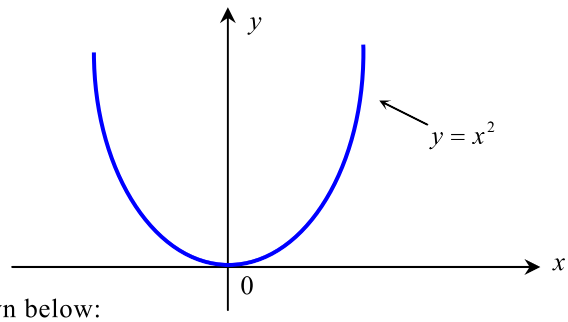
CHAPTER TWO

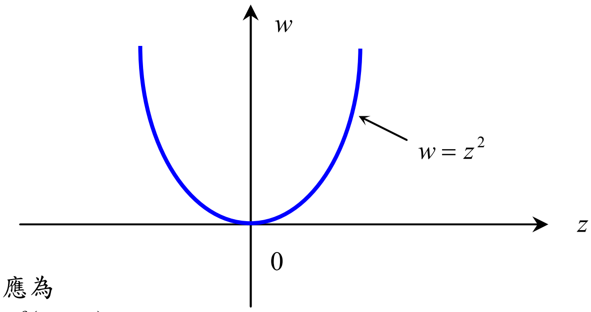
Analytic Function of Complex Variables

§2-1 Limits

1. If x and y are real variables, then
   $z = x + iy$ is a complex variable.
   And, $w = f(z)$ is a function of the complex variable of z.
   For example, $y = f(x) = x^2$, its graph is



But $w = f(z) = z^2$, its graph is not as shown below:



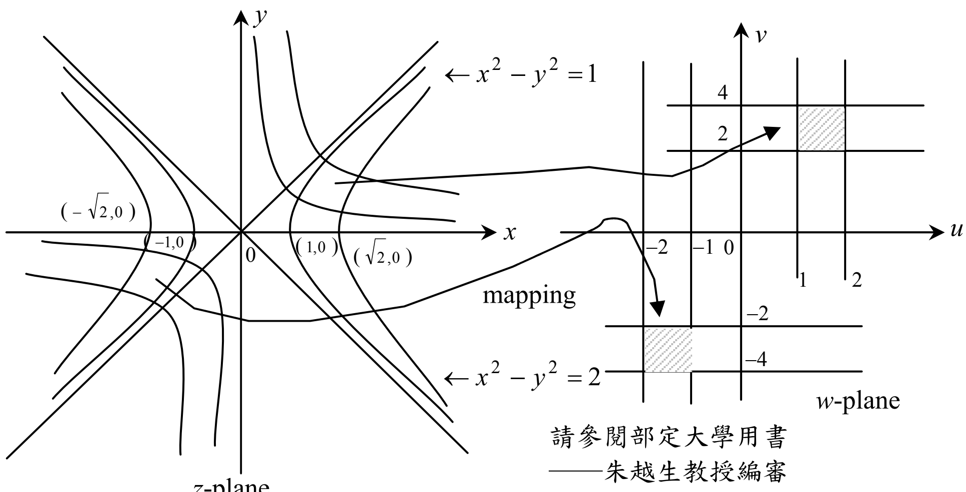
反而是函數關係應為

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

如上例所示：

$$\begin{aligned} w = f(z) &= z^2 \\ &= (x+iy)^2 \\ &= x^2 - y^2 + i2xy \end{aligned}$$

Hence, $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$，故其函數 graph 之相互映像應為



請參閱部定大學用書
——朱越生教授編審
p.1391~p.1392

♣ **Some useful Identities:**

1) $x = \dfrac{z+\bar{z}}{2}$, $y = \dfrac{1}{i}\dfrac{\left(z-\bar{z}\right)}{2}$

2) $z\bar{z} = x^2 + y^2$

Example 1 Express w directly in terms of z if $w(z) = 2x + iy + \dfrac{x-iy}{x^2+y^2}$.

<Sol.>

$$w(z) = \left(z+\bar{z}\right) + \frac{i\left(z+\bar{z}\right)}{i2} + \frac{\bar{z}}{z\bar{z}} = \frac{3z}{2} + \frac{\bar{z}}{2} + \frac{1}{z}$$

Example 2 With $z = x+iy$, and $w = i/z$, find the real functions $u(x,\ y)$ and $v(x,\ y)$ if $w(z) = u(x,\ y) + iv(x,\ y)$.

