CHAPTER THREE
Basic Transcendental Functions of Complex Variables

§3-1 The Exponential Function

1. If $z = x + iy$, the complex exponential e^z is given by

$$e^z = e^{x+iy} = e^x e^{iy}$$
$$= e^x(\cos y + i\sin y)$$

上述之定義中，使用到 Euler's Formula：

$$e^{i\theta} = \cos\theta + i\sin\theta \text{ ， } \theta \in R$$

在此須先證明 e^{iy} 是否亦符合 Euler's Formula，使得

$$e^{iy} = \cos y + i\sin y \quad \text{------} \quad (1)$$

所以，底下我們將先舉出在實變函數中 Exponential 函數的性質，而後再證明之。

* 在實變微積分中，若定義

$$f(x) = e^{ax}$$

則其有下列之性質

i) $e^{x_1+x_2} = e^{x_1}e^{x_2}$

ii) $f(0) = 1$

iii) $f'(x) = ae^{ax} = af(x)$

Using the above mentioned properties, we want to prove that the Equation (1) holds for any real number *y*.

$$e^{iy} = \cos y + i\sin y$$

Let $\quad f(y) = e^{iy}$

$$= g(y) + i\,h(y)$$

where both $g(y)$ and $h(y)$ are real functions and satisfy the properties mentioned above, i.e.,

$$f'(y) = i\,f(y)$$
$$f(0) = 1$$

Thus, when

$$f'(y) = g'(y) + i\,h'(y)$$
$$= i\,f(y)$$
$$= i[g(y) + i\,h(y)]$$
$$= i\,g(y) - h(y)$$

We know that when the equality holds any *y*, there must have

Real part = Real part and Im = Im

So, we obtain the following equations：

$$\begin{cases} g'(y) = -h(y) \text{ ------ } (2) \\ h'(y) = g(y) \text{ ------ } (3) \end{cases}$$

Differentiating equation (2) with respect to *y*, gives

$$g''(y) = -h'(y) = -g(y)$$

$\Rightarrow \quad g''(y) + g(y) = 0$

$\Rightarrow \quad g(y) = c_1\cos y + c_2\sin y \text{ ------ } (4)$

From equation (2), we know that

$$h(y) = -g'(y)$$
$$= c_1\sin y - c_2\cos y \text{ ------ } (5)$$

And then we use the property $f(0) = 1$:

$\Rightarrow \quad f(0) = g(0) + i\,h(0) = 1$

This implies that

$$\begin{cases} g(0)=1 \\ h(0)=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} c_1=1 \\ c_2=0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} g(y)=\cos y \\ h(y)=\sin y \end{cases}$$

$$\Rightarrow \quad e^{iy}=f(y)=g(y)+i\,h(y)$$
$$=\cos y+i\sin y$$

♣ **Some basic properties of the exponential function are discussed as below.**

2. If, $z_1=x_1+i\,y_1$ and $z_2=x_2+i\,y_2$ are two complex numbers, then

$$e^{z_1}e^{z_2}=e^{z_2+z_2}$$

<pf.>

$$e^{z_1}e^{z_2}=e^{(x_1+i\,y_1)}e^{(x_2+i\,y_{21})}$$
$$=e^{x_1}\left[\cos y_1+i\sin y_1\right]\cdot e^{x_2}\left[\cos y_2+i\sin y_2\right]$$
$$=e^{x_1+x_2}\left[(\cos y_1\cos y_2-\sin y_1\sin y_2)+i\,(\sin y_1\cos y_2+\sin y_2\cos y_1)\right]$$
$$=e^{x_1+x_2}\left[\cos(y_1+y_2)+i\sin(y_1+y_2)\right]$$
$$=e^{(x_1+x_2)}e^{i(y_1+y_2)}$$
$$=e^{(x_1+x_2)+i(y_1+y_2)}$$
$$=e^{z_1+z_2}$$

3. If $f(z)=e^x[\cos y+i\sin y]$ is analytic in the entire complex plane, then

$$f'(z)=e^z$$

＊ 為了證明上述結果，我們必須先了解下列的性質。

If $f(z)=u(x,y)+i\,v(x,y)$ ，且其

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}\ ,\quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

而且 $\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial u}{\partial y}$ 均為連續 $\Leftrightarrow$ $f(z)$ is analytic

<pf.> Since $f(z)=e^x[\cos y+i\sin y]$

$$\Rightarrow \quad u(x,y)=e^x\cos y$$
$$v(x,y)=e^x\sin y$$

It is easily seen that

$$\frac{\partial u}{\partial x}=e^x\cos y$$

$$\frac{\partial v}{\partial y}=e^x\cos y$$

$$\Rightarrow \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \quad \text{--------} \quad (1)$$

And $\frac{\partial v}{\partial x}=e^x\sin y$, $\frac{\partial u}{\partial y}=-e^x\sin y$

$$\Rightarrow \quad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y} \quad \text{--------} \quad (2)$$

Furthermore, $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous everywhere in the xy-plane, and satisfy the Cauchy-Riemann conditions for all finite values of z everywhere in this plane.

$\Rightarrow$ $f(z)$ is analytic for all z and is therefore an entire function.

The derivative $f'(z)$ is easily found as

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= e^x \cos y + i\, e^x \sin y \\ &= e^x[\cos y + i \sin y] \\ &= e^z \end{aligned}$$

♣ **Chain Rule:** If $g(z)$ is also an analytic function, we have $\frac{d}{dz}e^{(g(z))} = e^{(g(z))}g'(z)$.

4. e^x 和 e^z 之比較如下表所示：

	e^x	e^z
1)	$e^x > 0$	e^z 沒有大小
2)	$e^x \neq 0$	$e^z \neq 0 \quad (e^z \cdot e^{-z} = e^0 = 1)$
3)	$f(x) = e^x$ is one-to-one function	*** $f(z) = e^z$ is a periodic function. Its period is $2\pi i$

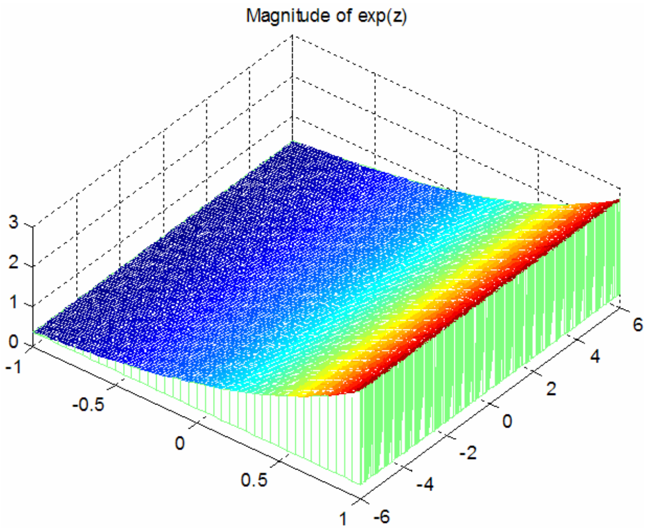
*** $$\begin{aligned} e^z &= e^x[\cos y + i \sin y] \\ &= e^x[\cos(y + 2n\pi) + i \sin(y + 2n\pi)] \\ &= e^x e^{i(y+2n\pi)} \\ &= e^{x+iy+2n\pi i} \\ &= e^{z+2n\pi i}, \quad n \in I \end{aligned}$$

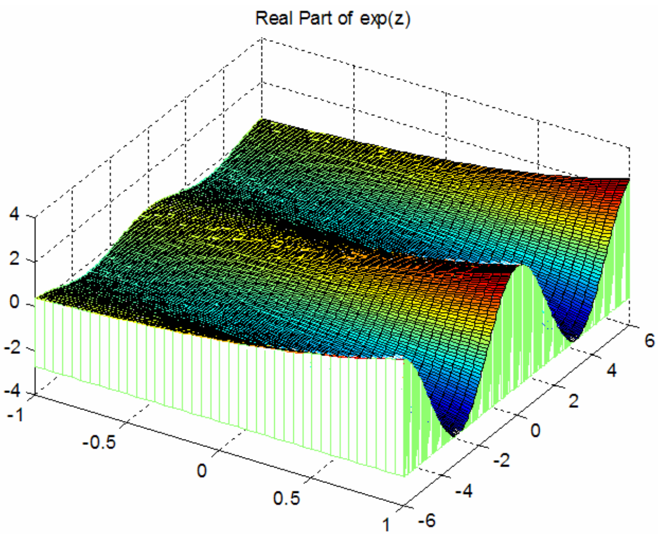
$T = 2\pi i$ (*imaginary period*)，故知 e^z 為週期函數。

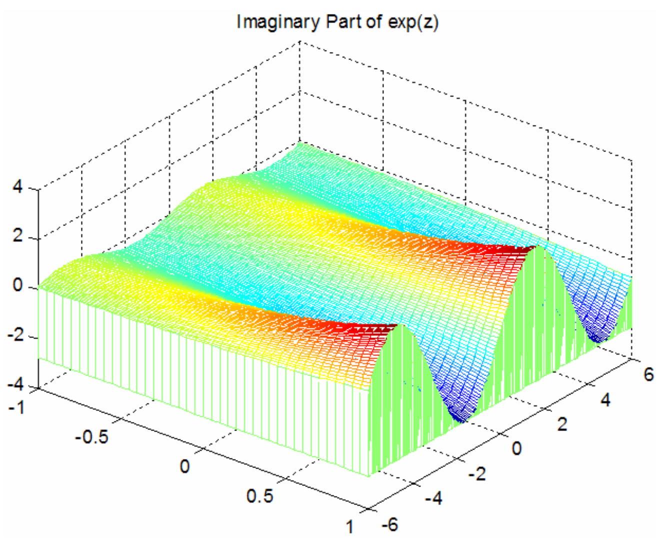
♣ **MATLAB Commands for plotting** $\left|e^z\right|$, $\mathrm{Re}\left(e^z\right)$, **and** $\mathrm{Im}\left(e^z\right)$**:**

```
% Magnitude of exp(z)
x=[-1:0.05:1];
y=[-6:0.05:6];
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=exp(Z);
wm=abs(w);
meshz(X,Y,wm);hold on
title('Magnitude of exp(z)')

% Real part of exp(z)
x=[-1:0.05:1];
y=[-6:0.05:6];
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=exp(Z);
wm=real(w);
meshz(X,Y,wm);hold on
surf(X,Y,wm)
title('Real Part of exp(z)')

% Imaginary part of exp(z)
x=[-1:0.05:1];
y=[-6:0.05:6];
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
```

```
w=exp(Z);
wm=imag(w);
meshz(X,Y,wm);hold on
title('Imaginary Part of exp(z)')
```







5. For $z = x + iy$, we have

1) $e^z \neq 0$

2) $\left|e^{iy}\right| = 1$ and $\left|e^z\right| = e^x$ (The magnitude of e^z is determined by the real part of z.)

3) If $e^z = 1 \Leftrightarrow z = 2k\pi i$, $k =$ constant.

4) If $e^{z_1} = e^{z_2} \Leftrightarrow z_1 - z_2 = 2k\pi i$

where k is an integer.

