sum_{k=1}^{n} k\Delta z_k = k[(z_1 - z_0) + (z_2 - z_1) + \cdots + (z_n - z_{n-1})] = k(z_n - z_0)$$

而得

$$\int_C k\,dz = \lim_{\Delta z_k \to 0} s_n = k(z_n - z_0)$$

可見線積分只與起點 z_0 及終點 z_n 有關，而與該線積分之路線無關。

若該曲線為封閉曲線則 $z_0 = z_n$ ，而有

$$\oint_C k\,dz = \lim_{\Delta z_k \to 0} s_n = k(z_n - z_0) = 0 \quad \text{or} \quad \oint_C dz = 0$$

2) 若 $f(z) \equiv z$ ，並令 C 為連接 z_n 及 z_0 之曲線。利用線積分之原始定義，首先寫出

$$s_n = \sum_{k=1}^{n} \xi_k \Delta z_k$$

若取 $\xi_k = z_k$ 及 $\xi_k = z_{k-1}$ ，得出

$$s_{n1} = \sum_{k=1}^{n} z_k \Delta z_k \quad \text{------------} \quad \xi_k = z_k \text{ 時}$$
$$= z_1(z_1 - z_0) + z_2(z_2 - z_1) + \cdots + z_n(z_n - z_{n-1})$$

同樣，

$$s_{n2}=\sum_{k=1}^{n} z_{k-1}\Delta z_k \quad \text{-------------} \quad \xi_k = z_{k-1} \text{ 時}$$

$$= z_0(z_1-z_0)+z_1(z_2-z_1)+\cdots+z_{n-1}(z_n-z_{n-1})$$

故 $s_{n1}+s_{n2}=z_n{}^2-z_0{}^2$

$\Rightarrow$ $\lim\limits_{n\to\infty}(s_{n1}+s_{n2})=2\int_C z dz = z_n{}^2-z_0{}^2$

因此，可得證

$$\int_{z_0}^{z_n} z dz=\frac{1}{2}\left(z_n{}^2-z_0{}^2\right)$$

故知積分只與起點與終點 z_0 ，z_n 有關，而與該線積分之路線無關。

若該曲線為閉合曲線，則 $z_0=z_n$ ，故

$$\oint_C z dz = 0$$

3) 有圓心在 z_0 ，半徑為 r 之圓 C，求 Contour Integral：

$$\oint_C \frac{dz}{(z-z_0)^{n+1}}$$ ，其中 n 為整數。

該積分路線 C: $z(\theta)=z-z_n=re^{i\theta}$

故 $dz = i re^{i\theta} d\theta$

因此，

$$\oint_C \frac{dz}{(z-z_0)^{n+1}}=\int_0^{2\pi}\frac{i re^{i\theta}d\theta}{[re^{i\theta}]^{n+1}} ,$$

$$=\frac{i}{r^n}\int_0^{2\pi} e^{-n\theta} d\theta$$

i) 若 $n=0$ ，可得

$$\oint_C \frac{dz}{z-z_0}=i\int_0^{2\pi} d\theta = 2\pi i$$

ii) 若 $n\neq 0$ ，則得

$$\oint_C \frac{dz}{(z-z_0)^{n+1}}=\frac{i}{r^n}\int_0^{2\pi} e^{-i\theta} d\theta=\frac{-i}{in\, r^n}e^{-i\theta}\Big|_0^{2\pi}=0$$

綜合上述之結果，可得今後經常使用之公式，如下所示。

a) $$\oint_C \frac{dz}{(z-z_0)^{n+1}}=\begin{cases}2\pi i, & n=0\\ 0, & n\neq 0 \text{ 之整數}\end{cases}$$

* 其中積分路線 C 為圓心在 z_0 ，半徑為 r 之圓，即

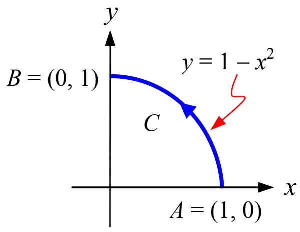
$$z-z_n=re^{i\theta}$$

b) $$\oint_C (z-z_0)^m dz=\begin{cases}2\pi i, & m=-1\\ 0, & m\neq -1 \text{ 之整數}\end{cases}$$

Example 3 Find an upper bound on the absolute value of $\int_{i+i0}^{0+i1} e^{1/z}dz$, where the integral is taken along the contour C, which is the quarter circle $|z|=1,\ 0\le \arg(z)\le \pi/2$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Example 4, Section 4.2, Pearson Education, Inc., 2005.】

<Sol.>



Let us first find M, an upper bound on $\left|e^{1/z}\right|$. We require that on C

$$\left|e^{1/z}\right|\le M \quad \text{-----------} \quad \text{(A)}$$

Notice that

$$e^{1/z}=e^{1/(x+iy)}=e^{x/(x^2+y^2)-iy/(x^2+y^2)}=e^{x/(x^2+y^2)}e^{i(-y)/(x^2+y^2)}$$

Hence

$$\left|e^{1/z}\right| = \left|e^{x/(x^2+y^2)}\right|\left|e^{i(-y)/(x^2+y^2)}\right| = e^{x/(x^2+y^2)}$$

Since $e^{x/(x^2+y^2)}$ is always positive, we can drop the magnitude sign on the right side of the preceding equation.
On contour C, $x^2+y^2=1$. Thus,

$$\left|e^{1/z}\right| = e^x \quad \text{on } C.$$

The maximum value achieved by e^x on the given quarter circle occurs when x is maximum, that is, at $x = 1$, $y = 0$.
On C, therefore, $e^x \le e$. Thus,

$$\left|e^{1/z}\right| = e \quad \text{on the given contour.}$$

A glance at Eq. (A) now shows that we can take M equal to e.
The length L of the path of integration is simply the circumference of the quarter circle, namely, $\pi/2$. Thus, applying the ML inequality,

$$\left|\int_{i+i0}^{0+i1} e^{1/z} dz\right| \le e\frac{\pi}{2}$$

H.W. 2 Consider $I = \int_1^i e^{i\,\mathrm{Ln}(\bar{z})} dz$ taken along the parabola $y = 1 - x^2$. Without doing the integration, show that $|I| \le 1.479 e^{\pi/2}$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 16, Exercise 4.2, Pearson Education, Inc., 2005.】

8. 複變函數之線積分的第二類型

$$\int_C P(x,y)dx + Q(x,y)dy$$

可改成 i) 定積分 ⇒ 非封閉曲線時使用之。

ii) 二重積分 ⇒ 封閉曲線時採用之。

a) 定積分型

C: $x = (t)$, $y = y(t)$, $\alpha \le t \le \beta$

$$\Rightarrow \int_C P(x,y)dx + Q(x,y)dy$$

$$= \int_\alpha^\beta \left\{P[x(t),y(t)]\frac{dx}{dt} + Q[x(t),y(t)]\frac{dy}{dt}\right\}dt$$

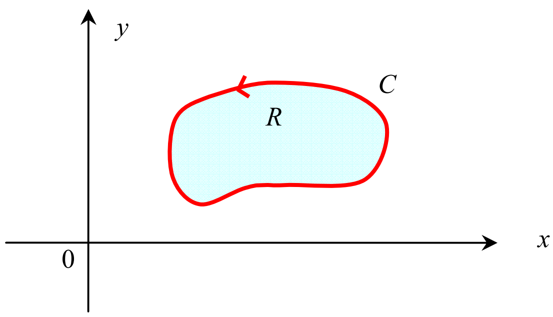
b) 二重積分型 = **Green's Theorem**

The conditions of Green's Theorem： If

1) C is the boundary of the region R.
2) $P(x,y)$, $Q(x,y)$ are continuous on C.
3) $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial p}{\partial y}$ are continuous in R.

$$\Rightarrow \int_C P(x,y)dx + Q(x,y)dy$$

$$= \int\!\!\int_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy$$



＊ 可改用複數表示法，如下所述：

設有在圍線 C 上及其區域 R 內之連續函數

$$B(z,\bar{z}) = P(x,y) + iQ(x,y)$$

故 $\oint_C B(z,\bar{z})dz$

$$= \oint_C (P+iQ)(dx+idy)$$

$$= \oint_C (Pdx - Qdy) + i\oint_C (Qdx + Pdy)$$

$$= -\int_R\int\left[\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right]dxdy + i\int_R\int\left[\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right]dxdy$$

$$= i\int_R\int\left\{\left[\frac{\partial P}{\partial x}-\frac{\partial Q}{\partial y}\right]+i\left[\frac{\partial P}{\partial y}+\frac{\partial Q}{\partial x}\right]\right\}dxdy$$

$$= 2i\int_R\int\frac{\partial B}{\partial z}dxdy$$

因此，可得 Green 氏平面定理之複數表示法：

$$\oint_C B(z,\bar{z})dz = 2i\int_R\int\frac{\partial B}{\partial z}dxdy$$

$$= 2i\int_R\int\frac{\partial B}{\partial z}ds \quad \text{------------} \quad (1)$$

其中，ds 代表微面積元素 $dxdy$。

＊ 若函數 $B(z,\bar{z})$ 及 $A(z,\bar{z})$ 在區域 R 及其周界 C 上為連續且具有連續之偏導數，則有另一型式之表示法：

$$\oint_C[B(z,\bar{z})dz + A(z,\bar{z})d\bar{z}] = 2i\int_R\int(\frac{\partial B}{\partial \bar{z}}-\frac{\partial A}{\partial z})ds \quad \text{-----------} \quad (4)$$

＊ 若 C 為簡單封閉曲線，所圍成之面積為 A，求證

① $A = \dfrac{1}{2i}\oint_C \bar{z}dz$

② $A = \dfrac{1}{4i}\oint_C(\bar{z}dz - zd\bar{z})$

<pf.> ① 利用式(1)，知

$$\oint_C \bar{z}dz = 2i\int_R\int\frac{\partial \bar{z}}{\partial \bar{z}}ds = 2i\int_R\int ds = 2iA$$

故知 $A = \dfrac{1}{2i}\oint_C \bar{z}dz$ ------------ 得證

② 利用式(4)，知

$$\oint_C(\bar{z}dz - zd\bar{z}) = 2i\int_R\int(\frac{\partial \bar{z}}{\partial \bar{z}}+\frac{\partial z}{\partial z})ds$$

$$= 4i\int_R\int ds = 4iA$$

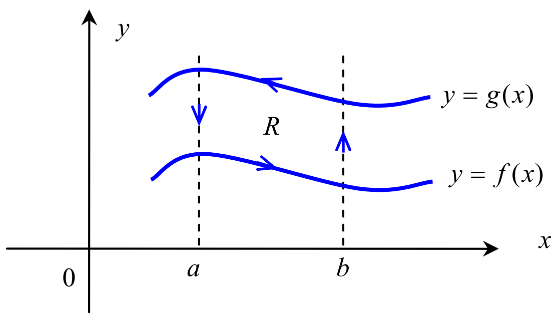
故知 $A = \dfrac{1}{4i}\oint_C(\bar{z}dz - zd\bar{z})$

＊ 如下圖所示，區域 R 之面積可由下列各方法求算之：

i) $A = \int_a^b[g(x)-f(x)]dx$

ii) $A = \int_R\int dxdy$ → Double Integral.

iii) $A = \oint_C xdy = -\oint_C ydx$ → Line Integral.