<Sol.>

$$u(x,\ y) = \mathrm{Re}(w) = \frac{y}{x^2+y^2} \quad \text{and} \quad v(x,\ y) = \mathrm{Im}(w) = \frac{x}{x^2+y^2}$$

H.W. 1 Using MATLAB obtain three-dimensional plots of $|f(z)|^2$, $\mathrm{Re}\left(f(z)\right)$, and $\mathrm{Im}\left(f(z)\right)$ for the function $f(z) = \dfrac{1}{(z-3i/2)}$ and allow z to assume values over a grid in the region of the complex plane defined by $-1 \le x \le 1$, $-1 \le y \le 1$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.1, Prob. 27, Pearson Education, Inc., 2005.】

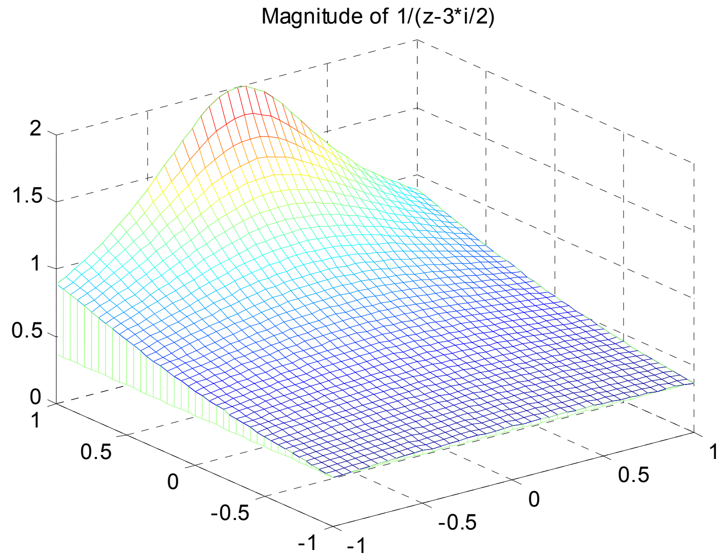
<Ans.>

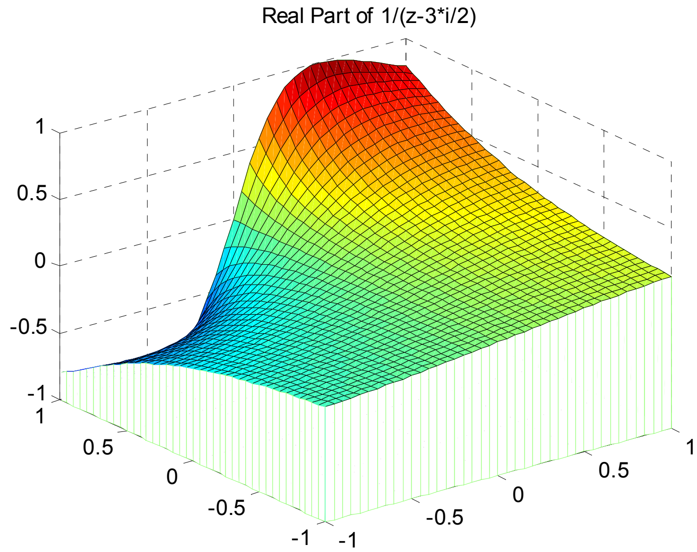
MATLAB Commands:

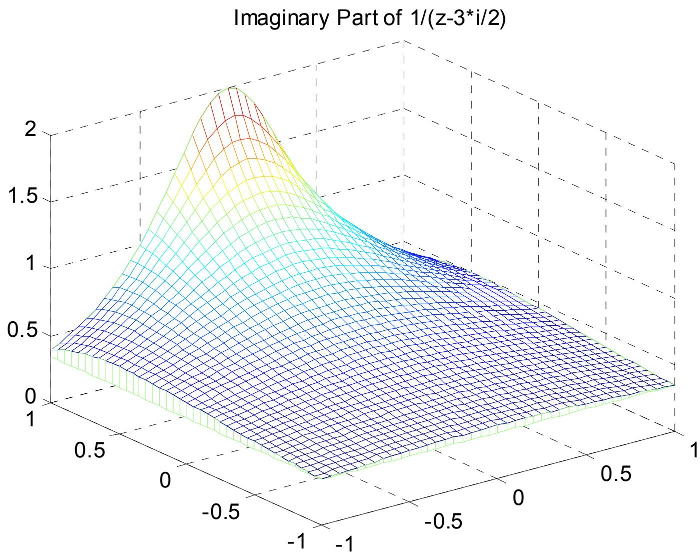
```
% H.W. 1, (a)
x=[-1:0.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./(Z-3*i/2);
wm=abs(w);
meshz(X,Y,wm);hold on
title('Magnitude of 1/(z-3*i/2)')

% H.W. 1, (b)
x=[-1:0.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./(Z-3*i/2);
wm=real(w);
meshz(X,Y,wm);hold on
surf(X,Y,wm)
title('Real Part of 1/(z-3*i/2)')

% H.W. 1, (c)
x=[-1:0.05:1];
y=x;
[X,Y]=meshgrid(x,y);
Z=X+i*Y;
w=1./(Z-3*i/2);
wm=imag(w);
meshz(X,Y,wm);hold on
title('Imaginary Part of 1/(z-3*i/2)')
```