<pf.> 1) From $e^z \cdot e^{-z} = e^0 = 1$, since the product is never zero, neither factor can be zero.

Therefore, $e^z \neq 0 \quad \forall \quad z \in C$.

2) $\left|e^{iy}\right| = \sqrt{\cos^2 y + \sin^2 y} = 1$

Since $e^z = e^x[\cos y + i\sin y]$,

then $\left|e^z\right| = \left|e^x\right|\left|\cos y + i\sin y\right|$

$$= e^x \cdot 1$$

$$= e^x$$

3) Let $e^z = e^x[\cos y + i\sin y] = 1$

thus $\begin{cases} e^x \cos y = 1 \\ e^x \sin y = 0 \end{cases}$

Since $e^x \neq 0 \quad \forall \quad x \in R$, we see that $\sin y = 0$.

Hence $y = n\pi$, $n \in I$

Because $e^x > 0 \Rightarrow \cos y = 1$ $(-1$ 不合$)$

$\Rightarrow$ $y = 2k\pi$, $k \in I$

Thus, we seek that $n = 2k$.

If we take $e^x = 1 \Rightarrow x = 0$

因此， $z = x + iy$

$= 0 + in\pi$

$= 2k\pi i$ ← 此為必要條件

現在，尚須證明其充分條件亦成立，故

Suppose that $z = 2k\pi i$, where $k \in I$.

Then, we obtain that

$$e^z = e^{2k\pi i}$$

$$= \cos 2k\pi + i\sin 2k\pi$$

$$= 1$$

4) Since $e^{z_1} = e^{z_2} \Leftrightarrow e^{z_1 - z_2} = 1$

Hence from the property 3), we have

$z_1 - z_2 = 2k\pi i$, where $k \in I$.

The theorem is thus established.

Since $e^z \neq 0 \quad \forall \quad z \in C$, and $e^z e^{-z} = 1$

We have that

$$e^{-z} = \frac{1}{e^z}$$

6. Let us show that $e^{\bar{z}} = \overline{e^z}$

<pf.> $e^{\bar{z}} = e^{\overline{x+iy}} = e^{x-iy}$

$= e^x e^{-iy}$

$= e^x(\cos y - i\sin y)$

$$\overline{e^z} = \overline{e^x(\cos y + i\sin y)} = e^x(\cos y - i\sin y)$$

Thus, we see that $e^{\bar{z}} = \overline{e^z}$

♣ **Some useful identities:**

1) $e^{i0} = 1,\quad e^{i\pi/2} = i,\quad e^{i\pi} = -1,\quad e^{i3\pi/2} = -i = e^{-i\pi/2}$

2) $e^{i\pi} + 1 = 0$

3) DeMoivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ or $\left(e^{i\theta}\right)^n = e^{in\theta}$.

4) Complex functions of a real variable: $f(t) = u(t) + iv(t)$, where $u(t)$ and $v(t)$ are real.
   $\Rightarrow \quad f'(t) = u'(t) + iv'(t)$

Example 1 Find $\dfrac{d^7\left[e^{2t}\cos(2t)\right]}{dt^7}$.

<Sol.>

1. Direct differentiation $\Rightarrow$ laborious !
2. Note that $e^{2t}\cos(2t) = \mathrm{Re}\left(e^{2(1+i)t}\right)$.

   Let $f(t) = e^{2(1+i)t}$. Then, we have

$$f^{(7)}(t) = 2^7(1+i)^7 e^{2(1+i)t} = 2^7 \times (-8i)(1+i)e^{2(1+i)t}$$

   Thus,

$$\frac{d^7\left[e^{2t}\cos(2t)\right]}{dt^7} = \mathrm{Re}\left[2^7 \times (-8i)(1+i)e^{2(1+i)t}\right] = \mathrm{Re}\left\{2^{10}e^{2t}\left[(\sin 2t + \cos 2t) + i(\sin 2t - \cos 2t)\right]\right\}$$
$$= 2^{10}e^{2t}(\sin 2t + \cos 2t)$$

H.W. 1 (a) Suppose we want the n-th derivative, with respect to t, of $f(t) = t/\left(t^2+1\right)$. Notice that $f(t) = \mathrm{Re}\left(\dfrac{1}{t-i}\right)$ and that the n-th derivative of the function in the brackets is exactly taken. Using the method of Example 1, as well as the binomial theorem (which perhaps should be reviewed), show that

$$f^{(n)}(t) = \frac{(-1)n!(n+1)!}{(t^2+1)^{n+1}}\sum_{k=0}^{(n+1)/2}\frac{(-1)^k t^{n+1-2k}}{(2k)!(n+1-2k)!}, \quad \text{for } n \text{ odd},$$

$$f^{(n)}(t) = \frac{n!(n+1)!}{(t^2+1)^{n+1}}\sum_{k=0}^{n/2}\frac{(-1)^k t^{n+1-2k}}{(2k)!(n+1-2k)!}, \quad \text{for } n \text{ even}.$$

(b) Using the method of part (a), find similar expressions for the nth derivative of $f(t) = 1/\left(t^2+1\right)$.

Note that this function is identical to $\mathrm{Im}\left(1/(t-i)\right)$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.1, Problem 24, Pearson Education, Inc., 2005.】

H.W. 2 The absolute magnitude of the expression

$$P = 1 + e^{i\psi} + e^{i2\psi} + \cdots + e^{i(N-1)\psi} = \sum_{n=0}^{N-1} e^{in\psi}$$

is of interest in many problems involving radiation from N identical physical elements (e.g., antennas, loudspeakers). Here ψ is a real quantity that depends on the separation of the elements and the position of an observer of the radiation. $|P|$ can tell us the strength of the radiation observed.

(a) Using the formula for the sum of a finite geometric series (see **H.W. 8** in page 6, lecture note of ch_01), show that

$$|P(\psi)| = \left|\frac{\sin N\psi/2}{\sin\psi/2}\right|$$

(b) Find $\lim_{\psi\to 0}|P(\psi)|$.

(c) Use a calculator or a simple computer program to plot $|P(\psi)|$ for $0 \le \psi \le 2\pi$ when $N = 3$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.1, Problem 25, Pearson Education, Inc., 2005.】

<Ans.>

(a) Let $z = e^{i\psi}$, then $P = 1 + z + z^2 + \cdots + z^{N-1} = \dfrac{1 - z^{N-1+1}}{1-z} = \dfrac{1-z^N}{1-z}$.

$$P(\psi) = \frac{1-e^{iN\psi}}{1-e^{i\psi}} = \frac{e^{iN\psi}-1}{e^{i\psi}-1} = \frac{e^{iN\psi/2}}{e^{i\psi/2}}\left[\frac{e^{iN\psi/2}-e^{-iN\psi/2}}{e^{i\psi/2}-e^{-i\psi/2}}\right]$$

$$\Rightarrow \quad = \frac{e^{iN\psi/2}}{e^{i\psi/2}}\left[\frac{(\cos(N\psi/2)+i\sin(N\psi/2))-(\cos(N\psi/2)-i\sin(N\psi/2))}{(\cos(\psi/2)+i\sin(\psi/2))-(\cos(\psi/2)+i\sin(\psi/2))}\right]$$

$$= \frac{e^{iN\psi/2}}{e^{i\psi/2}}\frac{2i\sin(N\psi/2)}{2i\sin(\psi/2)}$$

Thus, we have

$$|P(\psi)| = \left|\frac{e^{iN\psi/2}}{e^{i\psi/2}}\right|\left|\frac{2i\sin(N\psi/2)}{2i\sin(\psi/2)}\right| = \left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| \quad \text{------------} \quad \text{Q.E.D.}$$

(b) Consider

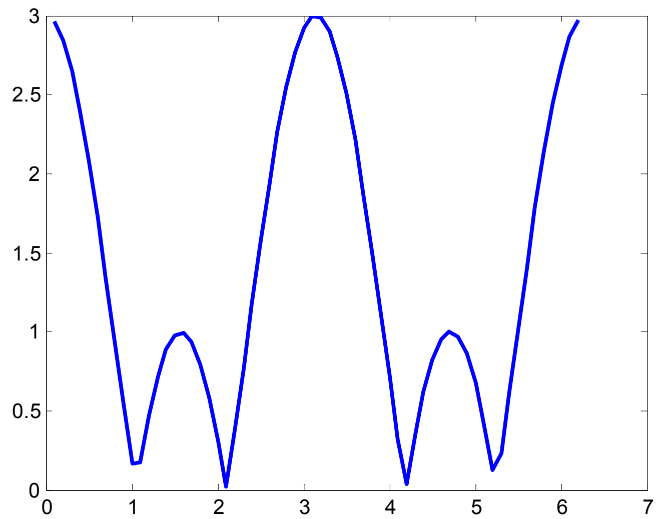
$$\lim_{\psi\to 0}\left|\frac{\sin(N\psi/2)}{\sin(\psi/2)}\right| = \frac{N/2}{1/2}\frac{\cos(N\psi/2)}{\cos(\psi/2)} = N$$

Thus, we have

$$\lim_{\psi\to 0}|P(\psi)| = N$$

(c) MATLAB commands:

```
% H.W.1 part (c)
N=3;
phi=[0:0.1:2*pi];
y=sin(N*phi)./sin(phi);
Y=abs(y);
plot(phi,Y)
```



§3-2 The Trigonometric Function

1. Since $e^{ix} = \cos x + i\sin x$

   $e^{-ix} = \cos x - i\sin x$

   then we see that

$$\begin{cases} \sin x = \dfrac{e^{ix} - e^{-ix}}{2i} \\ \cos x = \dfrac{e^{ix} + e^{-ix}}{2} \end{cases} \Rightarrow \text{屬於實數}$$

and

$$\begin{cases} \sinh x = \dfrac{e^{x} - e^{-x}}{2} \\ \cosh x = \dfrac{e^{x} + e^{-x}}{2} \end{cases} \Rightarrow \text{屬於實數}$$

1) Given any complex number z, we define

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

2) The functions $\sin z$ and $\cos z$ are analytic for all values of z. Moreover,

$$\frac{d}{dz}\sin z = \frac{i\,e^{iz} + i\,e^{-iz}}{2i} = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$

$$\frac{d}{dz}\cos z = \frac{i\,e^{iz} - i\,e^{-iz}}{2} = \frac{-(e^{iz} - e^{-iz})}{2i} = -\sin z$$