<pf.> 首先，令 $P=0$, $Q=x$

$\Rightarrow \quad \int_C xdy = \int_R\int dxdy = A$

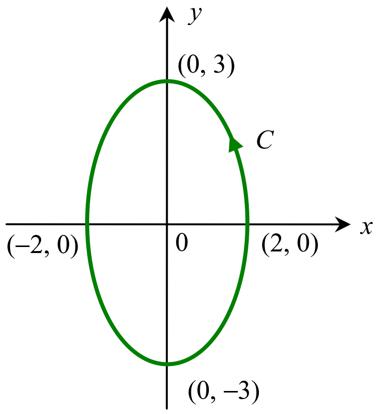
其次，取 $P=y$, $Q=0$

$\Rightarrow \quad \int_C ydx = -\int_R\int dxdy = -A$

$\Rightarrow \quad A = -\oint_C ydx$

＊ 此種方法最適合於 C 之 parameter form。

8. If $C: x=2\cos t,\quad y=3\sin t,\quad 0\le t\le 2\pi$,
find the area bounded by C.



1) 若用定積分，則

因 $9x^2+4y^2=36$

$$\Rightarrow \quad y=\frac{\sqrt{36-9x^2}}{2}$$

故 $$A=\int_{-2}^{2}2\cdot\frac{\sqrt{36-9x^2}}{2}dx$$
$$=\int_{-2}^{2}\sqrt{36-9x^2}\,dx$$
$$=6\pi$$

2) 若用二重積分法

$$A=\int_R\int dxdy$$

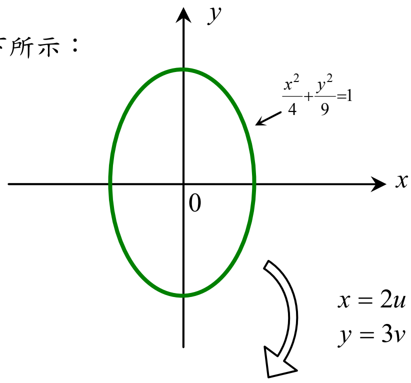
在此我們可以利用 Jacobian 積分來簡化問題，
其處理過程中所用之 Jacobian 積分變換公式如下所示：

When $x=x(u,v)$ ， $y=y(u,v)$

$$\Rightarrow \quad \int_{R_{xy}}\int f(x,y)dxdy$$
$$=\int_{R_{uv}}\int f[x(u,v),\,y(u,v)]\,|\,J\,|\,dudv$$

where

$$J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{vmatrix}$$



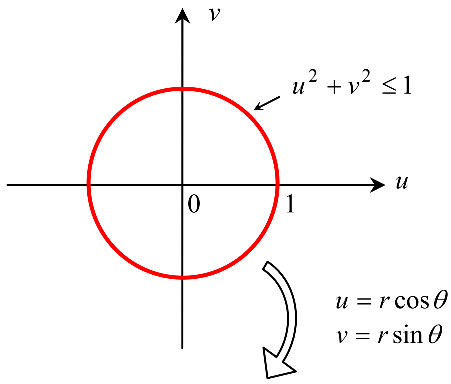
(見下圖)

因此，

$$A=\iint_{\frac{x^2}{4}+\frac{y^2}{9}=1}dxdy \qquad \text{---------} \qquad (1)$$

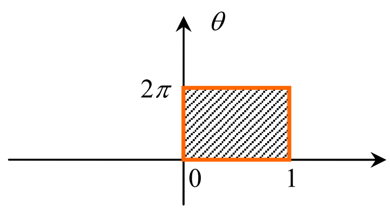
如圖所示：

$$J_1=\frac{\partial(x,y)}{\partial(u,v)} \qquad J_2=\frac{\partial(u,v)}{\partial(x,y)}$$
$$=\begin{vmatrix}2 & 0\\ 0 & 3\end{vmatrix} \quad \text{and} \quad =\begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix}$$
$$=6 \qquad\qquad =r$$



故(1)式 $\Rightarrow$

$$A=\iint_{u^2+v^2\le 1}6dudv$$
$$=\int_0^{2\pi}\int 6drd\theta=6\pi$$

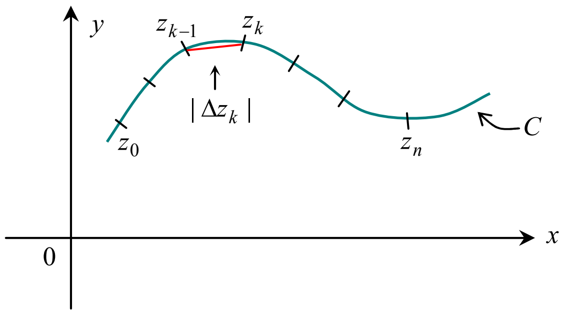


3) 若用 Line Integral 法

$C: x=2\cos t,\quad y=3\sin t,\quad 0\le t\le 2\pi$

$$\Rightarrow \quad A=\int_C xdy=\int_0^{2\pi}2\cos t\cdot 3\cos t\,dt$$
$$=6\int_0^{2\pi}\cos^2 t\,dt$$
$$=3\left(t+\frac{\sin 2t}{2}\right)\Bigg|_0^{2\pi}=6\pi$$

9. 由前面所述之定義知



$$\int_C f(z)dz = \lim_{\|\Delta\|\to 0} \sum_{k=1}^{n} f(\xi_x)\Delta z_k$$

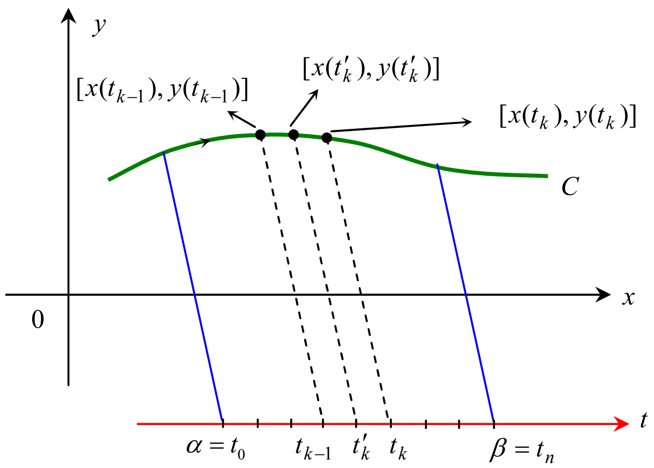
其中，$\Delta z_k = z_k - z_{k-1}$，且

$\|\Delta\| = \text{Max}\,|\Delta z_k|$，

則有下列之性質：

i) If $f(z)$ is continuous on C

$\Rightarrow \int_C f(z)dz$ exists



ii) $\int_a^b [f(t)+ig(t)]dt = \int_a^b f(t)dt + i\int_a^b g(t)dt$

iii) If $f(z)$ is continuous on C, and $z(t) = x(t) + i\,y(t)$, $\alpha \le t \le \beta$, and $x'^2(t) + y'^2(t) \ne 0$, (except for at finitely point), and $x'(t)$, $y'(t)$ are continuous on the interval $\alpha \le t \le \beta$

$$\Rightarrow \int_C f(z)dz = \int_\alpha^\beta f(z(t))z'(t)dt$$

此乃因 $dz = \dfrac{dz}{dt}\cdot dt = z'(t)dt$，且又因 $z'(t) = x'(t) + i\,y'(t)$，故知

$|z'(t)|^2 = x'^2(t) + i\,y'^2(t)$

亦即表示 $x'^2(t) + y'^2(t) \ne 0$，在此證明被省略。

10. **Cauchy-Goursat Theorem**

Let C be a simple closed contour and let $f(z)$ be a function that is analytic in the interior of C as well as on C itself. Then,

$$\oint_C f(z)dz = 0$$

♣ **Alternative statement of Cauchy-Goursat Theorem**

Let $f(z)$ be analytic in a simply connected domain D. Then, for any simple closed contour C in D, we have

$$\oint_C f(z)dz = 0\,.$$

1) $\oint_C z^n dz = 0,\quad n = 0,\ 1,\ 2,\ \cdots$, where C is any simple closed contour.

2) $\oint_C z^n dz = \begin{cases} 0, & n \ne -1 \\ 2\pi i, & n = -1 \end{cases}$, where C: $|z| = r$.

11. **Some Examples**

Example 1

Evaluate $\oint_C \frac{dz}{z-z_0}$, where $|z-z_0|=1$.

<Sol.> 解法 I:

$C: z(t)=z_0+\cos t+i\sin t, \quad 0\le t\le 2\pi$

$\Rightarrow \quad z'(t)=-\sin t+i\cos t$

故 $\oint_C \frac{1}{z-z_0}dz$

$=\int_0^{2\pi}\frac{1}{\cos t+i\sin t}\cdot(-\sin t+i\cos t)\,dt$

$=\int_0^{2\pi}\frac{i(\cos t+i\sin t)}{\cos t+i\sin t}dt=2\pi i$

解法 II:

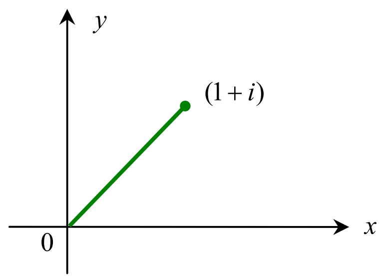
$C: z(t)=z_0+e^{it}, \quad 0\le t\le 2\pi$

$\Rightarrow \quad z'(t)=i\,e^{it}$

故 $\oint_C \frac{dz}{z-z_0}=\int_0^{2\pi}\frac{i\,e^{it}}{e^{it}}dt=\int_0^{2\pi} i\,dt=2\pi i$

Example 2

Find $\int_C \bar{z}\,dz$, where C is the line segment from 0 to $1+i$.



<Sol.> $C: z(t)=t+it, \quad 0\le t\le 1$

$\Rightarrow \quad z'=1+i$

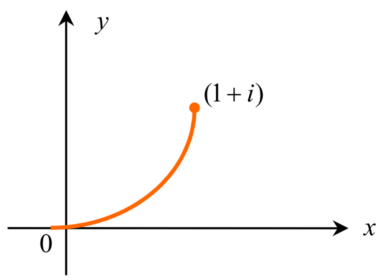
故知

$\int_C \bar{z}dz$

$=\int_0^1 (t-it)(1+i)dt=\int_0^1 2tdt=1$ #

Example 3

Find $\int_C |z|^2\,dz$, where $C: z=x+iy, \quad y=x^2$, z is from 0 to $1+i$.



<Sol.> $C: z(t)=t+it^2, \quad 0\le t\le 1$

故可知 $z'=1+i2t$

因此，

$\int_C |z|^2\,dz$

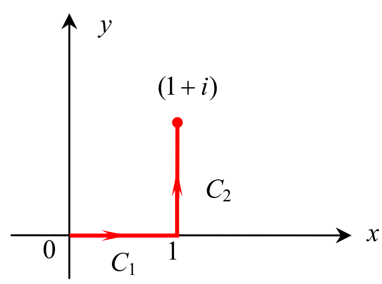
$=\int_0^1 (t^2+t^4)(1+i2t)dt$

$=\frac{8}{15}+\frac{5}{6}i$

Example 4

Find $\int_0^{1+2i}\text{Im}(z^2+1)\,dz$ where C is the polygonal line from

0 to 1 to $1+i$.



<Sol.>

C_1: $z_1(t)=t, \quad 0\le t\le 1$

C_2: $z_2(t)=1+i2t, \quad 0\le t\le 1$

Then, we have

$z_1'(t)=1$, and $z_2'(t)=2i$

So,

$$\int_0^{1+2i} \mathrm{Im}(z^2+1)\,dz$$

$$= \int_{C_1} \mathrm{Im}(z^2+1)\,dz + \int_{C_2} \mathrm{Im}(z^2+1)\,dz$$

$$= \int_0^1 \mathrm{Im}(t^2+1)\cdot 1\cdot dt + \int_0^{1+2i} \mathrm{Im}(1+4ti-4t^2+1)\cdot 2i\,dt$$

$$= 0 + \int_0^1 4t\cdot 2i\,dt$$

$$= 4i \quad \#$$

12. Conditions of Contour：

Contour ≡ piecewise regular curve.

If the contour C: $z(t) = x(t) + i\,y(t)$, $\alpha \le t \le \beta$, where the conditions of the contour C are described as following：

i) $x'(t)$, $y'(t)$ are continuous, and

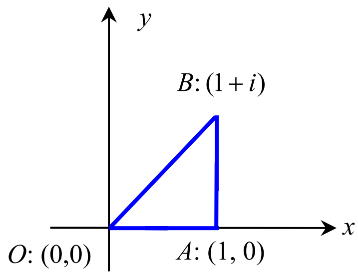
ii) $x'^2(t) + y'^2(t) \ne 0$, except for at $t=\alpha$, and $t=\beta$.

Here, let us find the values of the integral

$$\int_C z^2 dz$$

according to the integral paths shown below:

1) When the contour C is the contour OB from $z=0$ to $z=1+i$;
2) When the contour C is the contour OAB;
3) When the contour C is simple closed contour $OABO$.



<Sol.> 1) C∶ $z(t) = t + it$, $0 \le t \le 1$

We have $z'(t) = 1+i$

Then,

$$\int_C z^2 dz$$

$$= \int_0^1 (t+it)^2(1+i)\,dt$$

$$= \int_0^1 t^2(+i)^2(1+i)\,dt$$

$$= (-2+2i)\int_0^1 t^2\,dt$$

$$= \frac{1}{3}(-2+2i)$$

＊ 本例亦可由直接法求之：

$$\int_0^{1+i} z^2 dz = \frac{1}{3}z^3\Big|_0^{1+i} = \frac{1}{3}(-2+2i)$$

2) $$\int_C z^2 dz = \int_{OA} z^2 dz + \int_{AB} z^2 dz = -\frac{2}{3} + \frac{2}{3}i$$

3) $$\oint_C z^2 dz = \int_{OA+AB+BO} z^2 dz$$

$$= \int_{OA+AB} z^2 dz + \int_{BO} z^2 dz$$

$$= -\frac{2}{3} + \frac{2}{3}i - \int_{BO} z^2 dz = -\frac{2}{3} + \frac{2}{3}i - \frac{1}{3}(-2+2i)$$

Example 5

Find $\oint_C |z^2-1|^2 |dz|$, where the contour C: $|z| = 2$ described in the counterclockwise direction.