Magnitude of 1/(z-3*i/2)
2
1.5
1
0.5
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1


Real Part of 1/(z-3*i/2)
1
0.5
0
-0.5
-1
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1


Imaginary Part of 1/(z-3*i/2)
2
1.5
1
0.5
0
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1

2. **Definition of Limit**

Let function f be defined in a domain D except at point z_0 in D.
We say that the limit of function f equals to ℓ as z approaches z_0, and we write

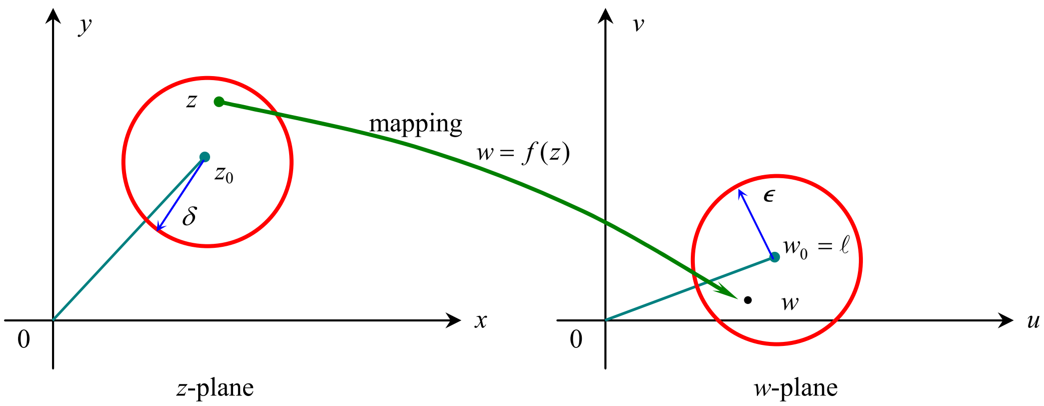
$$\lim_{z \to z_0} f(z) = \ell$$

$\Leftrightarrow$ For every $\epsilon > 0$, there exists a number $\delta > 0$, such that for all z satisfying

$$0 < |z - z_0| < \delta \qquad \text{(A)}$$

$$\Rightarrow \quad |f(z) - \ell| < \epsilon \qquad \text{(B)}$$

此項極限 ℓ 必須與 z 漸近於 z_0 之方式無關。一般，單值函數若有極限存在，則必為唯一之極限。但因 $f(z)$ 為多值函數，則當 $z \to z_0$ 時，該函數之極限值當隨所處之分支而定。極限定義之幾何含義如下圖所示。



在 w-plane 內以 w_0 點為圓心，而取任意指定之微小正數ϵ為半徑而畫一圓，則在 z-plane 內恒可求出以 z_0 為中心，δ為半徑之另一圓，而使該圓內之各點 z（除了 z_0）其像恒落入 w-plane 之圓內者，則稱

$$\lim_{z \to z_0} f(z) = \ell = w_0$$

與函數在 z_0 點之值 $f(z_0)$，其含義並不相同，自不必相等也。

♣ If $\lim_{z \to z_0} f(z)$ exists, $f(z)$ must tend toward the same complex value no matter which of the infinite number of paths of approach to z_0 is selected.

Example 1 As a simple example of this definition, show that

$$\lim_{z \to i} (z + i) = 2i$$

<pf.>

We have $f(z) = z + i$, $\ell = 2i$, $z_0 = i$. From Eq. (B), we need

$$|z + i - 2i| < \epsilon$$

or, equivalently,

$$|z - i| < \epsilon \qquad (1)$$

which according to Eq. (B) must hold for

$$0 < |z - i| < \delta \qquad (2)$$

Taking δ as, say, ϵ (this is not the only possible choice; e.g., $\epsilon/2$ will work), we see that Eq. (1) will be satisfied as long as z lies in the deleted neighborhood of i described in Eq. (2).