2. 底下是一些三角函數和 Hyperbolic Function 的性質：

1) $\cos^2 x + \sin^2 x = 1$,

   $\cosh^2 x - \sinh^2 x = 1$

2) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + c$,

   $\displaystyle\int \frac{1}{\sqrt{1+x^2}}\,dx = \sinh^{-1} x + c$

3) $\mathcal{L}\{\sin \omega t\} = \dfrac{\omega}{s^2 + \omega^2}$,

   $\mathcal{L}\{\sinh \omega t\} = \dfrac{\omega}{s^2 - \omega^2}$

4) $\mathcal{L}\{\cos \omega t\} = \dfrac{s}{s^2 + \omega^2}$,

   $\mathcal{L}\{\cosh \omega t\} = \dfrac{s}{s^2 - \omega^2}$

3. If $z = x + iy$, then

1) $\sin z = \sin x \cosh y + i \cos x \sinh y$

2) $\cos z = \cos x \cosh y - i \sin x \sinh y$

<pf.> 1) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

$$= \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i}$$

$$= \frac{e^{-y+ix} - e^{-y-ix}}{2i}$$

$$= \frac{e^{y}[\cos x + i\sin x] - e^{y}[\cos x - i\sin x]}{2i}$$

$$= \frac{\cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y})}{2i}$$

$$= i\cos\frac{e^{-y} - e^{y}}{2} + \sin x\frac{e^{-y} + e^{y}}{2}$$

$$= \sin x\cosh y + i\cos x\sinh y$$

2) 同理可證。在此另舉一證法：

由 Taylor's Series，知

$$\begin{cases} \sin z = z - \dfrac{z^3}{3!} + \dfrac{z^5}{5!} - + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\ \sinh z = z + \dfrac{z^3}{3!} + \dfrac{z^5}{5!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \end{cases}$$

比較上列二式可得

$$\begin{cases} \sinh iz = i\sin z \\ \sin iz = i\sinh z \end{cases}$$

同樣地，由下二式

$$\begin{cases} \cos z = 1 - \dfrac{z^2}{2!} + \dfrac{z^4}{4!} - + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ \cosh z = z + \dfrac{z^2}{2!} + \dfrac{z^4}{4!} + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \end{cases}$$

比較可得

$$\begin{cases} \cosh iz = \cos z \\ \cos iz = \cosh z \end{cases}$$

由上述結果我們便可證明 2) 之結果：

Since $\cos z = \cos(x + iy)$

$$= \cos x\cos iy - \sin x\sin iy$$

$$= \cos x\cosh y - i\sin x\sinh y$$

4. $|\sin x| \le 1$，在實變函數中成立；但在複變函數中，$|\sin z| \le 1$，不一定成立。

Since $\sin z = \sin x\cosh y + i\cos x\sinh y$，

thus $|\sin z|^2 = \sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y$

$$= \sin^2 x(1 + \sinh^2 y) + (1 - \sin^2 x)\sinh^2 y$$

$$= \sin^2 x + \sinh^2 y$$

If we take $x = \dfrac{\pi}{2}$，$y = 1$，then obtain

$\sin^2 x = 1$，$\sinh^2 y > 0$

$\Rightarrow$ $\sin^2 z > 1$

$\Rightarrow$ $|\sin z| \le 1$，不一定成立

5. If $\sin z = 0 \;\Rightarrow\; z = n\pi$，$n \in I$.

From $\quad \sin z = \dfrac{e^{iz} - e^{-iz}}{2i} = 0$

$\Rightarrow \quad e^{iz} = e^{-iz} \quad \Rightarrow \quad e^{2iz} = 1$

$\Rightarrow \quad e^{i(2x+i2y)} = 1$

$\Rightarrow \quad e^{-2y}[\cos 2x + i\sin 2x] = 1$

$\Rightarrow \quad \sin 2x = 0$

Since $e^{2y} > 0$，when $\sin 2x = 0$, we know $\cos 2x = \pm 1$

But in this case we must take $\cos 2x = +1$（如此 $e^{2y} > 0$ 才能配合題意之要求）.

$\Rightarrow \quad 2x = 2n\pi \quad \Rightarrow \quad x = n\pi$

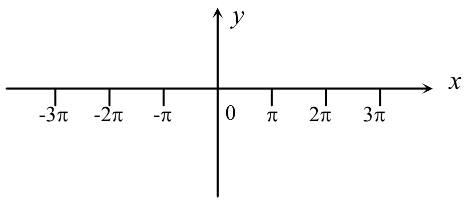
而因 $\cos 2x = 1$，故取 $e^{-2y} = 1$

故知 $\quad y = 0$

$\Rightarrow \quad z = x + iy$

$\quad = n\pi + i0$

$\quad = n\pi \quad , \quad n \in I$



6. If $\cos z = 0 \Rightarrow z = \left(n + \dfrac{1}{2}\right)\pi$ ， $n \in I$.

From $\quad \cos z = \dfrac{e^{iz} + e^{-iz}}{2} = 0$, we see that $e^{iz} = e^{-iz}$.

$\Rightarrow \quad e^{i2z} = -1$

$\Rightarrow \quad e^{-2y}[\cos 2x + i\sin 2x] = -1$

Then, we have $\sin 2x = +0$.

Since $e^{-2y} > 0$，when $\sin 2x = 0$, we know that $\cos 2x = \pm 1$.

But in this case we must take $\cos 2x = -1$ (such that they satisfy $e^{-2y} > 0$，$e^{-2y}\cos 2x = -1$).

$\Rightarrow \quad 2x = (2n+1)\pi$

$\Rightarrow \quad x = \left(n + \dfrac{1}{2}\right)\pi$

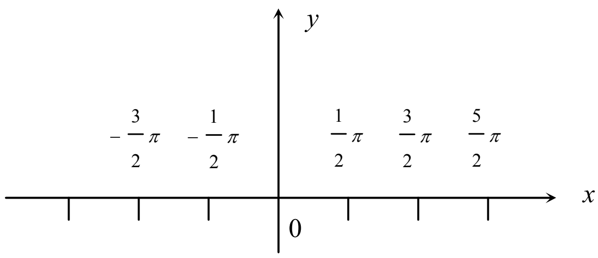
又因 $\cos 2x = -1$，故取 $e^{-2y} = 1$。

$\Rightarrow \quad -2y = 0 \quad \Rightarrow \quad y = 0$

故 $\quad z = x + iy$

$\quad = \left(n + \dfrac{1}{2}\right)\pi + i0$

$\quad = \left(n + \dfrac{1}{2}\right)\pi$



7. Given the complex number z, we define

1) $\tan z = \dfrac{\sin z}{\cos z}, \quad z \neq \left(n + \dfrac{1}{2}\right)\pi$

2) $\cot z = \dfrac{\cos z}{\sin z}, \quad z \neq n\pi$

3) $\sec z = \dfrac{1}{\cos z}, \quad z \neq \dfrac{\pi}{2} + n\pi$

4) $\csc z = \dfrac{1}{\sin z}, \quad z \neq n\pi$

where $n \in I$ in all above cases.

5) $\tan z$ and $\cot z$ both have a fundamental period of π.

6) $\sec z$ and $\csc z$ both have a fundamental period of 2π.

8. $\tan z$ ，$\cot z$ ，$\sec z$ and $\csc z$ are analytic functions of z except for the above mentioned

limiting values. We can define that

1) $\frac{d}{dz}(\tan z) = \sec^2 z$ ， $z \neq \frac{\pi}{2} + n\pi$

2) $\frac{d}{dz}(\cot z) = -\csc^2 z$ ， $z \neq n\pi$

3) $\frac{d}{dz}(\sec z) = \sec z \tan z$ ， $z \neq \frac{\pi}{2} + n\pi$

4) $\frac{d}{dz}(\csc z) = -\csc z \cot z$ ， $z \neq n\pi$

where in all cases , $n \in I$.

9. If $z = x + iy$, then

1) $\sin \bar{z} = \overline{\sin z}$ ， $\cos \bar{z} = \overline{\cos z}$

<pf.>

$$\sin \bar{z} = \sin(\overline{x+iy}) = \sin(x-iy)$$
$$= \sin x \cos iy - \cos x \sin iy$$
$$= \sin x \cosh y - i \cos x \sinh y$$

$\Rightarrow$ $\sin \bar{z} = \overline{\sin z}$

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$
$$\overline{\sin z} = \sin x \cosh y - i \cos x \sinh y$$

同理 $\cos \bar{z} = \overline{\cos z}$

2) $|\sin z|^2 = \sin^2 x + \sinh^2 y$

3) $|\cos z|^2 = \cos^2 x + \sinh^2 y$

4) $\sin^2 z + \cos^2 z = 1$

5) $\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

6) $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$

7) $\sin\left(\frac{\pi}{2} - z\right) = \cos z$

8) $\sin 2z = 2 \sin z \cos z$

9) $\cos 2z = \cos^2 z - \sin^2 z$

以上各式之證明均非常簡單，留待給讀者證明。

10. Let us show that

$$\tan(z_1 + z_2) = \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

where $z_1 + z_2 \neq \left(n + \frac{1}{2}\right)\pi$, $n \in I$.

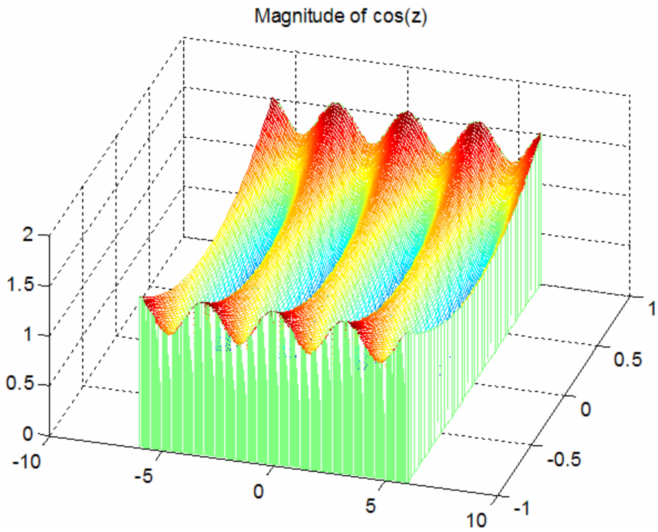
<pf.>

$$\tan(z_1 + z_2) = \frac{\sin(z_1 + z_2)}{\cos(z_1 + z_2)}$$
$$= \frac{\sin z_1 \cos z_2 + \cos z_1 \sin z_2}{\cos z_1 \cos z_2 - \sin z_1 \sin z_2}$$
$$= \frac{\tan z_1 + \tan z_2}{1 - \tan z_1 \tan z_2}$$

♣ **MATLAB Commands for plotting $|\cos(z)|$:**

```
% Magnitude of cos(z)
x=[-6:0.05:6];
y=[-1:0.05:1];
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=cos(Z);
```

```
wm=abs(w);
meshz(X,Y,wm);hold on
title('Magnitude of cos(z)')
```



H.W.1 Let $f(z)=\sin(1/z)$.

(a) Express this function in the form $u(x,y)+iv(x,y)$. Where in the complex plane is this function analytic?

(b) What is the derivative of $f(z)$? Where in the complex plane $f'(z)$ is analytic?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.2, Problem 22, Pearson Education, Inc., 2005.】

<Ans.>

(a) Since $f(z)=\sin(1/z)$, where $1/z$ is analytic except for $z=0$. But, sin$z$ is analytic for all z. Therefore, we have $\sin(1/z)$ is analytic for all $z\neq 0$.