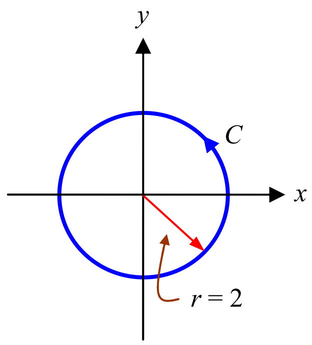
<Sol.> In this problem, $|dz|$ means ds, that is, s is the arc length.

Then, we have

$$\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

$$\Rightarrow \quad ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$

And let C: $z(t)=2(\cos t+i\sin t)$, $0\le t\le 2\pi$



Hence, we have

$$|z^2-1|=|4\cos^2 t+8i\sin t\cos t-4\sin^2 t-1|^2$$
$$=|4\cos 2t+i4\sin 2t-1|^2$$
$$=(4\cos 2t-1)^2+(4\sin 2t)^2$$
$$=17-8\cos 2t$$

and the arc length ds is

$$|dz|=ds=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt$$
$$=\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt$$
$$=2dt$$

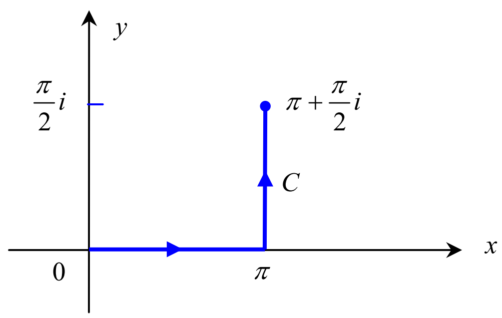
$$\Rightarrow \quad \oint_C |z^2-1|^2|dz|$$
$$=\int_0^{2\pi}(17-8\cos 25)\,dt$$
$$=68\pi \qquad \#$$

Example 6

Find the value of the integral

$$\int_C \cos z dz$$

where C is the polygonal line connecting, in the given order, the points $z=0$, $z=\pi$, and $z=\pi+\frac{\pi}{2}i$.



<Sol.> Since $f(z)=\cos z$ is analytic in the entire complex plane

$$\Rightarrow \quad \int_C \cos z dz$$
$$=\sin z\Big|_0^{\pi+\frac{\pi}{2}i}$$
$$=\sin(\pi+\frac{\pi}{2}i)$$
$$=-\sin\frac{\pi}{2}i=-\sinh\frac{\pi}{2}$$

§4-3 一些有關曲線，區域名稱之定義

【本節相關內容摘自：Dennis G. Zill and Patrick D. Shanahan, A First Course in *Complex Analysis with Applications*, Ch.1 and Ch. 4, Jones and Bartlett, Inc., 2003.】

1. **Definition of Piecewise Smooth Curve (Contour)**
   A *piecewise smooth curve* is a path made up of a finite number of smooth arcs connected end to end.

2. **Definition of Simple Closed Contour (Jordan Contour)**
   A *simple closed contour* is a contour that creates two domains, a bounded one and an unbounded one;

each domain has the contour for its boundary. The bounded domain is said to be the interior of the contour.

♣ The integration is said to be performed in the ***positive sense*** around the contour if the interior of the contour is on our **left** as we move along the contour in the direction of integration.

3. **Terminology**

Suppose a curve C: $z(t)=x(t)+iy(t)$, $a \le t \le b$, where $x(t)$ and $y(t)$ are continuous real function. Let the initial and terminal points of C be $A\big(x(a),\, y(a)\big)$ and $B\big(x(b),\, y(b)\big)$.

- ♣ We say a curve C in the complex plane is **smooth** if $z'(t)=x'(t)+iy'(t)$ is continuous and never zero in the interval $a \le t \le b$.
- ♣ A curve C in the complex plane is said to be a **simple** if $z(t_1) \ne z(t_2)$ for $t_1 \ne t_2$, except possibly for $t=a$ and $t=b$.
- ♣ C is a **closed curve** if $z(z)=z(b)$.

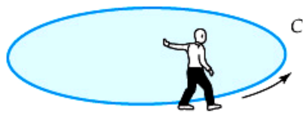
1) C is a *smooth curve* if x' and y' are continuous on the closed interval $[a,\, b]$ and simultaneously zero on the open interval $(a,\, b)$.

2) C is a *piecewise smooth curve* if it consists of a finite number of smooth curves $C_1, C_2, \ldots, C_n$ jointed end to end, that is, the terminal point of one curve C_k coinciding with the initial point of the next curve C_{k-1}.

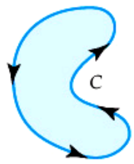
3) C is a *simple curve* if the curve C does not cross itself except possibly at $t=a$ and $t=(b)$.

4) C is a *closed curve* if $A = B$.

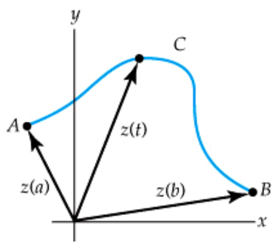
5) C *is simple closed* curve if the curve C does not cross itself and except possible $A = B$, hat is, C is simple and closed.

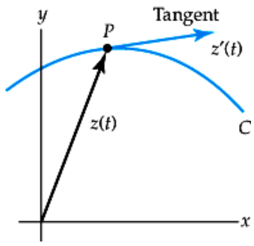


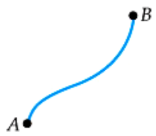
(a) Positive direction



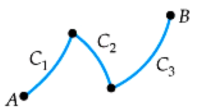
(b) Positive direction





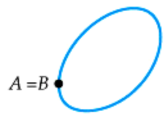


(a) Smooth curve and simple



(b) Piecewise smooth curve and simple

(c) Closed but not simple



(d) Simple closed curve

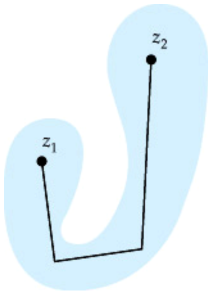
4. **Annulus, Domain, and Regions**

1) **Annulus** The set S_1 of points satisfying the inequality $\rho_1 < |z - z_0|$ lie exterior to the circle of radius ρ_1 centered at z_0, whereas the set S_2 of the points satisfying $|z - z_0| < \rho_2$ lies interior to the circle of radius ρ_2 centered at z_0. Thus, if $0 < \rho_1 < \rho_2$, the set of points satisfying the simultaneous inequality

$$\rho_1 < |z - z_0| < \rho_2 \quad \text{----------} \quad \text{(A)}$$

is the intersection of the sets S_1 and S_2. The set defined in (A) is called an open circular annulus.

2) **Domain** If any pair of points z_1 and z_2 in a set S can be connected by a polygonal line that consists of a finite number of line segments jointed end to end that lies entirely in the set, then the set S is said to be **connected**. An open connected set is called a **domain**.

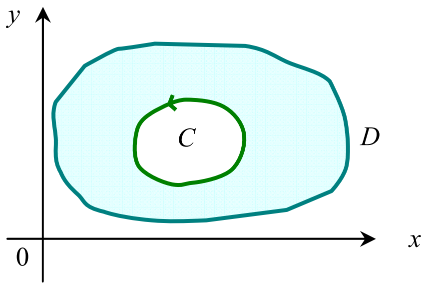


Connected set

3) **Regions** A region is a set of points in the complex plane with all, some or none of its boundary points. Since an ***open set*** does not contain any boundary points, it is automatically a **region**. A region that contains all its boundary points is said to be **closed**.

§4-4 Cauchy's Integral Theorem and Its Some Lemma

1. Cauchy's Integral Theorem



If 1) $f(z)$ is analytic in simply connected domain D.

2) C is a simple closed curve in D.

$\Rightarrow \oint_C f(z)dz = 0$

<pf.> 在此處，這個定理之證明並不是很嚴密：

Let $f(z) = u(x,y) + i\,v(x,y)$. Since $f(z)$ is analytic in D, then

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

在點 $(x,y) \in D$ 時均成立。

Furthermore,

$$\frac{\partial u}{\partial x},\ \frac{\partial v}{\partial y},\ \frac{\partial v}{\partial x},\ \frac{\partial u}{\partial y}$$

are continuous in D,

$$\begin{aligned}
\Rightarrow \oint_C f(z)dz &= \oint_C (u+iv)(dx+idy) \\
&= \oint_C [udx - vdy] + i\oint_C [vdx + udy] \\
&= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy - i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy \\
&= \iint_R 0\cdot dxdy + i\iint_R 0\cdot dxdy \\
&= 0
\end{aligned}$$

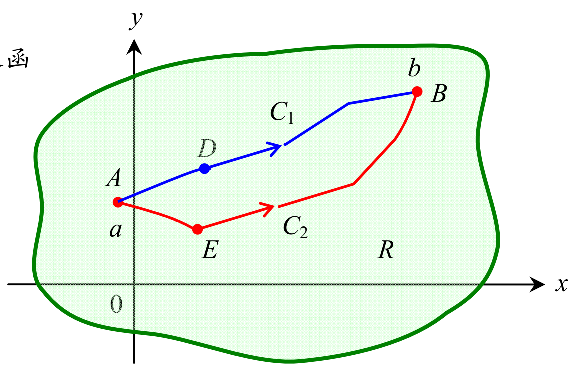
* 注意，此種使用 Green's Theorem 之證法，其逆證並不真確。

2. 由 Cauchy's Theorem 所得出之幾個之重要理論

1) 若 $f(z)$ 在簡連區域 R 內為單值且可解析之函數，則簡連區域 R 內兩點 a，b 之曲線 C 而作線積分

$$\int_C f(z)dz$$

與積分路線無關。



<pf.> From Cauchy's Theorem, we see that

$$\oint_{AEBDA} f(z)dz = 0$$

or $$\int_{AEB} f(z)dz + \int_{BDA} f(z)dz = 0$$

Hence,

$$\int_{AEB} f(z)dz = -\int_{BDA} f(z)dz = \int_{ADB} f(z)dz$$

So, we obtain

$$\int_{C_1} f(z)dz = \int_{C_2} f(z)dz$$

故知此種積分，只起點 a 及終點 b 有關，而與路線無關。

＊ 又 C_2 可認為 C_1 經由連續變形而成，只要在變形時不通過任意 $f(z)$ 不可解之點（奇異點）時，則上述推論仍屬成立，故此推論稱為路線變形原理。

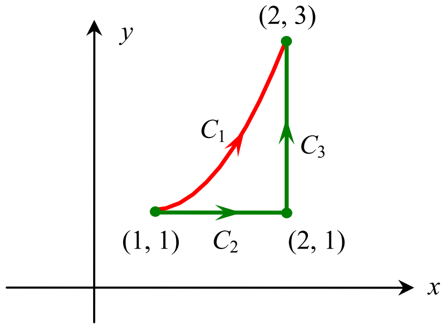
Example 1

求線積分

$$\int_C (12z^2 - 4iz)dz$$

其積分路線 C 為自(1, 1)至(2, 3)之曲線

$$z(t) = (t+1) + i(2t^2+1), \quad 0 \le t \le 1$$



<Sol.>

利用路線變形原理將 C_1 改由 C_2 和 C_3 所達成。

C_2：$z = x + i$

故知

$$dz = dx\,,\quad dy = 0$$

因此，

$$\int_{C_2} (12z^2 - 4iz)dz$$

$$= \int_1^2 [12(x+i)^2 - 4i(x+i)]dz$$

$$= 20 + 30i$$

而 C_3：$z = 2 + iy$

故知

$$dz = idy\,,\quad dx = 0$$

因此，

$$\int_{C_3} (12z^2 - 4iz)dz$$

$$= \int_1^3 [12(2+iy)^2 - 4i(2+iy)]idy$$

$$= -176 + 8i$$

所以，

$$\int_{C_1} (12z^2 - 4iz)dz$$

$$= \int_{C_2} + \int_{C_3} (12z^2 - 4iz)dz = -156 + 38i$$

2) 若 $f(z)$ 在 simple connected 區域 R 內為單值且可解析之函數，取 R 內兩點 a 及 z 作線積分

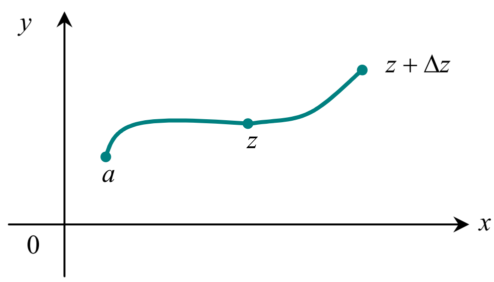
$$\int_C f(z)dz \quad \text{-----------} \quad (1)$$