Example 2 若已知 $f(z) = \begin{cases} z^2, & z \neq z_0 \\ 0, & z = z_0 \end{cases}$，試求極限 $\lim_{z \to z_0} f(z)$。

<pf.>

當 $z \to z_0$ 時，$f(z) \to {z_0}^2$。吾人可先推測該函數在 $z \to z_0$ 時的極限為 z_0^2，亦即

$$\lim_{z \to z_0} f(z) = z_0^2$$

然後，再證明其是否正確。

若 $\delta \le 1$，則由 $0 < |z - z_0| < \delta < 1$ 而得

$$\begin{aligned}
\left|f(z) - {z_0}^2\right| &\equiv \left|z^2 - {z_0}^2\right| \\
&= |z - z_0||z - z_0 + 2z_0| \\
&< \ \delta|z - z_0 + 2z_0| \\
&< \ \delta\left[\ |z - z_0| + 2|z_0|\right] \\
&< \ \delta\left[1 + 2|z_0|\ \right] \\
&< \ \epsilon
\end{aligned}$$

若取 $\delta = 1$ 或 $\delta = \epsilon / \left[1 + 2|z_0|\right]$ 中之較小者，則當 $|z - z_0| < \delta$ 時，必有 $\left|f(z) - {z_0}^2\right| < \epsilon$。

故知

$$\lim_{z \to z_0} f(z) = {z_0}^2 \quad \text{成立}$$

進而確知當 $z \to z_0$ 時，函數 $f(z)$ 之極限為 ${z_0}^2$，但該函數在 z_0 點之函數值，則由其定義知 $f(z) = 0$。

Example 3 若已知函數 $f(z) = \dfrac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i}$

求 $z \to i$ 時，函數之極限。

<pf.> 求 $z \to i$ 而不等於 i 時，

$$\begin{aligned}
f(z) &= \frac{3z^4 - 2z^3 + 8z^2 - 2z + 5}{z - i} \\
&= \frac{[3z^3 - (2 - 3i)z^2 + (5 - 2i)z + 5i](z - i)}{z - i} \\
&= 3z^3 - (2 - i \cdot 3)z^2 + (5 - 2i)z + 5i \\
&= 4 + 4i
\end{aligned}$$

故首先推測其極限為 $4+4i$。

若 $\delta \le 1$，則由 $0 < |z - z_0| < \delta < 1$，而得

$$\begin{aligned}
|f(z) - (4 + 4i)| &= |3z^3 - (2 - 3i)z^2 + (5 - 2i)z - (4 + i)| \\
&= |z - i||3z^2 + (6i - 2)z - 1 - 4i| \\
&= |z - i||3(z - i)^2 + (12i - 2)(z - i) - 10 - 6i| \\
&< \delta[3|z - i|^2 + |12i - 2||z - i| + |-10 - 6i|] \\
&< \delta[3 + 13 + 12] \\
&= 28\delta \\
&< \epsilon
\end{aligned}$$

故若取 $\delta = 1$ 或 $\delta = \epsilon / 28$ 中之較小者，而當 $|z - z_0| < \delta$ 時，有

$$|f(z) - 4 + 4i| < \epsilon$$

故知 $\lim_{z \to z_0} f(z) = 4 + 4i$ 成立。

但 $f(i)$ 並不存在。

Example 4 若 $\lim_{z \to z_0} f(z)$ 存在，試證此極限為唯一極限。此即求證，若 $\lim_{z \to z_0} f(z) = w_1$，及 $\lim_{z \to z_0} f(z) = w_2$ 時，

則恒有 $w_1 = w_2$ 。

<pf.> 根據極限之定義，對於任何指定之微小數 $\epsilon > 0$ ，恒有一數 $\delta > 0$ ，而使得

$$|f(z) - w_1| < \frac{\epsilon}{2} \text{，當 } 0 < |z - z_0| < \delta$$

$$|f(z) - w_2| < \frac{\epsilon}{2} \text{，當 } 0 < |z - z_0| < \delta$$

則

$$\begin{aligned} |w_1 - w_2| &= |w_1 - f(z) + f(z) - w_2| \\ &\le |w_1 - f(z)| + |f(z) - w_2| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