Since $z=x+iy$, we have

$$\sin(1/z)=\sin\left(\frac{x-iy}{x^2+y^2}\right)=\sin\left(\frac{x}{x^2+y^2}-i\frac{y}{x^2+y^2}\right)$$

$$=\sin\left(\frac{x}{x^2+y^2}\right)\cosh\left(\frac{y}{x^2+y^2}\right)-i\cos\left(\frac{x}{x^2+y^2}\right)\sinh\left(\frac{y}{x^2+y^2}\right)$$

where

$$u(x,y)=\sin\left(\frac{x}{x^2+y^2}\right)\cosh\left(\frac{y}{x^2+y^2}\right) \quad\text{and}\quad v(x,y)=\cos\left(\frac{x}{x^2+y^2}\right)\sinh\left(\frac{y}{x^2+y^2}\right)$$

(b) $\dfrac{d}{dz}\sin(1/z)=\cos(1/z)\left(-\dfrac{1}{z^2}\right)$ ----------- analytic for all $z\neq 0$.

H.W. 2 Show that $\tan z=\dfrac{\sin(2x)+i\sinh(2y)}{\cos(2x)+\cosh(2y)}$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.2, Problem 28, Pearson Education, Inc., 2005.】

<Ans.>

Since

$$\tan z=\frac{\sin z}{\cos z}=\frac{\sin x\cosh y+i\cos x\sinh y}{\cos x\cosh y-i\sin x\sinh y}$$

$$=\frac{(\sin x\cosh y+i\cos x\sinh y)(\cos x\cosh y-i\sin x\sinh y)}{\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y}\equiv\frac{N}{D}$$

Consider the denominator D:

$$\cos^2 x\cosh^2 y+\sin^2 x\sinh^2 y=\cosh^2 y\left(1-\sin^2 x\right)+\sin^2 x\left(\cosh^2 y-1\right)$$
$$=\cosh^2 y-\sin^2 x$$
$$=\frac{(1+\cosh 2y)-[1-\cos(2x)]}{2}$$
$$=\frac{1}{2}\left[\cosh(2y)+\cos(2x)\right]=D$$

Now, consider the numerator N: The real part of N is

$$\cosh^2 y\sin x\cos x-\sinh^2 y\sin x\cos x$$
$$=\sin x\cos x\left(\cosh^2 y-\sinh^2 y\right)=\sin x\cos x=\frac{1}{2}\sin(2x)$$

Then, consider the imaginary part of numerator:

$$\cos^2 x\cosh y\sinh y+\sin^2 x\cosh y\sinh y$$
$$=\sinh y\cosh y\left(\cos^2 x+\sin^2 x\right)=\sinh y\cosh y=\frac{1}{2}\sinh(2y)$$

Thus, we have

$$\frac{N}{D}=\frac{\frac{1}{2}\sin(2x)+i\frac{1}{2}\sinh(2y)}{\frac{1}{2}\left(\cosh 2y+\cos 2x\right)}$$

This implies that

$$\tan z=\frac{\sin(2x)+i\sinh(2y)}{\cos(2x)+\cosh(2y)} \qquad \text{----------} \qquad \text{Q.E.D.}$$

H.W. 3 (a) Since $\sin z=\sin x\cosh y+i\cos x\sinh y$ and $|\sinh y|\le\cosh y$, show that $|\sinh y|\le|\sin z|\le\cosh y$.

(b) Derive a comparable double inequality for $|\cos z|$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.2, Problem 30, Pearson Education, Inc., 2005.】

<Ans.>

(a) Since $\sin z=\sin x\cosh y+i\cos x\sinh y$, we have

$$|\sin z|=|\sin x\cosh y+i\cos x\sinh y|$$
$$=\sqrt{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}$$
$$\le\sqrt{\sin^2 x\cosh^2 y+\cos^2 x\cosh^2 y} \qquad \because\ |\sinh y|\le\cosh y$$
$$=\sqrt{\cosh^2 y\left(\sin^2 x+\cos^2 x\right)}$$
$$=\sqrt{\cosh^2 y}=|\cosh y|=\cosh y \quad \text{(since } \cosh y \text{ is positive!)}$$

$$\therefore \qquad |\sin z|\le\cosh y \qquad \text{-----------} \qquad (1)$$

On the other hand, we have

$$|\sin z|=\sqrt{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y}$$
$$\ge\sqrt{\sin^2 x\sinh^2 y+\cos^2 x\sinh^2 y} \qquad \because\ |\sinh y|\le\cosh y$$
$$=\sqrt{\sinh^2 y\left(\sin^2 x+\cos^2 x\right)}$$
$$=\sqrt{\sinh^2 y}=|\sinh y|$$

$$\Rightarrow \qquad |\sinh y|\le|\sin z| \qquad \text{------------} \qquad (2)$$

From Eqs.(1) and (2), we conclude that

$$|\sinh y|\le|\sin z|\le\cosh y \qquad \text{------------} \qquad \text{Q.E.D.}$$

(b) Since $\cos z=\cos x\cosh y-i\sin x\sinh y$, we have

$$
\begin{aligned}
|\cos z| &= |\cos x\cosh y - i\sin x\sinh y| \\
&= \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \\
&\le \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\cosh^2 y} \qquad \because\ |\sinh y| \le \cosh y \\
&= \sqrt{\cosh^2 y\left(\sin^2 x + \cos^2 x\right)} \\
&= \sqrt{\cosh^2 y} = |\cosh y| = \cosh y \quad (\text{since } \cosh y \text{ is positive!})
\end{aligned}
$$

$\therefore \qquad |\cos z| \le \cosh y \qquad$ ----------- (3)

Similarly, we have

$$
\begin{aligned}
|\cos z| &= \sqrt{\cos^2 x\cosh^2 y + \sin^2 x\sinh^2 y} \\
&\ge \sqrt{\sin^2 x\sinh^2 y + \cos^2 x\sinh^2 y} \qquad \because\ |\sinh y| \le \cosh y \\
&= \sqrt{\sinh^2 y\left(\sin^2 x + \cos^2 x\right)} \\
&= \sqrt{\sinh^2 y} = |\sinh y|
\end{aligned}
$$

$\Rightarrow \qquad |\sinh y| \le |\cos z| \qquad$ ------------ (4)

From Eqs.(3) and (4), we conclude that

$|\sinh y| \le |\cos z| \le \cosh y \qquad$ ------------ Q.E.D.

H.W. 4 Find all roots of the equation $\sin z = \cosh 4$.

<Ans.> $z = \left(\frac{\pi}{2} + 2n\pi\right) \pm 4i, \quad n = 0,\ \pm 1,\ \pm 2,\ \cdots$

【本題摘自：James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications*, 6rd ed., Exercise 24, Problem 16, McGraw-Hill, Inc., 2005.】

H.W. 5 Find all roots of the equation $\cos z = 2$.

<Ans.> $z = 2n\pi + i\cosh^{-1} 2$ or $2n\pi \pm i\ln\left(2+\sqrt{3}\right), \quad n = 0,\ \pm 1,\ \pm 2,\ \cdots$

【本題摘自：James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications*, 6rd ed., Exercise 24, Problem 17, McGraw-Hill, Inc., 2005.】

§3-3 The Hyperbolic Functions

1. Recall that when we study in the real variable, the hyperbolic functions have been defined as following:

$$\sinh x = \frac{e^x - e^{-x}}{2}, \text{ its graph is as shown in } \textbf{Fig. (a)}.$$

and $$\cosh x = \frac{e^x + e^{-x}}{2}, \text{ its graph is as shown in } \textbf{Fig. (b)}.$$

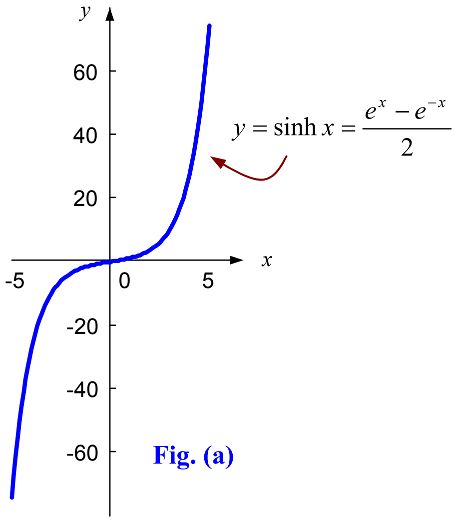


Fig. (a)

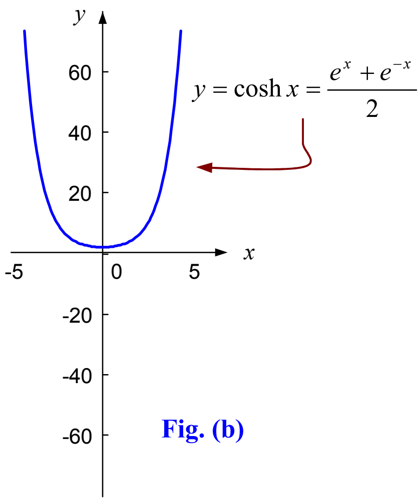


Fig. (b)

2. Given any complex number z, we define

$$\sinh z = \frac{e^z - e^{-z}}{2} \text{ , } \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$\cosh z = \frac{e^z + e^{-z}}{2} \text{ , } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

1) 若取 z 為實數，則上述定義與實變函數中的雙曲線函數之定義可相符合。

2) $\sinh z$ and $\cosh z$ both have a fundamental period of $2\pi i$.

3) $\sinh(-z) = -\sinh z$

$\cosh(-z) = \cosh z$

4) $\cosh^2 z - \sinh^2 z = 1$

5) $$\frac{d}{dz}\sinh z = \frac{d}{dz}\left(\frac{e^z - e^{-z}}{2}\right)$$

$$= \frac{e^z - (-)e^{-z}}{2} = \cosh z$$

$$\frac{d}{dz}\cosh z = \frac{d}{dz}\left(\frac{e^z + e^{-z}}{2}\right)$$

$$= \frac{e^z - e^{-z}}{2} = \sinh z$$

6) If $\sinh z = 0 \Rightarrow z = n\pi i$, $n \in I$.