則可寫出

$$F(z) \equiv \int_a^z f(\xi)d\xi \quad \text{------------} \quad (2)$$

此種寫法說明積分變數局限於所指定之積分路線，而 $F(z)$ 在該區域 R 內為 Analytic，且

$$F'(z) = f(z) \quad \text{-------------} \quad (3)$$



<pf.>

Since $f(z)$ is analytic, then the line integral

$$\int_a^z f(\xi)d\xi$$

與積分路徑無關，而純由起點與終點 a，z 所決定，a 為定點，z 為變點，則積分所得為 z 之函數，可寫成如(2)式。

則用(2)式可寫出

$$\frac{F(z+\Delta z)}{\Delta z}$$

$$=\frac{1}{\Delta z}\left[\int_a^{z+\Delta z} f(\xi)d\xi-\int_a^z f(\xi)d\xi\right]$$

$$=\frac{1}{\Delta z}\left[\int_a^z f(\xi)d\xi+\int_a^{z+\Delta z} f(\xi)d\xi-\int_a^z f(\xi)d\xi\right]$$

$$=\frac{1}{\Delta z}\int_z^{z+\Delta z} f(\xi)d\xi$$

$$=\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\xi)-f(z)+f(z)]d\xi$$

$$=\frac{1}{\Delta z}\left[f(z)\int_z^{z+\Delta z} d\xi\right]+\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\xi)-f(z)]d\xi]$$

但 $$\int_z^{z+\Delta z} d\xi=\Delta z$$

故 $$\frac{F(z+\Delta z)-F(z)}{\Delta z}$$

$$=f(z)+\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\xi)-f(z)]d\xi$$

若在路線 z 至 $z+\Delta z$ ，

$$M=|f(\xi)-f(z)|_{\max}$$

則

$$\left|\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\xi)-f(z)]d\xi\right|\le M$$

又因 $f(z)$ 具連續性，故當 $\Delta z\to 0$ ，$M\to 0$ 。

即

$$\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_z^{z+\Delta z} [f(\xi)-f(z)]d\xi=0$$

故

$$\lim_{\Delta z\to 0}\frac{1}{\Delta z}\int_z^{z+\Delta z}\ \frac{F(z+\Delta z)-F(z)}{\Delta z}=f(z)$$

與 $\Delta z\to 0$ 之方式無關。

故得證

$$F(z)=\int_a^{z+\Delta z} f(\xi)d\xi \quad \text{為解析函數，且} \quad F(z)=f(z) \qquad \#$$

Example 2

(a) Find the antiderivative of ze^z .

(b) Use the result of (a) to find $\int_i^z we^w dw$.

(c) Verify the above-mentioned theorem for the integral in part (b).

(d) Use the result of (a) to find $\int_i^1 ze^z dz$.

<Sol.>

(a) Using the integration by part, we have

$$\int ze^z dz=ze^z-e^z+C=F(z)$$

(b) Using the result of (b), we have

$$\int_i^z we^w d\,w = ze^z - e^z + C$$

We observe that the left side of this equation is zero when $z = i$. The right side will agree with the left at $z = i$ if we put $C = -ie^i + e^i$. Thus,

$$\int_i^z we^w d\,w = ze^z - e^z + -ie^i + e^i$$

(c) The above theorem asserts that

$$\frac{d}{dz}\left(\int_i^z we^w d\,w\right) = \frac{d}{dz}\left(ze^z - e^z + -ie^i + e^i\right) = ze^z$$

(d) With $F(z) = ze^z - e^z + C$, since $dF/dz = ze^z$ throughout the z-plane, we have

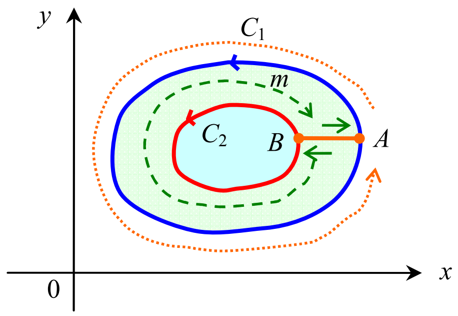
$$\int_i^1 ze^z d\,z = ze^z - e^z + C\Big|_i^1 = -ie^i + e^i$$

3) If $f(z)$ is analytic on C_1 and C_2, and the m is the annulus bounded by C_1 and C_2.

$\Rightarrow \oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$

<pf.> Let $C = C_1 + AB - C_2 + BA$

$\Rightarrow$ C is also a closed curve (see Figure).



From Cauchy's Integral Theorem, we see that

$$0 = \oint_C f(z)dz$$

$$= \oint_{C_1} f(z)dz + \int_{AB} f(z)dz - \oint_{C_2} f(z)dz + \int_{BA} f(z)dz$$

Hence, we obtained that

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

此一定理又可稱為圍線變形原理，擇要述之如下：

解析函數 $f(z)$ 沿任意封閉曲線 C_1 之圍線積分，與沿其他由 C_1 連續變形而得之封閉曲線 C_2 之圍線積分值完全相等，只要變形時並未通過 $f(z)$ 之異點者。

Example 3

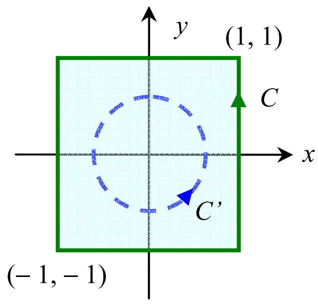
What is the value of $\oint_C dz/z$, where the contour C is the square shown in the figure?

<Sol.>

Using the principle of deformation of contours, we have

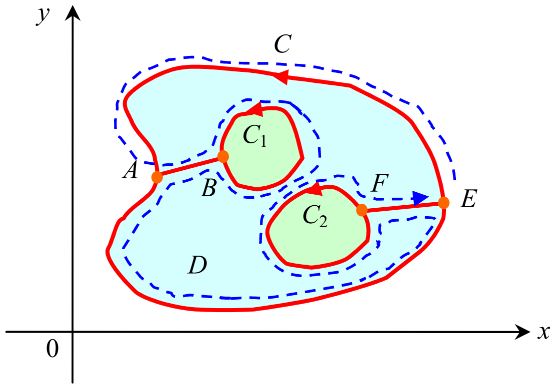
$$\oint_C dz/z$$

$$= \oint_{C'} dz/z = 2\pi i$$



4) If $f(z)$ is analytic on C, C_1, C_2, in the region R.

$\Rightarrow \quad \oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$



<pf.> Let $C' = C + AB - C_1 + BA + EF - C_2 + FE$

$\Rightarrow$ C' is a closed curve (see the above Figure)

From Cauchy's Integral Theorem, we see that

$$0 = \oint_{C'} f(z)dz$$

$$= \oint_C f(z)dz + \int_{AB} f(z)dz - \oint_{C_1} f(z)dz + \oint_{BA} f(z)dz$$

$$+ \oint_{EF} f(z)dz - \oint_{C_2} f(z)dz + \oint_{FE} f(z)dz$$

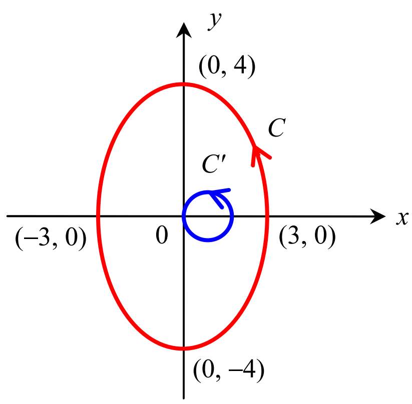
$\Rightarrow \quad \oint_C f(z)dz = \oint_{C_1} f(z)dz + \oint_{C_2} f(z)dz$

3. Some important Examples

Example 1

Evaluate $\oint_C \frac{dz}{z-1}$

where $C: z(t) = 3\cos t + i\sin t$, $0 \le t \le 2\pi$.



<Sol.> Let $C': z(t) = 1 + e^{it}$, $0 \le t \le 2\pi$

$$\Rightarrow \oint_C \frac{dz}{z-1}$$

$$= \oint_{C_1} \frac{dz}{z-1}$$

$$= \int_0^{2\pi} \frac{ie^{it}}{e^{it}} dt$$

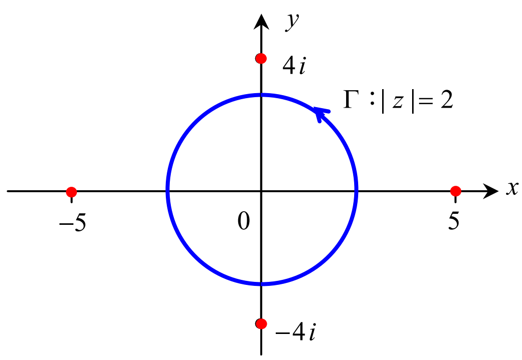
$$= 2\pi i$$

Example 2

Show that

$$\oint_\Gamma \frac{\cos z + \cosh \frac{z}{2}}{(z^2+16)(z^2-25)} dz = 0$$

where the simple closed contour $\Gamma : |z| = 2$.



<pf.>

Since the function

$$f(z)=\frac{\cos z+\cosh\frac{z}{2}}{(z^2+16)(z^2-25)}$$

is analytic for every point in the region of $|z|\le 2$,

$$\Rightarrow \oint_\Gamma \frac{\cos z+\cosh\frac{z}{2}}{(z^2+16)(z^2-25)}dz=0$$

Example 3

Show that $\oint_\Gamma \frac{\sinh z}{\cos^2 z}dz=0$

where the simple closed contour $\Gamma : |z|=1$.

<pf.> Since $\cos z=0$,

$$\Rightarrow \quad z=\frac{\pi}{2}+n\pi\,,\quad n=0,\ \pm1,\ \pm2,\ \cdots$$

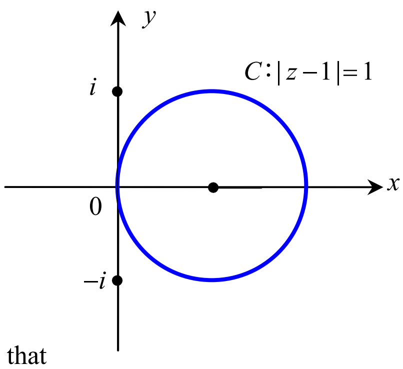
$\Rightarrow \quad f(z)=\frac{\sinh z}{\cos^2 z}$ is analytic on Γ and the inside of Γ.

By the Cauchy's Integral Theorem, we see that

$$\oint_\Gamma \frac{\sinh z}{\cos^2 z}dz=0$$

Example 4

Evaluate $\oint_C \frac{dz}{z+1}=?$ where $c : |z-1|=1$.



<Sol.> Since $f(z)=\frac{1}{z^2+1}$

is analytic in the region of $|z-1|\le 1$,

by the Cauchy's Integral Theorem we see that

$$\oint_C \frac{dz}{z^2+1}=0$$

Example 5

Evaluated $\oint_C \frac{z^2+4}{z}dz$, where $C : |z|=1$.

<Sol.> Since $C : z(t)=e^{it}$, $0\le t\le 2\pi$,

we have

$$\oint_C \frac{z^2+4}{z}dz$$

$$=\int_0^{2\pi}\frac{e^{i2t}+4}{e^{it}}\cdot i\,e^{it}\,dt$$

$$=i\int_0^{2\pi}(e^{i2t}+4)\,dt$$

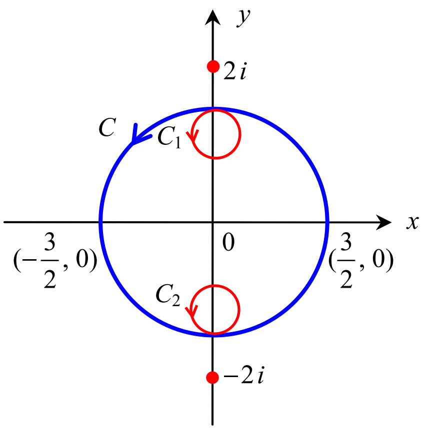
$$=i\left[\int_0^{2\pi}(\cos 2t+i\sin 2t+4)\,dt\right]$$

$$=i\left[\frac{\sin 2t}{2}\Big|_0^{2\pi}-i\frac{\cos 2t}{2}\Big|_0^{2\pi}+4t\Big|_0^{2\pi}\right]$$

$$=8\pi i$$

Example 6

Evaluate $\oint_C \frac{dz}{(z^2+1)(z^2+4)}$, where $C: |z| = \frac{3}{2}$.



<Sol.> Since

$$\oint_C \frac{dz}{(z^2+1)(z^2+4)} = \frac{1}{3}\oint_C \left[\frac{-1}{z^2+4} + \frac{1}{z^2+1}\right] dz = \frac{1}{3}\oint_C \frac{dz}{z^2+1} - \underbrace{\frac{1}{3}\oint_C \frac{dz}{z^2+4}}_{=0}$$

thus, we have

$$\frac{1}{3}\oint_C \frac{1}{z^2+1} = \frac{1}{3}\oint_{C_1} \frac{dz}{z^2+1} + \frac{1}{3}\oint_{C_2} \frac{dz}{z^2+1} = \frac{1}{3}\left[\oint_{C_1} \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz + \oint_{C_2} \frac{1}{2i}\left(\frac{1}{z-i} - \frac{1}{z+i}\right)dz\right]$$
$$= \frac{1}{6i}\left[\oint_{C_1} \frac{dz}{z-i} - \oint_{C_2} \frac{dz}{z+i}\right] = \frac{1}{6i}[2\pi i - 2\pi i] = 0$$

Hence,

$$\oint_C \frac{dz}{(z^2+1)(z^2+4)} = \frac{1}{3}\oint_C \frac{dz}{z^2+1} - \frac{1}{3}\oint_C \frac{dz}{z^2+4} = 0$$

H.W. 1 (a) Find $\oint_C e^z dz$, where $C: |z| = 1$.

(b) Show that $\int_0^{2\pi} e^{\cos\theta}\left[\cos(\sin\theta + \theta)\right] d\theta = 0$ and $\int_0^{2\pi} e^{\cos\theta}\left[\sin(\sin\theta + \theta)\right] d\theta = 0$

(c) Show that $\int_0^{2\pi} e^{\sin n\theta} \cos(\theta - \cos n\theta) d\theta = 0$ and $\int_0^{2\pi} e^{\sin n\theta} \sin(\theta - \cos n\theta) d\theta = 0$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 12-15, Section 4.3, Pearson Education, Inc., 2005.】

<Ans.> (a) $\oint_C e^z dz = 0$

H.W. 2 The contour is the square centered at the origin with the corners at $\pm(2 \pm 2i)$. Find

(a) $\oint_C \frac{dz}{(z-i)^4}$; (b) $\oint_C \frac{(z+1)^m}{z^m} dz$, where $m \geq 0$ is an integer; (c) $\oint_C \frac{z^m}{(z-1)^m} dz$, where $m \geq 0$ is an integer

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 19 and 21-22, Section 4.3, Pearson Education, Inc., 2005.】

<Ans.> (a) $\oint_C \frac{dz}{(z-i)^4} = -2\pi$; (b) $\oint_C \frac{(z+1)^m}{z^m} dz = 2\pi i m$; (c) $\oint_C \frac{z^m}{(z-1)^m} dz = 2\pi i m$

4. Special Topics

【本小節相關內容摘自：James Ward Brown and Ruel V. Churchill, *Complex Variable and Applications*, 7th ed., Sec. 79, McGraw-Hill, Inc., 2004.】

♣ A function $f(z)$ is said to be meromorphic in a domain *D* if it is analytic throughout *D* except for poles.

1) Basic concept:
Consider a function $f(z)$ that is analytic and nonzero everywhere on a simple closed contour *C*. Assume that $f(z)$ is analytic in the domain inside C, except possibly at a finite number of pole singularity.

2) Suppose we write

$$f(z) = |f(z)|e^{i(\arg f(z))} \quad \text{----------} \quad (1)$$

3) Define $\Delta_C \arg f(z) \equiv$ the increase in argument of $f(z)$ (final minus initial value) as the contour *C* is negotiated once in the positive sense.

Principle of the argument:

Suppose that (a) a function $f(z)$ is meromorphic in the domain interior to a positively oriented simple closed contour *C*;
(b) $f(z)$ is analytic and nonzero on *C*;
(c) counting multiplicities, *Z* is the number of zeros and *P* is the number of poles of $f(z)$ inside *C*.

Then

$$\frac{1}{2\pi}\Delta_C \arg f(z) \equiv Z - P = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$$

<pf.> Let $z = z(t)$, $a \le t \le b$ be a parametric representation for *C*, so that

$$\oint_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{f'[z(t)]z'(t)}{f[z(t)]}dt$$

Since, under the transformation $w = f(z)$, the image Γ of *C* never passes through the origin in the w-plane, the image of any point $z = z(t)$ on *C* can be expressed in exponential form as $w = \rho(t)\exp[i\phi(t)]$. Thus,

$$f[z(t)] = \rho(t)\exp[i\phi(t)], \quad a \le t \le b$$

$$\Rightarrow \quad f'[z(t)]z(t) = \frac{d}{dz}f[z(t)] = \frac{d}{dz}\{\rho(t)\exp[i\phi(t)]\} = \rho'(t)e^{i\phi(t)} + i\rho(t)e^{i\phi(t)}\phi'(t)$$

$$\Rightarrow \quad \oint_C \frac{f'(z)}{f(z)}dz = \int_a^b \frac{\rho'(t)}{\rho(t)}dt + i\int_a^b \phi'(t)\,dt = \ln\rho(t)\Big|_a^b + i\phi(t)\Big|_a^b$$

But

$$\rho(a) = \rho(b) \quad \text{and} \quad \phi(b) - \phi(a) = \Delta_C \arg f(z)$$

Hence

$$\oint_C \frac{f'(z)}{f(z)}dz = i\Delta_C \arg f(z) \quad \text{-----------} \quad (2)$$

If $f(z)$ ha s a zero of order m_0 at z_0, then

$$f(z) = (z - z_0)^m g(z)$$

where $g(z)$ is analytic and nonzero at z_0. Hence

$$f'(z_0) = m_0(z - z_0)^{m_0 - 1}g(z) + (z - z_0)^{m_0}g'(z)$$

$$\Rightarrow \quad \frac{f'(z)}{f(z)} = \frac{m_0}{z - z_0} + \frac{g'(z)}{g(z)} \quad \text{-----------} \quad (3)$$

Since $g'(z)/g(z)$ is analytic at z_0, it ahs a Taylor series representation about that point; Eq. (3) tell us that $f'(z)/f(z)$ has a simple pole at z_0 with residue m_0. If $f(z)$ ha s a pole of order m_p at z_0, then

$$f(z) = (z - z_0)^{-m_p}\phi(z) \quad \text{----------} \quad (4)$$

where $\phi(z)$ is analytic and nonzero at z_0. Similarly, in this case $f'(z)/f(z)$ has a simple pole at z_0 with residue $-m_p$. Applying the residue theorem, then, we find that

$$\oint_C \frac{f'(z)}{f(z)}dz = i(Z - P)$$

$$\Rightarrow \quad \frac{1}{2\pi}\Delta_C \arg f(z) \equiv Z - P = \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}dz$$

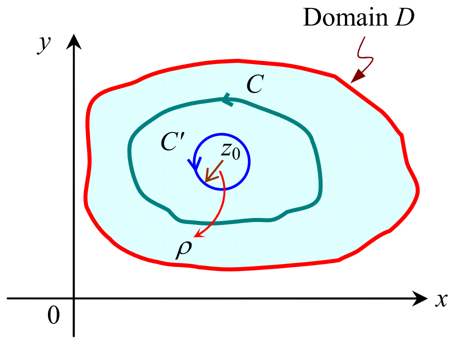
§4-5 Cauchy's Integral Formulas

1. ***Cauchy's Integral Formula***

If

1) $f(z)$ is analytic in simply connected domain D.
2) C is a simple closed curve in D.
3) z_0 is an interior point of C

$$\Rightarrow \quad \oint_C \frac{f(z)}{z - z_0} dz = 2\pi i\, f(z_0)$$



<pf.> Since

$$\frac{f(z)}{z - z_0} = \frac{f(z) - f(z_0)}{z - z_0} + \frac{f(z_0)}{z - z_0}$$

$$\Rightarrow \quad \oint_C \frac{f(z)}{z - z_0} dz$$

$$= \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz + \oint_C \frac{f(z_0)}{z - z_0} dz$$

$$= \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz + 2\pi i f(z_0)$$

Here, we need to prove that

$$\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0$$

Since $f(z)$ is analytic at $z = z_0$, then

$\Rightarrow$ $f(z)$ is continuous at $z = z_0$

$\Rightarrow$ For every $\epsilon > 0$, there exists $\delta > 0$ such that

$$|z - z_0| < \delta$$

We have

$$|f(z) - f(z_0)| < \epsilon$$

Then, we take a circle whose radius is ρ, and let

$$0 < \rho < \delta$$

Hence, $\Rightarrow$ $C': |z - z_0| = \rho$

$$\Rightarrow \quad \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$= \oint_{C'} \frac{f(z) - f(z_0)}{z - z_0} dz$$

$$\Rightarrow \quad \left| \oint_C \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$$= \left| \oint_{C'} \frac{f(z) - f(z_0)}{z - z_0} dz \right|$$

$$\le \frac{\epsilon}{\rho} \cdot 2\pi\rho = 2\pi\epsilon$$

Since ϵ is an arbitrary small constant, and $f(z)$ is continuous, we can take $\epsilon \to 0$, such that

$$\oint_C \frac{f(z) - f(z_0)}{z - z_0} dz = 0$$

Therefore, we obtain that

$$\oint_C \frac{f(z)}{z-z_0}dz$$

$$= \oint_C \frac{f(z)-f(z_0)}{z-z_0}dz + \oint_C \frac{f(z_0)}{z-z_0}dz$$

$$= 0 + 2\pi i\, f(z_0) = 2\pi i\, f(z_0)$$

That is,

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz \qquad \#$$

2. Some Examples

Example 1

Show that $\oint_C \frac{2\sinh^2 z + 3\cosh 3z}{z}dz = 6\pi i$,

where C is a simple closed contour having $z=0$ in its interior.

<pf.> Since $f(z) = 2\sinh^2 z + 3\cosh 3z$ is analytic in the entire complex plane, from Cauchy's integral formula we see that

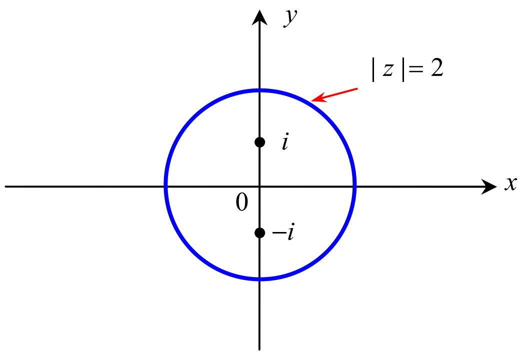
$$\oint_C \frac{2\sinh^2 z + 3\cosh 3z}{z}dz$$

$$= 2\pi i \cdot f(z_0) = 2\pi i \cdot 3 = 6\pi i$$

Example 2

Show that $\oint_C \frac{9z^2 - iz + 4}{z(z^2+1)}dz = 18\pi i$,

where $C: |z| = 2$.



<pf.>

Since

$$\frac{9z^2 - iz + 4}{z(z^2+1)} = \frac{9z^2 - iz + 4}{z(z+i)(z-i)}$$

$$= \frac{4}{z} + \frac{3}{z+i} + \frac{2}{z-i}$$

$$\Rightarrow \quad \oint_C \frac{9z^2 - iz + 4}{z(z^2+1)}dz$$

$$= \oint_C \frac{4}{z}dz + \oint_C \frac{3}{z+i}dz + \oint_C \frac{2}{z-i}dz$$

$$= (4+3+2)\cdot 2\pi i$$

$$= 18\pi i$$

*** Find $\mathscr{L}^{-1}\left\{\frac{3s+1}{(s-1)^2(s-2)}\right\}$

<Sol.> Since

$$\frac{3s+1}{(s-1)^2(s-2)}$$

$$= \frac{A}{s-1} + \frac{B}{(s-1)^2} + \frac{C}{s-2} \qquad \text{----------} \qquad (1)$$

$$\Rightarrow \quad B = \left[\frac{3s+1}{s-2}\right]\Bigg|_{s=1} = -4$$

$$C = \left[\frac{3s+1}{(s-1)^2}\right]\Bigg|_{s=2} = 7$$

but $A = \frac{d}{ds}\left[\frac{3s+1}{s-2}\right]\Big|_{s=1}$

$$= \frac{(s-2)\cdot 3-(3s+1)\cdot 1}{(s-2)^2}\Big|_{s=1}$$

$$= -7$$

Here, we can also use another method to find A. When we find A, we can multiply by $(s-1)$ in both sides of equation (1) and then let $s \to \infty$.