<pf.> Since $\sinh z = 0$

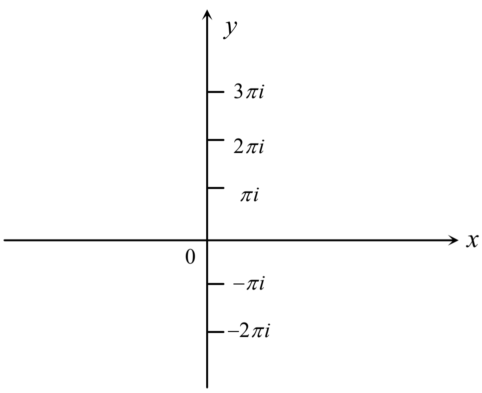
$$\Rightarrow \left(\frac{e^z - e^{-z}}{2}\right) = 0$$

$$\Rightarrow e^z = e^{-z}$$

Then, we have

$$e^{2x+i2y} = 1$$

This means that



$$e^{2x}(\cos 2y + i\sin 2y) = 1$$

利用實部=實部，虛部=虛部，我們可得知

$$\sin 2y = 0 \quad \text{----------} \quad ①$$

and $\quad e^{2x}\cdot\cos 2y = 1 \quad \text{----------} \quad ②$

Since $\quad e^{2x} > 0 \quad \Rightarrow \quad \cos 2y = 1 \quad$ （-1 不合）

From ①, we can obtain that

$$2y = 2n\pi \quad \Rightarrow \quad y = n\pi$$

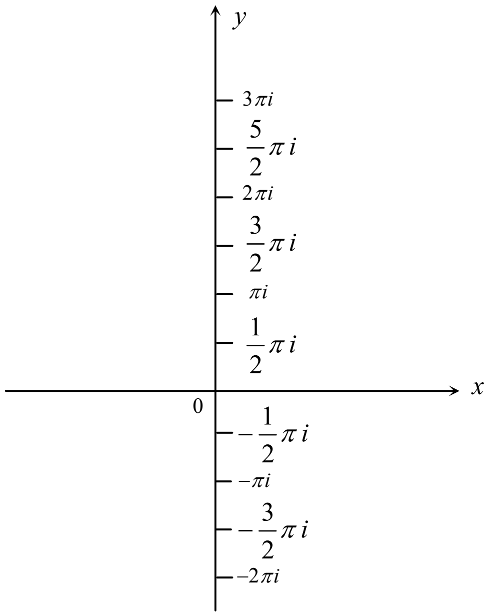
Again, we need

$$e^{2x} = 1 \quad \Rightarrow \quad x = 0$$

So, if $\sinh z = 0$, we conclude that

$$z = x + iy = 0 + in\pi = n\pi i$$

7) If $\quad \cosh z = 0 \quad \Rightarrow \quad z = \left(n + \frac{1}{2}\right)\pi i, \quad n \in I$



<pf.> Since $\cosh z = \dfrac{e^z + e^{-z}}{2} = 0$,

$$\Rightarrow \quad e^z = -e^{-z} \quad \Rightarrow \quad e^{2z} = -1$$

$$\Rightarrow \quad e^{-2x}[\cos 2y + i\sin 2y] = -1$$

$$\Rightarrow \quad \sin 2y = 0$$

Since $\quad e^{2x} > 0 \quad \Rightarrow \quad \cos 2y = -1$

$$\Rightarrow \quad 2y = (2n+1)\pi$$

故 $\quad y = \left(n + \frac{1}{2}\right)\pi, \quad n \in I$.

Again we need

$$e^{2x} = -1 \quad \Rightarrow \quad x = 0$$

So, if $\sinh z = 0$, we conclude that

$$z = x + iy = \left(n + \frac{1}{2}\right)\pi i$$

* 由(6)及(7)知，只有純虛數才能使 $\sinh z$ 和 $\cosh z$ 等於 0。

3. Given the complex number z, we define the other hyperbolic functions as followings:

1) $\tanh z = \dfrac{\sinh z}{\cosh z}, \quad z \neq \left(n + \frac{1}{2}\right)\pi i$.

2) $\coth z = \dfrac{\cosh z}{\sinh z}, \quad z \neq n\pi i$.

3) $\operatorname{sech} z = \dfrac{1}{\cosh z}, \quad z \neq \left(n + \frac{1}{2}\right)\pi i$.

4) $\operatorname{csch} z = \dfrac{1}{\sinh z}, \quad z \neq n\pi i$.

where $n \in I$ in all cases.

5) Both $\tanh z$ and $\coth z$ have a fundamental period of πi.

6) Both $\operatorname{sech} z$ and $\operatorname{csch} z$ have a fundamental period of $2\pi i$.

4. $\tanh z$, $\coth z$, $\operatorname{sech} z$ and $\operatorname{csch} z$ are analytic functions of z，其中 z 值不可為上述重點 3 所限制的各不允許值，則

1) $\dfrac{d}{dz}(\tanh z) = \operatorname{sech}^2 z, \quad z \neq \left(n + \frac{1}{2}\right)\pi i$

2) $\dfrac{d}{dz}(\coth z) = -\operatorname{csch}^2 z, \quad z \neq n\pi i$

3) $\dfrac{d}{dz}(\operatorname{sech} z) = \operatorname{sech} z \tanh z$， $z \neq \left(n+\dfrac{1}{2}\right)\pi i$

4) $\dfrac{d}{dz}(\operatorname{csch} z) = -\operatorname{csch} z \coth z$， $z \neq n\pi i$

where $n \in I$ in all cases.

5. If $z = x + iy$, then

1) $\sinh z = \cos y \sinh x + i \sin y \cosh x$

2) $\cosh z = \cos y \cosh y + i \sin y \sinh x$

<pf.> 1)

$$\begin{aligned}\sinh z &= \frac{e^z - e^{-z}}{2} \\ &= \frac{e^{(x+iy)} - e^{-(x+iy)}}{2} \\ &= \frac{1}{2}[e^x(\cos x + i\sin y) - e^{-x}(\cos y - i\sin y)] \\ &= \cos y \frac{e^x - e^{-x}}{2} + i\sin y \frac{e^x + e^{-x}}{2} \\ &= \cos y \sinh x + i \sin y \cosh x\end{aligned}$$

2) $\cosh z$ 同理可證，亦可仿 §3-2 3.之 2) 證之。

6. If $z = x + iy$, then

a) $\sinh(iz) = i\sin z$ ， $\sin(iz) = i\sinh z$

b) $\cosh(iz) = \cos z$ ， $\cos(iz) = \cosh z$

c) $\sinh \overline{z} = \overline{\sinh z}$ ， $\cosh \overline{z} = \overline{\cosh z}$

d) $|\sinh z|^2 = \sin^2 y + \sinh^2 x$

$|\cosh z|^2 = \cos^2 y + \sinh^2 x$

H.W. 1 (a) Where on the line $x = y$ is the equation $\sin z + i\sinh z = 0$ satisfied?

(b) Using MATLAB, obtain a three-dimensional plot of $|\sin z + i\sinh z|$ and verify that the surface obtained has zero height at points in Part (a). Include $z = 0$ and at least one other solution, on the line, of the given equation.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.3, Problem 20, Pearson Education, Inc., 2005.】

<Ans.>

(a) Since $\sin z + i\sinh z = 0$, we have

$\sin x \cosh y + i\cos x \sinh y + i[\sinh x \cos y + i\cosh x \sin y] = 0$ ---------- (A)

Using Re = Re and Im =Im, gives

$\sin x \cosh y - \cosh x \sin y = 0$

Put $x = y$ in the preceding equation, yields

$\sin x \cosh x = \cosh x \sin x$ ---------- ***satisfied!***

Equating the imaginary part in Eq.(A), gives

$\cos x \sinh y + \sinh x \cos y = 0$

Put $x = y$ in the preceding equation, yields

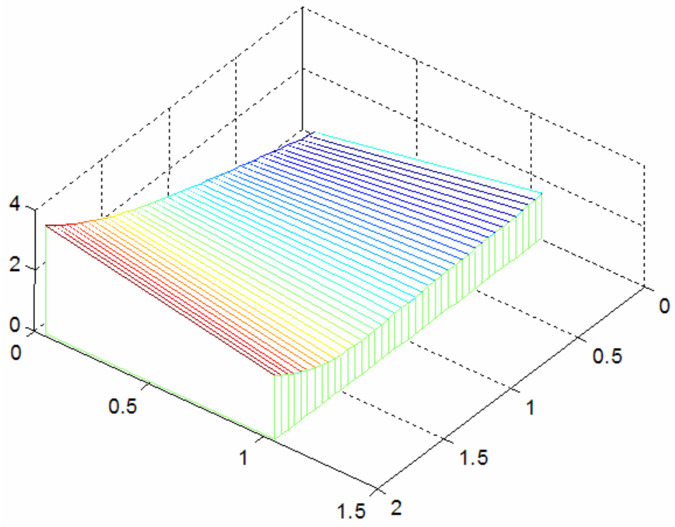
$2\sinh x \cos x = 0$

$\Rightarrow$ $x = 0$ or $x = \pm\left[n\pi + \frac{\pi}{2}\right]$, $n = 0, 1, 2, 3, \cdots$. Also, $y = x$ in this case.

(b) MATLAB commands:

```
x=[0:0.05:2];
y=[0:0.05:2] ;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
```

```
w=sin(Z)+i*sinh(Z);
wm=abs(w);
meshz(X,Y,wm);view(150,70)
```



H.W. 2 Find all roots of the equation (a) $\cosh z = \frac{1}{2}$, (b) $\sinh z = i$, and (c) $\cosh z = -2$.

【本題摘自：James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications*, 6rd ed., Exercise 25, Problem 14, Pearson Education, Inc., 2005.】

<Ans.>

(a) $\left(2n \pm \frac{1}{3}\right)\pi i, \quad n = 0, \pm 1, \pm 2, \cdots$

(b) $\left(2n + \frac{1}{2}\right)\pi i, \quad n = 0, \pm 1, \pm 2, \cdots$

(c) $\pm \ln\left(2+\sqrt{3}\right) + (2n+1)\pi i, \quad n = 0, \pm 1, \pm 2, \cdots$

§3-4 The Logarithmic Function

1. For any complex number $z \neq 0$, there exists complex numbers w such that $e^w = z \neq 0$. In particular, one of such w's is the complex number as shown below:

$$w = \ln|z| + i\theta$$

<pf.> Let $z = r e^{i\theta}$, where $|z| = r$.

Hence, we have

$$\begin{aligned} e^w &= e^{\ln|z| + i\theta} \\ &= e^{\ln r + i\theta} \\ &= e^{\ln r} \cdot e^{i\theta} \\ &= r e^{i\theta} \\ &= z \end{aligned}$$

Thus, we can define that

$$\begin{aligned} \ln z &= \ln|z| + i\theta \\ &= \ln|z| + i \arg z \end{aligned}$$

1) Define $\operatorname{Ln} z = \ln|z| + i\theta$, $-\pi < \theta \leq \pi$.

$\Rightarrow$ $\operatorname{Ln} z$ is the principal value of $\ln z$ so that the $\operatorname{Ln} z$ is a single-valued function and $\ln z$ is a multiple-valued function.

Hence,

$$\begin{aligned} \ln z &= \operatorname{Ln}|z| + 2n\pi i, \\ &= \ln|z| + i\theta + 2n\pi i, \end{aligned} \quad n \in I.$$

2) If $x < 0$，$y = 0$ $\Rightarrow$ $\arg z$ is not continuous.

Since

$$\begin{cases} \lim\limits_{y \to 0^+} \arg z = \pi \\ \lim\limits_{y \to 0^-} \arg z = -\pi \end{cases}$$

$\Rightarrow$ $\lim\limits_{y \to 0} \arg z$ does not exist.

$\Rightarrow$ $\arg z$ is not continuous for $x < 0$，$y = 0$.

$\Rightarrow$ $f(z)$ is not analytic for $x < 0$，$y = 0$.