Hence,

$$\frac{3s+1}{(s-1)^2(s-2)}\Big|_{s\to\infty} = A + \frac{B}{s-1} + \frac{C(s-1)}{s-2}\Big|_{s\to\infty}$$

$\Rightarrow$ $A = -C = -7$

故 $\mathcal{L}^{-1}\left\{\frac{3s+1}{(s-1)^2(s-2)}\right\}$

$$= \mathcal{L}^{-1}\left\{\frac{-7}{s-1} + \frac{-4}{(s-1)^2} + \frac{7}{s-2}\right\}$$

$$= -7e^t - 4te^t + 7e^{+2t}$$

Example 3

Let $C: z = e^{i\theta}$ be the unit circle, where $-\pi \le \theta \le \pi$.

a) Show that

$$\oint_C \frac{e^{Kz}}{z} dz = 2\pi i \quad , \quad K \in R$$

b) Using the result of part (a), show that

$$\int_0^{\pi} e^{K\cos\theta} \cdot \cos(K\sin\theta) d\theta = \pi$$

<pf.>

a) Since $f(z) = e^{Kz}$ is analytic in the entire complex plane and $z = 0$ is an interior point of C.

By Cauchy's integral formula, we see that

$$\oint_C \frac{e^{Kz}}{z} dz = 2\pi i \cdot [e^{Kz}]\Big|_{z=0}$$

$$= 2\pi i$$

b) Since

$$2\pi i = \oint_C \frac{e^{Kz}}{z} dz$$

$$\Rightarrow \oint_C \frac{e^{Kz}}{z} dz = \int_{-\pi}^{\pi} \frac{e^{K\cdot e^{i\theta}}}{e^{i\theta}} \cdot ie^{i\theta} d\theta$$

$$= i\int_{-\pi}^{\pi} e^{K(\cos\theta + i\sin\theta)} d\theta$$

$$= i\int_{-\pi}^{\pi} e^{K\cdot\cos\theta}[\cos(K\sin\theta) + i\sin(K\sin\theta)] d\theta$$

$$= i\int_{-\pi}^{\pi} e^{K\cos\theta} \cdot \cos(K\sin\theta) d\theta - \int_{-\pi}^{\pi} e^{K\cos\theta} \cdot \sin(K\sin\theta) d\theta$$

Since $\sin(K\sin\theta)$ is an odd function, and $e^{K\cos\theta}$ is an even function, hence

$$e^{K\cos\theta} \cdot \sin(K\sin\theta)$$

is an odd function still.

Thus, we have

$$\int_{-\pi}^{\pi} e^{K\cos\theta} \cdot \sin(K\sin\theta) d\theta = 0$$

Then,

$$\Rightarrow \quad \int_{-\pi}^{\pi} e^{K\cos\theta}\cdot\cos(K\sin\theta)d\theta = 2\pi$$

i.e. $$2\int_{0}^{\pi} e^{K\cos\theta}\cdot\cos(K\sin\theta)d\theta = 2\pi$$

$$\Rightarrow \quad \int_{0}^{\pi} e^{K\cos\theta}\cdot\cos(K\sin\theta)d\theta = \pi$$

H.W.1 Evaluate $\frac{1}{2\pi i}\oint_C \frac{\mathrm{Ln}(z)}{z^2+9}dz$ around $|z-4i|=3$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 7, Exercise 4.5, Pearson Education, Inc., 2005.】

<Ans.> $\frac{1}{2\pi i}\oint_C \frac{\mathrm{Ln}(z)}{z^2+9}dz = \frac{\pi}{12} - i\frac{1}{6}\mathrm{Ln}\,3$

§4-6 Derivatives of Analytic Functions

1. If
   i) $f(z)$ is analytic in a simply connected Domain D.
   ii) C is a simple closed curve in D.
   iii) z_0 is an interior point of C.

$$\Rightarrow \quad f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

證明此定理時須使用到下述之定理，即

If $F(z) = \int_C f(z,\xi)d\xi \quad \Rightarrow \quad F'(z) = \int_C \left[\frac{\partial}{\partial z}f(z,\xi)\right]d\xi$

<pf.> Since $f(z)$ is analytic in a simply connected domain and z_0 is an interior point of C, where C is a simple closed curve in D.
By Cauchy's integral formula,

$$\Rightarrow \quad f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-z_0}dz$$

$$\Rightarrow \quad f'(z_0) = \frac{1}{2\pi i}\oint_C \frac{\partial}{\partial z_0}\left[\frac{f(z)}{z-z_0}\right]dz$$

$$= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}dz$$

$$\Rightarrow \quad f''(z_0) = \frac{2!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^3}dz$$

$$f'''(z_0) = \frac{3!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^4}dz$$

$$\vdots$$

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz$$

** 以上之證明缺乏數學上之嚴密性，只能算是一個說明而已，請參閱課本說明。

2. **Some Examples**

Example 1

Show that

$$\oint_C \frac{e^{3z}+3\cosh z}{\left(z-\frac{\pi}{2}i\right)^4}dz$$

where C is any simple closed contour containing $z_0=\frac{\pi}{2}i$ in its interior and the integral along C is taken in the positive direction.

<Sol.>

$$\oint_C \frac{e^{3z}+3\cosh z}{\left(z-\frac{\pi}{2}i\right)^4}dz$$

$$=\frac{2\pi i}{3!}\frac{d^3}{dz^3}\left[e^{3z}+3\cosh z\right]\Big|_{z=\frac{\pi}{2}i}$$

$$=\frac{\pi i}{3}\left[27e^{3z}+3\sinh z\right]\Big|_{z=\frac{\pi}{2}i}$$

$$=\frac{\pi i}{3}\left[27\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)+3\sinh\left(\frac{3\pi i}{2}\right)\right]$$

$$=\frac{\pi i}{3}\left[-27i+i3\sin\frac{\pi}{2}\right]$$

$$=\frac{\pi i}{3}(-24i)$$

$$=8\pi$$

H.W. 1 (a) Show that $\oint_C \frac{e^{az}}{z^{n+1}}dz=\frac{a^n 2\pi i}{n!}$, where C: $|z|=1$.

(b) Show that $\int_0^{2\pi} e^{a\cos\theta}\cos(a\sin\theta-n\theta)d\theta=2\pi a^n/n!$ and $\int_0^{2\pi} e^{a\cos\theta}\sin(a\sin\theta-n\theta)d\theta=0$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 15, Exercise 4.5, Pearson Education, Inc., 2005.】

H.W. 2 (a) If a is a real number and $|a|=1$ show that

$$\int_0^{2\pi}\frac{1-a\cos\theta}{1-2a\cos\theta+a^2}d\theta=2\pi$$

(b) Evaluate the above integral for the case $|a|>1$.

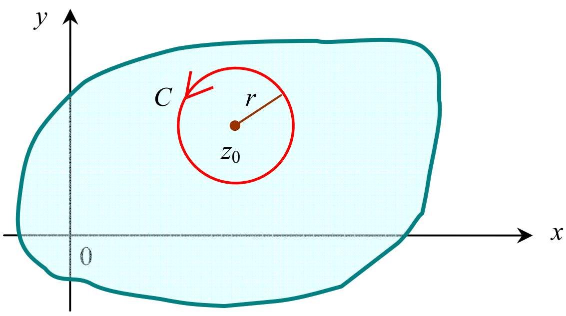
【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problem 17, Exercise 4.5, Pearson Education, Inc., 2005.】

3. Cauchy's Inequality For $f^{(n)}(z_0)$

Let C be a circle with center at z_0 and radius r, and let function f be analytic in an open set D containing C and its interior. Then, we have

$$f^{(n)}(z_0)\le\frac{Mn!}{r^n}\quad,\quad n=0,\ 1,\ 2,\cdots$$

where M is the least upper bound of $|f(z)|$.



<pf.>

The equation of the circle C is given by

$$C:|z-z_0|=re^{i\theta},\quad 0\le\theta\le 2\pi$$

Since $\quad f^{(n)}(z_0)=\frac{n!}{2\pi i}\oint_C\frac{f(z)}{(z-z_0)^{n+1}}dz$

Clearly, such a constant M (> 0) exist.

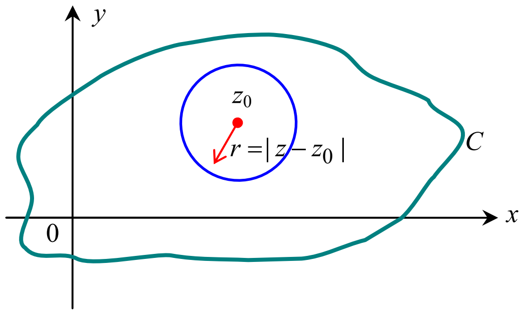
$$\Rightarrow \quad \left|f^{(n)}(z_0)\right| = \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$
$$= \frac{n!}{2\pi}\left|\oint_C \frac{f(z)}{(z-z_0)^{n+1}}dz\right|$$
$$\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\cdot 2\pi r$$
$$= \frac{Mn!}{r^n}$$

i.e. $\quad \left|f^{(n)}(z_0)\right| \le \dfrac{n!\,M}{r}$

4. Liouville's Theorem

If function f is analytic and bounded for all values of z in the complex plane (C), then f is a constant.

<pf.>



Since f is bounded $\forall\ z \in C$, there exists a constant $M > 0$ such that

$$|f(z)| \le M \qquad \forall\ z \in C$$

Let z_0 be any point in C, and let Γ be a circle with center z_0 and radius r.

Clearly, Γ and its interior are contained in C.

Since f is analytic $\forall\ z \in C$, then by Cauchy's Inequality for $f^{(n)}(z_0)$ with $n = 1$

$\Rightarrow \quad |f'(z_0)| \le \dfrac{M}{r}$

Let $r \to \infty$, then $|f'(z_0)| = 0$

$\Rightarrow \quad f'(z_0) = 0$

Since z_0 is any point in C (entire complex plane).

$\Rightarrow \quad f'(z_0) = 0$ for all z.

$\Rightarrow \quad f(z_0) = C$ is a constant.

5. Morera's Theorem

If $f(z)$ is continuous in a simply connected domain D, and if

$$\int_\Gamma f(z)dz = 0$$

for every closed contour Γ in D.

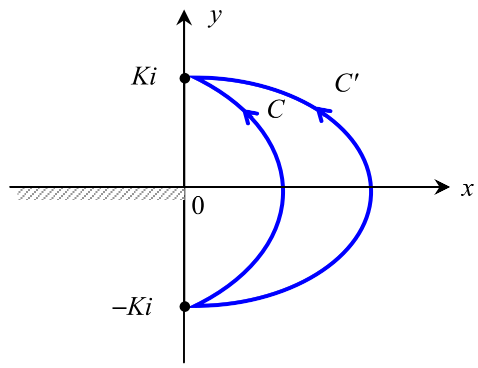
$\Rightarrow \quad f(z)$ is analytic in D.

6. **Some Examples**

Example 2

Find $\displaystyle\int_{-Ki}^{Ki} \frac{dz}{z}$ for any contour joining the points $-Ki$ to Ki $(K > 0)$ and lying in the domain of $D = C - \{z \mid z = x + 0i\ , x \le 0\}$.

<Sol.> **Method I:**



$$\int_{-Ki}^{Ki} \frac{dz}{z} = \operatorname{Ln} z \Big|_{-Ki}^{Ki}$$

$$= \operatorname{Ln} Ki - \operatorname{Ln}(Ki)$$

$$= \ln K + \frac{\pi}{2} i - \ln K + \frac{\pi}{2} i$$

$$= \pi i$$

Method II:

Since $f(z) = \dfrac{1}{z}$ is analytic in D,

$\Rightarrow$ $\displaystyle\int_{-Ki}^{Ki} \frac{dz}{z}$ is independent of the path, then we choose that

$$C': z(t) = K e^{it}, \quad -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

Hence, we have

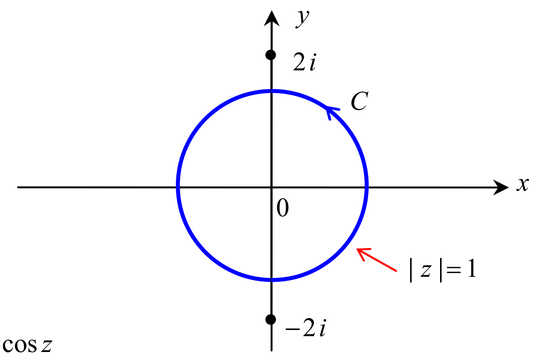
$$\int_{-Ki}^{Ki} \frac{1}{z} dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{K e^{it}} i e^{it} \cdot K dt$$

$$= i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dt = \pi i$$

Example 3

Evaluate $\displaystyle\oint_C \frac{\cos z}{z(z^2+4)} dz$, $C: |z| = 1$.

<Sol.>



Let $f(z) = \dfrac{\cos z}{z^2+4}$.

$\Rightarrow$ $f(z)$ is analytic on C and interior of C, then by Cauchy's Integral Formula, we obtain that

$$\oint_C \frac{\cos z}{z(z^2+4)} dz$$

$$= 2\pi i \cdot f(0)$$

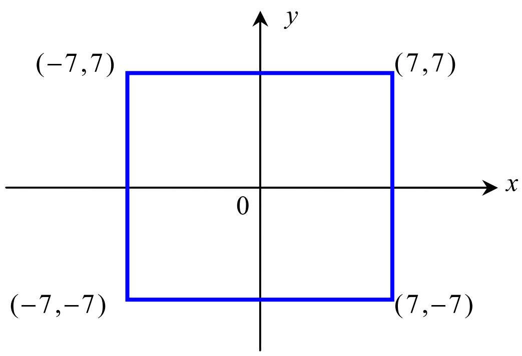
$$= 2\pi i \left[\frac{\cos z}{z(z^2+4)} \right] \Bigg|_{z=0}$$

$$= \frac{\pi}{2} i$$

Example 4

Find $\displaystyle\frac{1}{2\pi i} \int_\Gamma \frac{a^n e^{az}}{n!(z)^{n+1}} dz$, $a \in C$.

The given simple closed contour Γ is taken as the boundary of a square whose sides lies along the line $x = \pm 7$ and $y = \pm 7$, and is described in the positive direction.



<Sol.>

$$\frac{1}{2\pi i}\oint_{\Gamma}\frac{a^n e^{az}}{n!(z)^{n+1}}dz$$

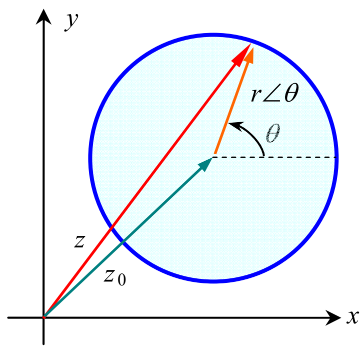
$$=\frac{1}{2\pi i}\cdot\frac{2\pi i}{n!}\cdot\frac{a^n}{n!}\cdot\frac{dn}{dz^n}\left(e^{az}\right)\Big|_{z=0}$$

$$=\frac{a^{2n}}{(n!)^2}$$

7. **Gauss' Mean Value Theorem**

Let $f(z)$ be analytic in a simply connected domain. Consider any circle lying in this domain. The value assumed by $f(z)$ at the center of the circle equal s the average of the values assumed by $f(z)$ on its circumference. If z_0 is the center of the circle and *r* its radius, this is equivalent to

$$f(z_o)=\frac{1}{2\pi}\int_0^{2\pi} f(z_o+re^{i\theta})d\theta.$$



<pf.> C: $z=z_0+re^{i\theta}$, $0\le\theta\le 2\pi$ $\Rightarrow$ $dz=re^{i\theta}d\theta$

From Cauchy Integral Theorem, we have

$$f(z_0)=\frac{1}{2\pi i}\oint_C\frac{f(z)}{z-z_0}dz$$

$$\Rightarrow\quad f(z_o)=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f(z_o+re^{i\theta})}{re^{i\theta}}ire^{i\theta}d\theta$$

$$\Rightarrow\quad f(z_0)=\frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})d\theta$$

The expression on the right of the above Equation is the arithmetic mean (average value) of $f(z)$ on the circumference of the circle.

If let $f(z)=u(x,y)+iv(x,y)$, it follows

$$u(z_o)+iv(z_o)=\frac{1}{2\pi}\int_0^{2\pi}u(z_o+re^{i\theta})d\theta+\frac{i}{2\pi}\int_0^{2\pi}v(z_o+re^{i\theta})d\theta$$

Taking $z_0=x_0+iy_0$ and equating corresponding parts of each side of above equation, we obtain

$$u(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi}u(z_0+re^{i\theta})d\theta.$$

$$v(x_0,y_0)=\frac{1}{2\pi}\int_0^{2\pi}v(z_0+re^{i\theta})d\theta.$$

Example 5 Using Gauss' mean value theorem, evaluate

$$\int_0^{2\pi}\cos(\cos\theta+i\sin\theta)d\theta$$

By identifying the real and imaginary parts of the integrand, what identities are obtained?

<Sol.>

Since $C:\ z=\cos\theta+i\sin\theta$ is a unit circle, we can take $z_0=1$ and $r=1$. Then, we have

$$\frac{1}{2\pi}\int_0^{2\pi}\cos(e^{i\theta})d\theta=\cos z\big|_{z=0}$$

$$\Rightarrow\qquad \int_0^{2\pi}\cos(\cos\theta+i\sin\theta)d\theta=2\pi$$

Since $\cos z=\cos x\cosh y-i\sin x\sinh y$, the last equation can be rewritten as

$$\int_0^{2\pi}\left[\cos(\cos\theta)\cosh(\sin\theta)-i\sin(\cos\theta)\sinh(\sin\theta)\right]d\theta=2\pi$$

Equating corresponding parts (real and imaginary) on both sides of the equation, we have

$$\int_0^{2\pi}\cos(\cos\theta)\cosh(\sin\theta)d\theta=2\pi$$

and $$\int_0^{2\pi}\sin(\cos\theta)\sinh(\sin\theta)d\theta=0$$

Also, we have

$$\int_{-\pi}^{\pi}\cos(\cos\theta)\cosh(\sin\theta)d\theta=2\pi$$

and $$\int_{-\pi}^{\pi}\sin(\cos\theta)\sinh(\sin\theta)d\theta=0$$

The last result can be checked if we verify the odd symmetry of the integrand.

H.W. 3 Using gauss' mean value theorem, prove the following:

(a) $\int_{-\pi}^{\pi}e^{\cos\theta}\cos(\sin\theta)d\theta=2\pi$

(b) $\int_{-\pi}^{\pi}\frac{a+\cos n\theta}{a^2+1+2a\cos n\theta}d\theta=\frac{2\pi}{a}$, where $a>1$, n integer.

(c) $\int_0^{2\pi}\mathrm{Ln}\left[a^2+1+2a\cos(n\theta)\right]d\theta=4\pi\,\mathrm{Ln}\,a$, where $a>1$, n integer.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 2 and 4-5 , Exercise 4.6, Pearson Education, Inc., 2005.】

<Hint> (a) Using $\frac{1}{2\pi}\int_0^{2\pi}e^{e^{i\theta}}d\theta=1$; (b) Using $f(z)=\frac{1}{z^n+a}$; (c) Using $a^2+1+2a\cos(n\theta)=\left|a+e^{in\theta}\right|^2$

H.W. 4 In this problem, we derive one of the four Wallis formulas. They allow one to evaluate

$$\int_0^{\pi/2}\left[f(\theta)\right]^m d\theta$$

where $m\ge 0$ is an integer and $f(\theta)=\sin\theta$ or $\cos\theta$. The cases of odd and even m must be considered separately. We will consider m even.

(a) Using the binomial theorem, show that

$$z^{-1}\left(z+\tfrac{1}{z}\right)^{2n}=\sum_{k=0}^{2n}\frac{(2n)!\,z^{2n-2k-1}}{(2n-k)!\,k!},\quad n=0,\ 1,\ 2,\ \cdots$$

(b) Using the above result, a term-by-term integration, and the extended Cauchy integral formula (i.e., **Derivatives of Analytic Functions**), show that

$$\oint z^{-1}\left(z+\tfrac{1}{z}\right)^{2n}dz=2\pi i\frac{(2n)!}{(n!)^2}$$

where the integration is around $|z|=1$.

(c) With $z=e^{i\theta}$, $0\le\theta\le 2\pi$, on the unit circle, show from (b) that

$$\int_0^{2\pi}(2\cos\theta)^{2n}d\theta=2\pi\frac{(2n)!}{(n!)^2}$$

(d) Noting the symmetry of $\cos\theta$, and that $2n$ is even ($n=0,\ 1,\ 2,\ \cdots$), explain why

$$\int_0^{\pi/2}(\cos\theta)^{2n}d\theta=\frac{\pi}{2}\frac{(2n)!}{(n!)^2 2^n}$$

This is one of Wallis's formulas.

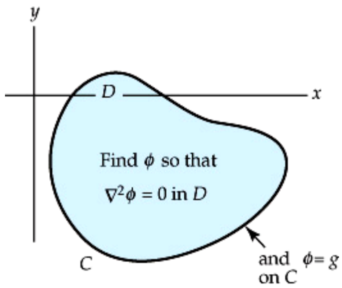
(e) Find $\int_0^{\pi/2} (\sin\theta)^{2n} d\theta$, where $n = 0,\ 1,\ 2,\ \cdots$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 17 , Exercise 4.6, Pearson Education, Inc., 2005.】

§4-7 Introduction to Dirichlet Problems: The Poisson Integral Formula for the Circle and Half Plane

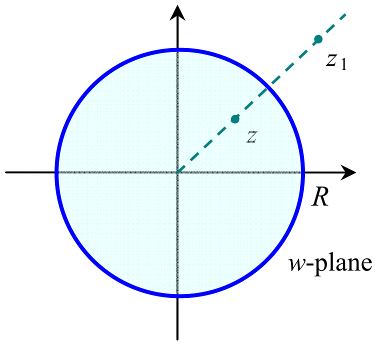
♣ **Dirichlet Problem**

Suppose that D is a domain in the plane and that g is a function defined on the boundary C of D. The problem of finding a function $\phi(x, y)$, which satisfies Laplace's equation in D and which equals g on the boundary C of D is called a **Dirichlet problem**.



1. **The Dirichlet Problem for a Circle**
   1) Consider a circle of radius R whose center lies at the origin of the complex w-plane.
   2) Let $f(w)$ be a function that is analytic on and throughout the interior of this circle.
   3) Cauchy Integral formula:

$$f(z) = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z}dw. \quad \text{------------} \quad (1)$$



Suppose $f(z) = U(x, y) + iV(x, y)$. We would like to use the preceding integral to obtain the explicit expressions for U and V.

Define the point z_1 as $z_1 = R^2/\bar{z}$. We have

$$|z_1| = \frac{R^2}{|\bar{z}|} = \frac{R}{|z|}R.$$

Since $|z| < R$ (z is an arbitrary interior point of the circle), we have $|z_1| > R$ (exterior point). However, $\arg z_1 = \arg z$. The function $f(w)/(w - z_1)$ is analytic in the w-plane on and inside the given circle. Hence, from the Cauchy integral formula, we have

$$0 = \frac{1}{2\pi i}\oint \frac{f(w)}{w-z_1}dw = \frac{1}{2\pi i}\oint \frac{f(w)}{w-(R^2/\bar{z})}dw \quad \text{------------} \quad (2)$$

Subtracting Eq. (2) from Eq. (1), we obtain

$$f(z) = \frac{1}{2\pi i}\oint f(w)\left[\frac{1}{w-z} - \frac{1}{w-(R^2/\bar{z})}\right]dw$$

$$= \frac{1}{2\pi i}\oint f(w)\left[\frac{z-(R^2/\bar{z})}{(w-z)\left(w-(R^2/\bar{z})\right)}\right]dw. \quad \text{------------} \quad (3)$$

Changing to polar coordinate, let $w = Re^{i\phi}$ and $z = re^{i\theta}$. Thus, $\bar{z} = re^{-i\theta}$. Along the path of integration, $dw = Re^{i\phi}d\phi$ and $0 \le \phi \le 2\pi$. Rewriting the right side of Eq. (3), we have

$$f(r,\theta) = \frac{1}{2\pi i}\int_0^{2\pi} f(R,\phi)\left[\frac{re^{i\theta} - \left(R^2/re^{-i\theta}\right)}{(Re^{i\phi} - re^{i\theta})\left(Re^{i\phi} - \left(R^2/re^{-i\theta}\right)\right)}\right]Re^{i\phi}id\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(R,\phi)\left[\frac{\left(re^{i\theta} - \left(R^2e^{i\theta}/r\right)\right)Re^{i\phi}}{(Re^{i\phi} - re^{i\theta})\left(Re^{i\phi} - \left(R^2e^{i\theta}/r\right)\right)}\right]d\phi$$

If we multiply the two terms in the denominator of the preceding integral together and then multiply numerator and denominator by $(-r/R)e^{-i(\theta+\phi)}$, we can show, with the aid of Euler's identity, that

$$f(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{f(R,\phi)(R^2-r^2)d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)}. \quad \text{------------} \quad (4)$$

Since $f(R,\phi)=U(R,\phi)+iV(R,\phi)$ and $f(r,\theta)=U(r,\theta)+iV(r,\theta)$, Eq. (4) becomes

$$U(r,\theta)+iV(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\left[U(R,\phi)+iV(R,\phi)\right]\left[R^2-r^2\right]d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)}. \quad \text{-------------} \quad (5)$$

By equating the real parts on either side of this equation, we obtain the following formula:

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{U(R,\phi)\left[R^2-r^2\right]d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)}. \quad \text{--------------} \quad (6)$$

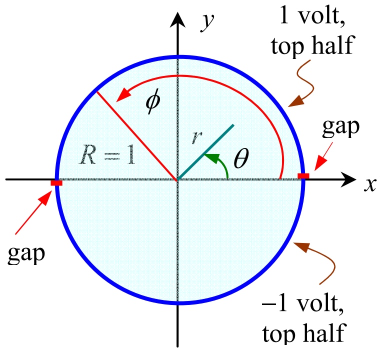
$$V(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}\frac{V(R,\phi)\left[R^2-r^2\right]d\phi}{R^2+r^2-2Rr\cos(\phi-\theta)}.$$

Poisson Integral Formula (For interior of a circle)

The formula yields the value of the harmonic function $U(r,\theta)$ (or $V(r,\theta)$) everywhere inside a circle of the radius *R*, provided we know the values $U(R,\phi)$ (or $V(R,\phi)$) assumed by *U* (or *V*) on the circumference of the circle.