3) The function $\operatorname{Ln} z$ is analytic in the domain D consisting of all point of complex plane except those lying on the negative real axis.

$$f(z) = \operatorname{Ln} z = \ln|z| + i\theta$$

$$\Rightarrow \quad f'(z) = \frac{1}{z}$$

<pf.> We can use two ways to show the above theorem.

i) Rectangular coordinate method：

Suppose $f(z) = \operatorname{Ln} z = \ln|z| + i\theta$,

and let $f(z) = u(x, y) + i\,v(x, y)$

$$\Rightarrow \begin{cases} u = \ln|z| = \ln\sqrt{x^2 + y^2} \\ v = \theta = \tan^{-1}\dfrac{y}{x} \end{cases}$$

Note that $f(z)$ is analytic in the domain D consisting of all point of complex plane except for those lying on the negative real axis, then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{1}{\sqrt{x^2 + y^2}} \cdot \frac{2x}{2\sqrt{x^2 + y^2}} + i\frac{-\dfrac{y}{x^2}}{1 + \left(\dfrac{y}{x}\right)^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$
$$= \frac{x-iy}{x^2+y^2} = \frac{\bar{z}}{z\cdot\bar{z}} = \frac{1}{z}\#$$

ii) Polar coordinate method：

Let $w = f(z)$ and $z = x+iy = r(\cos\theta + i\sin\theta)$, then

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

In Chapter 2, we have derived the formula as shown below:

$$\frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$

Note that

$$w = f(z) = \mathrm{Ln}\, z = \ln|z| + i\theta$$
$$= \ln r + i\theta$$

$$\Rightarrow \quad \frac{dw}{dz} = (\cos\theta - i\sin\theta)\frac{\partial w}{\partial r}$$
$$= (\cos\theta - i\sin\theta)\cdot\frac{1}{r}$$
$$= \frac{1}{r\cdot(\cos\theta + i\sin\theta)} \qquad \text{-----------} \qquad \text{Q.E.D.}$$
$$= \frac{1}{z}$$

2. In the real variable analysis, we know that if $x_1 > 0$ and $x_2 > 0$, then

$$\ln x_1 \cdot x_2 = \ln x_1 + \ln x_2$$

However, in the complex variable analysis, the following

$$\mathrm{Ln}\, z_1 \cdot z_2 = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2$$

may be not true.

For example:

Let $z_1 = i$, $z_2 = -1+i$

then $z_1 \cdot z_2 = -1-i$

First, we have

$$\mathrm{Ln}\, z_1 = \mathrm{Ln}\, i$$
$$= \ln|i| + \frac{\pi}{2}i$$
$$= \frac{\pi}{2}i$$

and $\mathrm{Ln}\, z_2 = \mathrm{Ln}(-1+i)$

$$= \ln|-1+i| + \frac{3}{4}\pi i$$
$$= \ln\sqrt{2} + \frac{3}{4}\pi i$$

But $\mathrm{Ln}(z_1 \cdot z_2) = \mathrm{Ln}(-1-i)$

$$= \ln|-1-i| + (-\frac{3}{4}\pi)i$$
$$= \ln\sqrt{2} - \frac{3}{4}\pi i$$

Hence, $\mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 = \frac{\pi}{2}i + \ln\sqrt{2} + \frac{3}{4}\pi i$

$$= \ln\sqrt{2} + \frac{5}{4}\pi i$$

$$\neq \mathrm{Ln}(z_1 \cdot z_2) \quad \#$$

♣ **But, the following theorems are still satisfied.**

3. If the complex numbers z_1, z_2, z_3 are different from zero, then the principal values of the arguments and logarithms of the product, quotient, and powers among these complex numbers are given by

1) $\mathrm{Arg}(z_1 \cdot z_2) = \arg z_1 + \arg z_1 + 2\pi\, n_1(z_1, z_2) \quad -(a)$

$\mathrm{Log}(z_1 \cdot z_2) = \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 + 2\pi\, i\, n_1(z_1, z_2) \quad -(b)$

where n_1 is assumed to be the values of $-1,\ 0,\ 1$ as following：

$$n_1(z_1, z_2) = \begin{cases} -1, & \text{if} \quad \pi < \arg z_1 + \arg z_2 \le 2\pi \\ 0, & \text{if} \quad -\pi < \arg z_1 + \arg z_2 \le \pi \\ 1, & \text{if} \quad -2\pi < \arg z_1 + \arg z_2 \le -\pi \end{cases}$$

<pf.> Let $z_1 = |z_1| e^{i\theta_1}$ and $z_2 = |z_2| e^{i\theta_2}$,

where $-\pi < \theta_1 \le \pi$ ， $-\pi < \theta_2 \le \pi$

This gives

$$-2\pi < \theta_1 + \theta_2 \le 2\pi$$

Note that

$$z_1 \cdot z_2 = |z_1||z_2| e^{i(\theta_1+\theta_2)}$$
$$= |z_1||z_2| e^{i(\theta_1+\theta_2+2n_1\pi)}$$

where n_1 is an integer such that

$$-\pi < \theta_1 + \theta_2 + 2n_1\pi \le \pi \quad \text{-----------} \quad (1)$$

i) Suppose $\pi < \theta_1 + \theta_2 \le 2\pi$, then

$$-\pi < \theta_1 + \theta_2 - 2\pi \le 0$$

Thus, $-\pi < \theta_1 + \theta_2 - 2\pi \le \pi$

Comparing this inequality with equation (1), we obtain

$$n_1(z_1, z_2) = -1$$

ii) Suppose $-\pi < \theta_1 + \theta_2 \le \pi$, comparing this inequality with Eq. (1), gives

$$n_1(z_1, z_2) = 0$$

iii) Suppose $-2\pi < \theta_1 + \theta_2 \le -\pi$, we have

$$0 < \theta_1 + \theta_2 + 2\pi \le \pi$$

So, $-\pi < \theta_1 + \theta_2 + 2\pi \le \pi$

Comparing this inequality with Eq. (1), we have

$$n_1(z_1, z_2) = 1 \quad \# \quad \text{已證得}(a)$$

Let $z = z_1 \cdot z_2$, then

$$\begin{aligned} \mathrm{Ln}\, z &= \mathrm{Ln}\, z_1 \cdot z_2 \\ &= \ln|z_1 \cdot z_2| + i\arg(z_1 \cdot z_2) \\ &= \ln[|z_1| \cdot |z_2|] + i[\arg z_1 + \arg z_2 + 2\pi\, n_1(z_1 \cdot z_2) \\ &= \ln|z_1| + \ln|z_2| + i\arg z_1 + i\arg z_2 + 2\pi\, i n_1(z_1 \cdot z_2) \\ &= [\ln|z_1| + i\arg z_1] + [\ln|z_2| + i\arg z_2] + 2\pi\, i n_1(z_1 \cdot z_2) \\ &= \mathrm{Ln}\, z_1 + \mathrm{Ln}\, z_2 + 2\pi\, i n_1(z_1 \cdot z_2) \end{aligned}$$

2) 基本自然對數函數尚具有底下之各項性質：

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2 + 2\pi n_2(z_1, z_2)$$

$$\mathrm{Ln}\left(\frac{z_1}{z_2}\right) = \mathrm{Ln}\, z_1 - \mathrm{Ln}\, z_2 + 2\pi i n_2(z_1, z_2)$$

where n_2 is assumed to be the values of $-1,\ 0,\ 1$ as followings：

$$n_2(z_1,z_2)=\begin{cases}-1, & \text{if} \quad \pi<\arg z_1-\arg z_2\le 2\pi \\ 0, & \text{if} \quad -\pi<\arg z_1-\arg z_2\le \pi \\ 1, & \text{if} \quad -2\pi<\arg z_1-\arg z_2\le -\pi\end{cases}$$

3) $\arg(z^n)=n\cdot\arg z+2\pi k(z,n)$

$\mathrm{Ln}(z^n)=n\cdot\mathrm{Ln}\,z+2\pi i\,k(z,n)$

where n is any integer, and k is the integer given by the ***bracket function***:

$$k(z,n)=\left[\frac{1}{2}-\frac{n}{2\pi}\arg(z)\right]$$

4. If $z\neq 0$ and w is any complex number, we define

$$z^w=e^{\ln z^w}=e^{w\ln z}$$

where $e^{w\ln z}$ is a multiple-valued function.

That is,

$$e^{w\ln z}=e^{w[\ln|z|+i\arg z+2n\pi i]}$$

where $n=0,\ \pm1,\ \pm2,\cdots$. This means that $e^{w\ln z}$ has infinite number of computed values.

Here, we define the principal value of z^w as following：

$z^w=e^{w\mathrm{Ln}\,z}$ ---------- is called single-valued function.

1) If $z_1\neq 0$, $z_2\neq 0$, and w is any complex number, then

$$(z_1\cdot z_2)^w=z_1{}^w\cdot z_2{}^w\cdot e^{2\pi iwn_1(z_1,z_2)} \quad \text{----------} \quad (1)$$

$$\left[\frac{z_1}{z_2}\right]^w=\frac{z_1{}^w}{z_2{}^w}e^{2\pi iwn_1(z_1,z_2)} \quad \text{-----------} \quad (2)$$

where $n_1(z_1,z_2)$ and $n_2(z_1,z_2)$ are the integers defined in preceding discussions.

<pf.> Use the definitions and theorems that we have discussed：

$$\Rightarrow \quad (z_1z_2)^w=e^{w_1\mathrm{Ln}(z_1z_2)}$$
$$=e^{w[\mathrm{Ln}\,z_1+\mathrm{Ln}\,z_2+2\pi in_1(z_1,z_2)]}$$
$$=e^{w\mathrm{Ln}\,z_1}\cdot e^{w\mathrm{Ln}\,z_2}\cdot e^{2\pi iwn_1(z_1,z_2)}$$
$$=z_1{}^w\cdot z_2{}^w\cdot e^{2\pi iwn_1(z_1,z_2)}$$

Example 1 Find the principal value of i^i.

<Sol.>

$$i^i=e^{i\mathrm{Ln}\,i}=e^{i(\ln|i|+\frac{\pi}{2}i)}$$
$$=e^{i\left(\frac{\pi}{2}i\right)}$$
$$=e^{-\frac{\pi}{2}}\in R$$

Example 2 Find the principal value of $(-1+i)^i$.

<Sol.>

$$(-1+i)^i=e^{i\,\mathrm{Ln}(-1+i)}$$
$$=e^{i(\ln\sqrt{2}+3\pi i/4)}$$
$$=e^{-3\pi/4+i\ln\sqrt{2}}$$
$$=e^{-3\pi/4}[\cos\ln\sqrt{2}+i\sin\ln\sqrt{2}]$$
$$=e^{-3\pi/4}\cos\ln\sqrt{2}+ie^{-3\pi/4}\sin\ln\sqrt{2}$$

Example 3 Find $\mathrm{Ln}(\sqrt{2}+i\sqrt{2})$.

<Sol.>

$$\begin{aligned}&\mathrm{Ln}(\sqrt{2}+i\sqrt{2})\\&=\ln|\sqrt{2}+i\sqrt{2}|+i\arg(\sqrt{2}+i\sqrt{2})\\&=\ln 2+i\frac{\pi}{4}\end{aligned}$$

Example 4 Find the principal value of $\left[\frac{e}{2}(-1-i\sqrt{3})\right]^{3\pi i}$.

<Sol.>

$$\begin{aligned}&\left[\frac{e}{2}(-1-i\sqrt{3})\right]^{3\pi i}\\&=\left[e(\frac{-1-i\sqrt{3}}{2})\right]^{3\pi i}\\&=e^{3\pi i}\left(\frac{-1-i\sqrt{3}}{2}\right)^{3\pi i}\\&=[\cos 3\pi+i\sin 3\pi]e^{3\pi i\left[\ln\left|\frac{-1-i\sqrt{3}}{2}\right|+i\arg\left(\frac{-1-i\sqrt{3}}{2}\right)\right]}\\&=(-1)\cdot e^{3\pi i\left[\ln 1+i\left(-\frac{2\pi}{3}\right)\right]}\\&=-e^{3\pi i\left(-\frac{2\pi}{3}i\right)}\\&=-e^{2\pi^2}\end{aligned}$$

5. If $z\neq 0$, w and λ are any complex number, then

$$\mathrm{Ln}(z^w)=w\,\mathrm{Ln}\,z+2\pi ik \quad ----------- \quad (1)$$

$$(z^w)^\lambda=z^{w\lambda}\cdot e^{2\pi i\lambda k} \quad ----------- \quad (2)$$

where k is the integer given by bracket function

$$k=\left[\frac{1}{2}-\frac{\mathrm{Im}(w)\ln|z|+\mathrm{Re}(w)\,\mathrm{Arg}\,z}{2\pi}\right]$$

<pf.> Denote z^w by α, $\arg z$ by θ, and let $w=u+iv$, then

$$\begin{aligned}\alpha=z^w&=e^{w\mathrm{Ln}z}\\&=e^{[u+iv][\ln|z|+i\theta]}\\&=e^{u\ln|z|-v\theta}\cdot e^{i[v\ln|z|+u\theta]}\end{aligned}$$

Thus,

$$\begin{cases}|\alpha|=e^{u\ln|z|-v\theta}\\ \arg z=v\ln|z|+u\theta+2\pi k\end{cases} \quad ------------ \quad (3)$$

where k is the integer such that

$$-\pi<v\ln|z|+u\theta+2\pi k\leq\pi \quad ------------ \quad (4)$$

Solving for k in Eq. (4), we obtain

$$t<k\leq t+1$$

where

$$t=-\frac{1}{2}-\frac{v\,\mathrm{Ln}|z|+u\theta}{2\pi}$$

Therefore,

$$k=[t+1]=\left[\frac{1}{2}-\frac{v\,\mathrm{Ln}\,|z|+u\theta}{2\pi}\right]$$

$$=\left[\frac{1}{2}-\frac{\mathrm{Im}(w)\,\mathrm{Ln}\,|z|+\mathrm{Re}(w)\arg z}{2\pi}\right]$$

Now, using Eq. (3), we see that

$$\mathrm{Ln}\,(z^w)=\mathrm{Ln}\,\alpha$$
$$=\ln|\alpha|+i\arg\alpha$$
$$=\mathrm{Ln}[e^{u\ln|z|-v\theta}]+i[v\ln|z|+u\theta+2k\pi]$$
$$=u\ln|z|-v\theta+i[v\ln|z|+u\theta+2k\pi]$$
$$=(u+iv)(\ln|z|+i\theta)+2\pi ik$$
$$=w\,\mathrm{Ln}\,z+2\pi ik$$

故(1)式已被證得。
同理，可證得(2)式。

H.W. 1 Consider the identity $\ln z^n=n\ln z$, where n is an integer, which is valid for appropriate choice of the logarithms on each side of the equation. Let $z=1+i$ and $n=5$.

(a) Find the values of $\ln z^n$ and $\ln z$ that satisfy $\ln z^n=n\ln z$.

(b) For the given z and n, is $\ln z^n=n\ln z$ satisfied?

(c) Suppose $n=2$ and **z** is unchanged. Is $\ln z^n=n\ln z$ then satisfied?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3.4, Problem 26, Pearson Education, Inc., 2005.】

<Ans.>

(a) $5\ln z=\ln z^5=5\left[\mathrm{Ln}\sqrt{2}+i\frac{\pi}{4}\right]$

(b) No.

(c) Yes.

6. **Definition (Branch)：**

A *branch* of a multivalued function is a single-valued function ***analytic*** in some domain. At every point of the domain, the single-valued function must assume exactly one of the various possible values that the multivalued function can assume.

Definition (Branch cut)

A line used to create a domain of analyticity is called a *branch line* or *branch cut*.

Definition (Branch point)

Any point that must lie on a branch cut — no matter what branch used — is called a *branch point* of a multivalued function.

Ex. (1) $\mathrm{Ln}\,z\equiv$ principal value of $\ln z$, which is defined for all z except $z=0$.

(2) $\mathrm{Ln}\,z$ is also used to denote the principal branch of the logarithm function, which is defined for all z except $z=0$ and values of z on the negative real axis.

(3) $f(z)=\mathrm{Ln}\,r+i\theta$, where $-3\pi/2<\theta\le\pi/2$.

$\Rightarrow$ It is discontinuous at the origin and at all points on the positive imaginary axis.

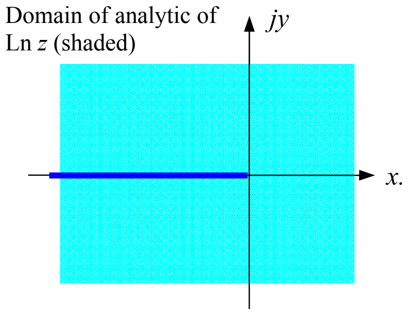
(4) $f(z)=\mathrm{Ln}\,r+i\theta$, where $-3\pi/2+2k\pi<\theta\le\pi/2+2k\pi$, $k=0,\ \pm1,\ \pm2,\ \cdots$, are, for each k, analytic branches, provided z is confined to the domain D_1.

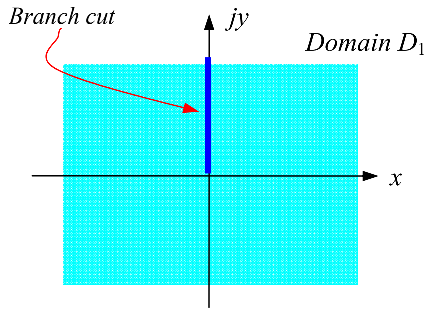
Example 5 (a) Find the largest domain of analyticity of $f(z)=\mathrm{Ln}[z-(3+4i)]$.

(b) Find the numerical value of $f(0)$.

<Sol.>

(a) For the given function, the non-analytic points are located at：





$$\operatorname{Im} w = 0, \quad \operatorname{Re} w \le 0$$

If $w = z - (3+4i)$, these two conditions can be rewritten as

$$\operatorname{Im}\left[(x+iy)-(3+4i)\right] = 0 \quad \Rightarrow \quad y = 4$$

$$\operatorname{Re}\left[(x+iy)-(3+4i)\right] \le 0 \quad \Rightarrow \quad x \le 3$$

So, the full domain of analyticity is shown in the following **Fig. (a)**.

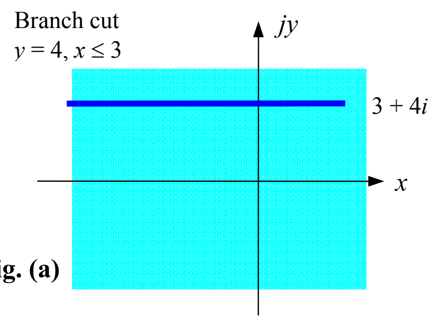


Fig. (a)

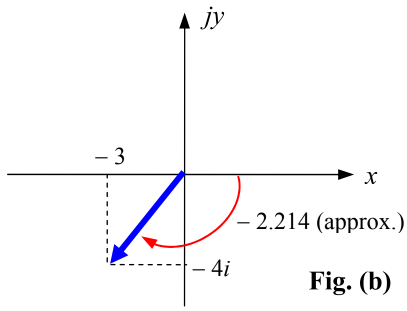


Fig. (b)

(b) $f(0) = \operatorname{Ln}(-3-4i) = \operatorname{Ln}(5) + i\arg(-3-4i)$, where $-\pi < \arg(-3-4i) < \pi$.

$\Rightarrow \quad f(0) = \operatorname{Ln}(5) - i2.214$

$\Rightarrow$ **Fig. (b)**.

H.W. 2 (a) Show that $\operatorname{Ln}\left(\operatorname{Ln}(z)\right)$ is analytic in the domain consisting of the z-plane with a branch cut along the line $y = 0, \quad x \le 1$. (See below)

(b) Find $d\operatorname{Ln}\left(\operatorname{Ln}(z)\right)/dz$ within the domain of analyticity found in part (a).

(c) What branch cut should be used to create the maximum domain of analyticity for $\operatorname{Ln}\left[\operatorname{Ln}\left(\operatorname{Ln}(z)\right)\right]$?

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3-5, Problem 15, Pearson Education, Inc., 2005.】

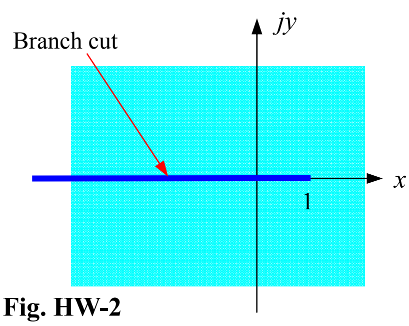


Fig. HW-2

H.W. 3 The complex electrostatic potential $\Phi(x,y) = \phi + i\psi = \operatorname{Ln}(1/z)$, where $z \ne 0$, can be created by an electric line charge located at $z = 0$ and lying perpendicular to the *xy*-plane.

(a) Sketch the streamlines for this potential.

(b) Sketch the equipotentials for $\phi = -1, \ 0, \ 1, \ 2$.

(c) Find the components of the electric field at an arbitrary point (x, y).

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3-5, Problem 16, Pearson Education, Inc., 2005.】

<Ans.>

(a) Since $\Phi(x,y)=\phi+i\psi=\mathrm{Ln}(1/z)=-\mathrm{Ln}\,z$

$\Rightarrow\quad \phi(x,y)=-\mathrm{Ln}\left(\sqrt{x^2+y^2}\right),\ \psi(x,y)=-\tan^{-1}\left(\frac{y}{x}\right)$, where ψ 's are ramp emanating from origin.