Example 1

An electrically conducting tube of unit radius is separated into two halves by means of infinitesimal slits. The top half of the tube ($R=1,\ 0<\phi<\pi$) is maintained at an electrical potential of 1 volt while the bottom half ($R=1,\ \pi<\phi<2\pi$) is maintained at -1 volt. Find the potential at an arbitrary point (r,θ) inside the tube (see the shown figure). Assume there is a dielectric material inside the tube.



<Sol.>

Since the electrostatic potential is a harmonic function, the Poisson integral formula is applicable. From above discussion (Eq. (6)), with $R=1$, we have

$$U(r,\theta)=\frac{1}{2\pi}\int_0^{\pi}\frac{(1-r^2)d\phi}{1+r^2-2r\cos(\phi-\theta)}-\frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)d\phi}{1+r^2-2r\cos(\phi-\theta)} \quad (7)$$

In each integral, we make the change of variable $x=\phi-\theta$; from a standard table of integrals, we find the following formula, which is valid for $a^2>b^2\geq 0$:

$$\int\frac{dx}{a+b\cos x}=\frac{2}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(x/2)}{a+b}\right].$$

Using this formula in Eq. (7) with $a=1+r^2$, $b=-2r$, we obtain

$$U(r,\theta)=\frac{1}{\pi}\left[2\tan^{-1}\left(\frac{1+r}{1-r}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right)-\tan^{-1}\left(\frac{1+r}{1-r}\tan\left(\pi-\frac{\theta}{2}\right)\right)\right.$$
$$\left.-\tan^{-1}\left(\frac{1+r}{1-r}\tan\left(-\frac{\theta}{2}\right)\right)\right] \quad (8)$$

Conclude that $-1\leq U(r,\theta)\leq 1$ when $r\leq 1$.

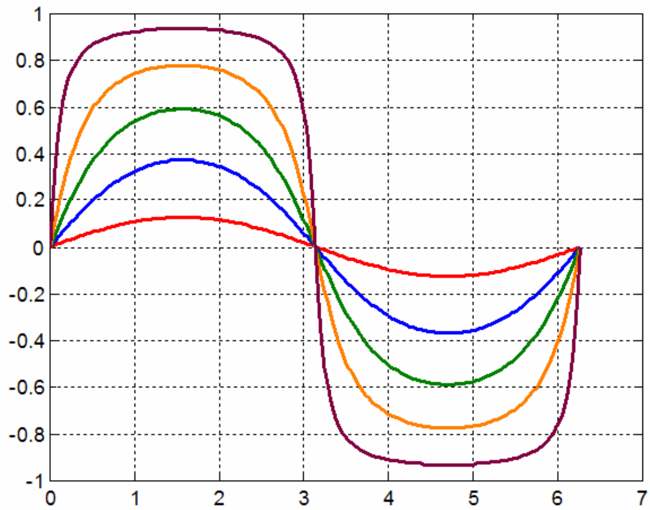
♣ Recall that $\tan(n\pi+\theta)=\tan\theta$, where θ is any angle and *n* any integer. We can recast eq. (8) as follows:

$$U(r,\theta)=\frac{2}{\pi}\left[\tan^{-1}\left[\frac{1+r}{1-r}\tan\left(\frac{\pi}{2}-\frac{\theta}{2}\right)\right]+\tan^{-1}\left[\frac{1+r}{1-r}\tan\frac{\theta}{2}\right]\pm\frac{\pi}{2}\right]. \quad (9)$$

where the minus sign is to be used in the front of $\pi/2$ when $0<\theta<\pi$ and the plus sign when $\pi<\pi<2\pi$.

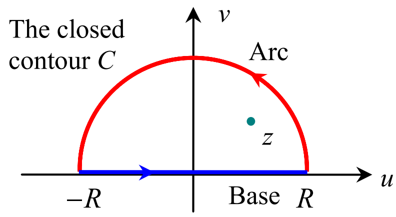
♣ **MATLAB Commands:**

```
% for figure of Example 1
clear fig
clear
r=[.1 .3 .5 .7 .9];
thet=linspace(0,2*pi,200);
for j=1:length(r)
    q=(1+r(j))/(1-r(j));
    p2=pi/2;
    u1=2/pi*(atan(q*tan(p2-thet/2)))
    u2=2/pi*(atan(q*tan(thet/2)))
    u3=sign(thet-pi);
    u=u1+u2+u3
    plot(thet,u);hold on;
end
grid on
```



2. **The Dirichlet Problem for a half plane (Infinite Line Boundary)**

1) Find a function $\phi(u,v)$ that is harmonic in the upper half complex w-plane (the domain $v>0$).
2) B.C. of $\phi(x,y)$: $\phi(u,0)$ at $v=0$.
3) Let $f(w)=\phi(u,v)+i\psi(u,v)$ be a function that is analytic for $v\geq 0$. Also, let z be a point inside this semicircle.
4) Cauchy Integral formula:



$$f(z)=\frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}dw. \quad \text{------------} \quad (10)$$

Since z lies inside the semicircle, the point $\bar{z}$ must lie outside the semicircle. Hence the function $f(w)/(w-\bar{z})$ is analytic on and interior to the contour C. We have

$$0=\frac{1}{2\pi i}\oint_C \frac{f(w)}{w-\bar{z}}dw. \quad \text{-------------} \quad (11)$$

Let us subtract Eq. (11) from eq. (10):

$$f(z)=\frac{1}{2\pi i}\oint_C f(w)\left(\frac{1}{w-z}-\frac{1}{w-\bar{z}}\right)dw=\frac{1}{2\pi i}\oint_C \frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}dw. \quad (12)$$

Note that the contour $C\equiv$ the base ($v=0$, $-R\leq u\leq R$, denoted by $\rightarrow$) + the arc with radius of R (denoted by $\frown$). Thus, Eq. (12) can be expressed as

$$f(z)=\frac{1}{2\pi i}\int_{\rightarrow}\frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}dw+\frac{1}{2\pi i}\int_{\frown}\frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}dw. \quad \text{--------------} \quad (13)$$

Consider the Cartesian representations $z=x+iy$, $\bar{z}=x-iy$, and $w=u+iv$. We see that $z-\bar{z}=2iy$ and $w=u$ along the base. Hence the first integrand on the right in the preceding equation can be rewritten as

$$\frac{(z-\bar{z})f(w)}{(w-z)(w-\bar{z})}=\frac{2iyf(u)}{[u-(x+iy)][u-(x-iy)]}=\frac{2iyf(u)}{(u-x)^2+y^2}.$$

so that Eq. (13) becomes

$$f(z)=\frac{y}{\pi}\int_{-R}^{R}\frac{f(u)du}{(u-x)^2+y^2}+\frac{y}{\pi}\int_{\frown}\frac{f(w)dw}{(w-z)(w-\bar{z})}. \quad \text{-----------} \quad (14)$$

As $R\rightarrow\infty$, we have

$$\int_{\frown}\frac{f(w)dw}{(w-z)(w-\bar{z})} \quad \rightarrow \quad 0$$

Thus, Eq.(14) simplifies to

$$f(z) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{f(u)du}{(u-x)^2 + y^2}. \quad \text{------------} \quad (15)$$

With $f(z) = \phi(x, y) + i\psi(x, y)$ and $f(w) = \phi(u, v) + i\psi(u, v)$, we obtain, from Eq. (15),

$$\phi(x, y) + i\psi(x, y) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(u,0) + i\psi(u,0)}{(u-x)^2 + y^2} du.$$

By equating the real parts on either side of this equation, we obtain the following formula:

$$\phi(x, y) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\phi(u,0)}{(u-x)^2 + y^2} du. \quad \text{-----------} \quad (16)$$

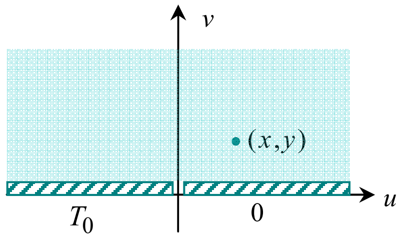
$$\psi(x, y) = \frac{y}{\pi}\int_{-\infty}^{+\infty} \frac{\psi(u,0)}{(u-x)^2 + y^2} du.$$

Poisson Integral Formula (For the Upper half plane)

Eq. (16) will yield the value of a harmonic function $\phi(x, y)$ anywhere in the upper half plane provided ϕ is already known over the entire real axis.

Example 2

As indicated in the figure, the upper half-space $\operatorname{Im} w > 0$ is filled with a heat-conducting material. The boundary $v = 0,\ u > 0$ is maintained at a temperature of 0 while the boundary $v = 0,\ u < 0$ is kept at temperature at T_0. Find the steady-state distribution of temperature $\phi(x, y)$ throughout the conducting material.



<Sol.>

Since the temperature is a harmonic function, the Poisson integral formula is directly applicable. We have $\phi(u,0) = T_0,\ u < 0$ and $\phi(u,0) = 0,\ u > 0$. Thus,

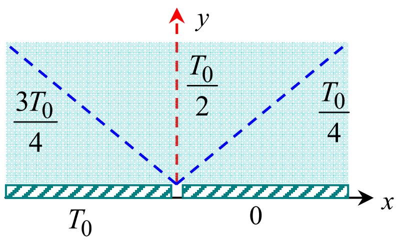
$$\phi(x, y) = \frac{y}{\pi}\int_{-\infty}^{0} \frac{T_0 du}{(u-x)^2 + y^2} + \frac{y}{\pi}\int_{0}^{\infty} \frac{0 du}{(u-x)^2 + y^2}$$

In the first term, make the change of variables $p = x - u$. Thus,

$$\phi(x, y) = \frac{T_0 y}{\pi}\int_{x}^{\infty} \frac{dp}{p^2 + x^2} = \frac{T_0}{\pi}\tan^{-1}\frac{p}{y}\Big|_x^{\infty} = \frac{T_0}{\pi}\left[\frac{\pi}{2} - \tan^{-1}\frac{x}{y}\right]. \quad \text{--------------} \quad (17)$$

From the trigonometric identity $\tan^{-1}\theta = \pi/2 - \tan^{-1}(1/\theta)$, we that the expression in the brackets on the right side of Eq. (17) is $\tan^{-1}(y/x) = \theta$, where θ is the polar angle associated with the point (x, y). If we take θ as the principal polar angle in the space $y \geq 0$. Thus,

$$\phi(x, y) = \frac{T_0}{\pi}\theta, \quad 0 \leq \theta \leq \pi$$



H.W. 1 (a) An electrically conducting metal sheet is perpendicular to the y-axis and passes through $y = 0$, as shown in **Fig. A**. The right half of the sheet, $x > 0$, is maintained at an electrical potential of V_0 volts while the left half, $x < 0$, is maintained at a voltage $-V_0$. Show that in the half space, $y \geq 0$, the electrostatic potential is given by

$$\phi(x, y) = V_0 - \frac{2V_0}{\pi}\tan^{-1}\left(\frac{y}{x}\right) = V_0 - \frac{2V_0}{\pi}\operatorname{Im}\left(\operatorname{Ln} z\right)$$

where $0 \leq \tan^{-1}(y/x) \leq \pi$.

(b) Sketch the equipotential lines (or surfaces) on which

$$\phi(x,y) = V_0/2, \quad \phi(x,y) = 0, \quad \phi(x,y) = -V_0/2$$

(c) Find the components of the electric field Ex and Ey at x =1, y = 1, and draw a vector representing the field at this point.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Problems 6 , Exercise 4.7, Pearson Education, Inc., 2005.】

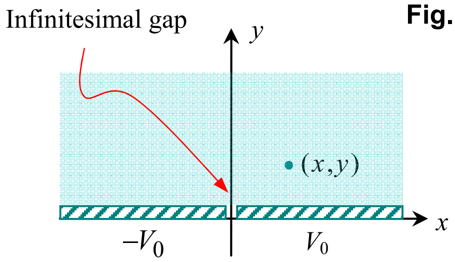


Fig. A