(b) Since $-1=-\mathrm{Ln}\left(\sqrt{x^2+y^2}\right)=\phi=-1$, therefore

$$\sqrt{x^2+y^2}=e \quad \text{if} \quad \phi=-1$$

Also, when $0=-\mathrm{Ln}\left(\sqrt{x^2+y^2}\right)=\phi=0$, therefore

$$\sqrt{x^2+y^2}=1 \quad \text{if} \quad \phi=0$$

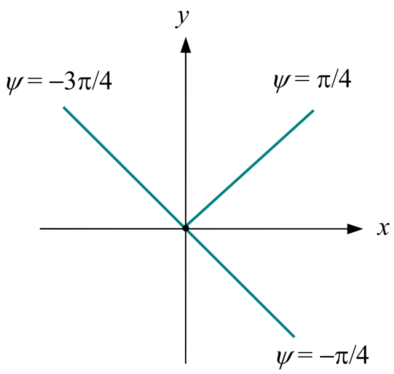
When $1=-\mathrm{Ln}\left(\sqrt{x^2+y^2}\right)=\phi=1$, therefore

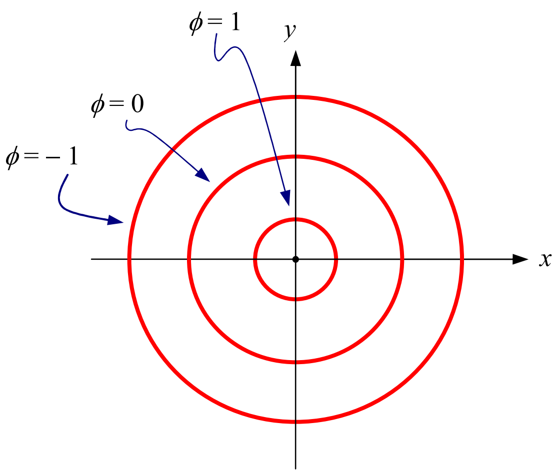
$$\sqrt{x^2+y^2}=1/e \quad \text{if} \quad \phi=1$$

The electric field is

$$E_x+iE_y=-\overline{\left(\frac{d\Phi}{dz}\right)}$$

$$\Rightarrow\quad E_x=\frac{x}{x^2+y^2} \quad \text{and} \quad E_y=\frac{y}{x^2+y^2}$$





(c) MATLAB commands:

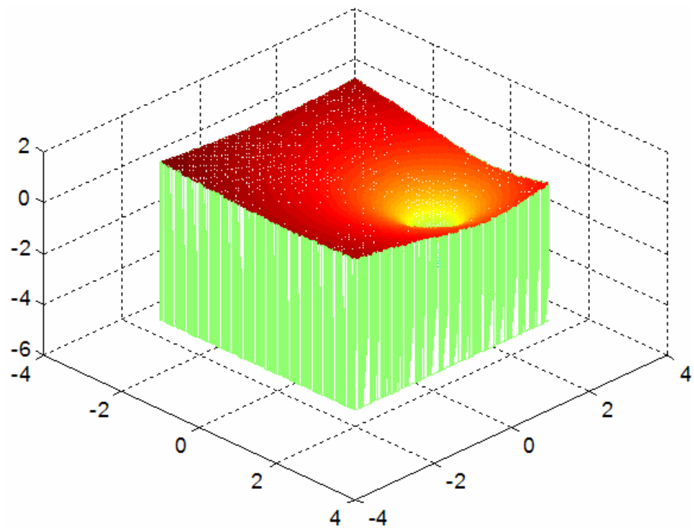
```
% HW (c)
x=[-2.5:0.02+0.001:2.5];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=log(Z-1-1*i);
%wm=real(w);  %   for real part
wm=imag(w); % for imag part
Meshz(X,Y,wm); % for imag part and real
View(45,45);
```

H.W. 4 Let $f(z)=10^{(z^3)}$. This function is evaluated such that $f'(z)$ is real when $z=1$. Find $f'(1+i)$. Where in the complex plane is $f(z)$ analytic?

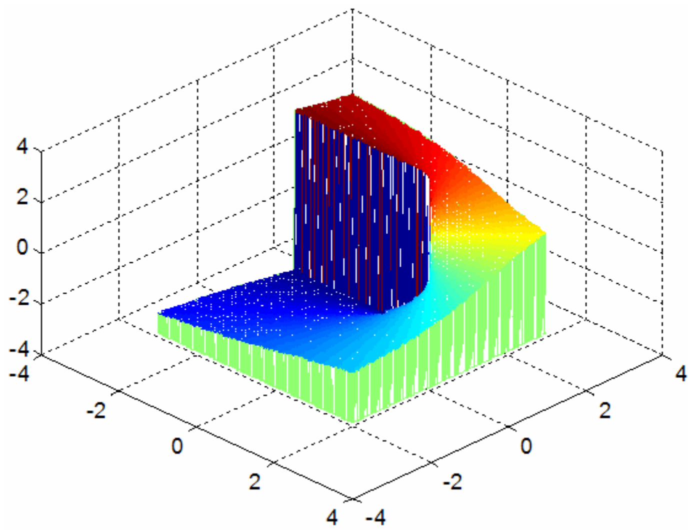
【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3-6, Problem 24, Pearson Education, Inc., 2005.】

<Ans.>

$f'(1+i) = 0.137 - i0.0148$; $f(z)$ is an entire function.



(a) real part



(b) imaginary part

H.W. 5 Let $f(z) = 10^{(e^z)}$. This function is evaluated such that $|f(i\pi/2)| = e^{-2\pi}$. Find $f'(z)$ and $f'(i\pi/2)$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3-6, Problem 24, Pearson Education, Inc., 2005.】

<Ans.>

$f'(z) = e^{[\mathrm{Ln}(10)+i2\pi]e^z}\left[\mathrm{Ln}(10) + i2\pi\right]e^z$ and $f'(i\pi/2) = 0.00464 - i0.01$

§3-5 Inverse Trigonometric and Hyperbolic Functions

1. 1) Let $z = \sin w$, then $\sin^{-1} z = -i \ln\left[zi + \left(1 - z^2\right)^{1/2} \right]$

<pf.> Since $z = \dfrac{e^{iw} - e^{-iw}}{2i}$, assume that $p = e^{iw}$ and $1/p = e^{-iw}$, then we have

$$z = \frac{p - 1/p}{2i}$$

$$\Rightarrow \quad 2izp = p^2 - 1$$

$$\Rightarrow \quad p^2 - 2izp - 1 = 0$$

Solve this equation for p:

$$p = zi + \left(1 - z^2\right)^{1/2} \quad \text{or} \quad e^{iw} = zi + \left(1 - z^2\right)^{1/2}$$

$$\Rightarrow \quad w = \frac{1}{i} \ln\left[zi + \left(1 - z^2\right)^{1/2} \right]$$

Since $w = \sin^{-1} z$, we have

$$\sin^{-1} z = \frac{1}{i} \ln\left[zi + \left(1 - z^2\right)^{1/2} \right]$$

2) Other inverse trigonometric functions:

$$\cos^{-1} z = \frac{1}{i} \ln\left[z + \left(1 - z^2\right)^{1/2} \right]$$

$$\tan^{-1} z = \frac{i}{2} \ln\left(\frac{i + z}{i - z} \right)$$

2. Inverse of Hyperbolic function

1) $$\sinh^{-1} z = \ln\left[z + \left(z^2 + 1\right)^{1/2} \right]$$

2) $$\cosh^{-1} z = \ln\left[z + \left(z^2 - 1\right)^{1/2} \right]$$

3) $$\tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1 + z}{1 - z} \right)$$

3. Derivative of inverse trigonometric functions:

1) $$\frac{d}{dz} \sin^{-1} z = \frac{d}{dz} \left\{ \frac{1}{i} \ln\left[zi + \left(1 - z^2\right)^{1/2} \right] \right\} = \frac{1}{\left(1 - z^2\right)^{1/2}}$$

2) $$\frac{d}{dz} \cos^{-1} z = \frac{-1}{\left(1 - z^2\right)^{1/2}}$$

3) $$\frac{d}{dz} \tan^{-1} z = \frac{1}{\left(1 + z^2\right)}$$

4. Derivative of inverse hyperbolic functions:

1) $$\frac{d}{dz} \sinh^{-1} z = \frac{1}{\left(1 + z^2\right)^{1/2}}$$

2) $$\frac{d}{dz} \cosh^{-1} z = \frac{1}{\left(z^2 - 1\right)^{1/2}}$$

3) $$\frac{d}{dz}\tanh^{-1} z = \frac{1}{\left(1-z^2\right)}$$

Example 1 Find $\sin^{-1}(1/2)$.

<Sol.>

$$\sin^{-1}(1/2) = \frac{1}{i}\ln\left[\frac{i}{2} + \left(\frac{3}{4}\right)^{1/2}\right]$$

Taking $(3/4)^{1/2} = \sqrt{3}/2$, the above expression becomes

$$\sin^{-1}(1/2) = \frac{1}{i}\ln\left[\frac{i}{2} + \frac{\sqrt{3}}{2}\right] = -i\ln\left(1\angle(\pi/6)\right) = \frac{\pi}{6} + 2k\pi, \quad k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

Taking $(3/4)^{1/2} = -\sqrt{3}/2$, the above expression becomes

$$\sin^{-1}(1/2) = \frac{1}{i}\ln\left[\frac{i}{2} - \frac{\sqrt{3}}{2}\right] = -i\ln\left(1\angle(5\pi/6)\right) = \frac{5\pi}{6} + 2k\pi, \quad k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

Example 2 Find all the numbers whose sine is 2.

<Sol.>

This question is to solve $\sin z = 2$. Thus, we have

$$\sin^{-1}(2) = \frac{1}{i}\ln\left[2i + (-3)^{1/2}\right]$$

Taking $(-3)^{1/2} = +\sqrt{3}i$, the above expression becomes

$$\sin^{-1}(2) = \frac{1}{i}\ln\left[2i + \sqrt{3}i\right] = -i\left[\mathrm{Ln}\left(2+\sqrt{3}\right) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$
$$= \left(\frac{\pi}{2} + 2k\pi\right) - i1.317, \quad k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

Taking $(-3)^{1/2} = -\sqrt{3}i$, the above expression becomes

$$\sin^{-1}(2) = \frac{1}{i}\ln\left[2i - \sqrt{3}i\right] = -i\left[\mathrm{Ln}\left(2-\sqrt{3}\right) + i\left(\frac{\pi}{2} + 2k\pi\right)\right]$$
$$= \left(\frac{\pi}{2} + 2k\pi\right) + i1.317, \quad k = 0,\ \pm 1,\ \pm 2,\ \cdots$$

Example 3 Find all the numbers whose sine is i.

<Sol.>

This question is to solve $\sin z = i$. Thus, we have

$$\sin^{-1}(i) = \frac{1}{i}\ln\left[i^2 + (2)^{1/2}\right] = -i\ln(1 \pm \sqrt{2})$$

Taking $(-3)^{1/2} = +\sqrt{3}i$, the above expression becomes

$$\sin^{-1}(i) = -i\ln\left[-1 \pm \sqrt{2}\right] = \begin{cases} -i\left[\mathrm{Ln}\left(\sqrt{2}-1\right) + i2k\pi\right] = 2k\pi - i\,\mathrm{Ln}\left(\sqrt{2}-1\right) \\ -i\left[\mathrm{Ln}\left(\sqrt{2}+1\right) + i\left(\pi + 2k\pi\right)\right] = \pi + 2k\pi - i\,\mathrm{Ln}\left(\sqrt{2}+1\right) \end{cases}$$

where $k = 0,\ \pm 1,\ \pm 2,\ \cdots$.

H.W. 1 Show that $\sin^{-1}(i) = n\pi + i(-1)^{n+1}\ln\left(1+\sqrt{2}\right)$.

【本題摘自：James Ward Brown and Ruel V. Churchill, *Complex Variables and Applications*, 6rd ed., Section 29, Example 1, McGraw-Hill, Inc., 2005.】

H.W. 2 Show that $\tanh^{-1}\left(e^{i\theta}\right) = (1/2)\ln\left[i\cot\left(\theta/2\right)\right]$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 3-7, Problem 16,

Pearson Education, Inc., 2005.】