即 $|w_1 - w_2| < \epsilon$

此即說明 $|w_1 - w_2|$ 較任何微小之指定正數 ϵ 為小，故必須為 0，進而證得 $w_1 = w_2$ 。

因此，函數 $f(z)$ 在 $z \to z_0$ 時，若有極限存在，則該極限具唯一性。

Example 5 試證 $\lim\limits_{z \to 0} \dfrac{\bar{z}}{z}$ ，並不存在。

<pf.> 若該極限存在，則必須與 z 漸近於 0 之方式無關。若想像 x 維持不變，而令 $y \to 0$ ，然後再令 $x \to 0$ ，則得

$$\begin{aligned} \lim_{z \to 0} \frac{\bar{z}}{z} &= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy} \\ &= \lim_{x \to 0} \frac{x}{x} \\ &= 1 \end{aligned}$$

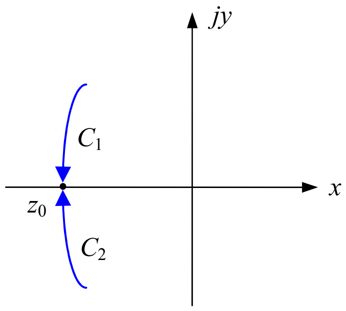
若想像 y 維持不變，而令 $x \to 0$ ，然後再令 $y \to 0$ ，則得

$$\begin{aligned} \lim_{z \to 0} \frac{\bar{z}}{z} &= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x - iy}{x + iy} \\ &= \lim_{y \to 0} \frac{-iy}{iy} \\ &= -1 \end{aligned}$$

因採取兩種不同路線來使 $z \to 0$ ，而得出兩個不一樣之結果，故知該函數在 $z \to 0$ 時，無極限存在。

Example 6 Let $f(z) = \arg(z)$ (principal value). Show that $f(z)$ fails to possess a limit on the negative real axis.

<pf.> Consider a point z_0 on the negative real axis. Refer to the following figure. Approaching z_0 on two different paths such as C_1 and C_2, we see that $\arg(z)$ tends to two different values, π (along C_1) and $-\pi$ (along C_2), respectively. Therefore, $\arg(z)$ fails to possess a limit at z_0.



H.W. 2 Let $f(z)=\frac{x^2+x}{x+y}+i\frac{y^2+y}{x+y}$. This function is undefined at $z = 0$. Show that $\lim_{z\to 0} f(z)$ fails to exist.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Section 2.1, Example 3, Pearson Education, Inc., 2005.】

3. **Definition of Limit at Infinity**

Let $f(z)$ be a complex function of the complex variable z, and let f_0 be c complex constant. If for every real number ϵ there exists a real number γ such that $|f(z)-f_0|<\epsilon$ for all $|z|>\gamma$, then we may say that $\lim_{z\to\infty} f(z)=f_0$.

♣ **Usually γ depends on ϵ.**

Ex. $\lim_{z\to\infty}(1/z^2)=0$

Ex. $\lim_{z\to\infty}(1+z^{-1})=1$

Example 7 Show that $\lim_{z\to\infty}(1/z)=0$.

<pf.>

函數 $1/z$ 之值，當 z 之值愈小則 $1/z$ 之值愈大，故首先推測其極限為 0。若取

$$|f(z)|=\left|\frac{1}{z}\right|=\frac{1}{|z|}<\epsilon$$

則 $$|z|>\frac{1}{\epsilon}$$

因此，由任一微小之指定正數 ϵ 恆可決定一正數 $\gamma=1/\epsilon$，使得當 $|z|>\gamma$ 時，有 $|f(z)-0|=\left|\frac{1}{z}-0\right|<\epsilon$。

故可得證

$$\lim_{z\to\infty}(1/z)=0$$

H.W. 3 In this problem we prove rigorously, using the definition of the limit at infinity, that

$$\lim_{z\to\infty}\frac{z}{1+z}=1$$

(a) Explain why, given $\epsilon > 0$, we must find a function $\gamma(\epsilon)$ such that $|1/(z+1)|<\epsilon$ for all $|z|>\gamma$.

(b) Using one of the triangle inequalities, show that the preceding inequality is satisfied if we take $\gamma > 1 + (1/\epsilon)$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.1, Problem 12, Pearson Education, Inc., 2005.】

4. **Theorem**

Let $\lim_{z\to z_0} f(z)=f_0$ and $\lim_{z\to z_0} g(z)=g_0$. Then, we have

(1) $\lim_{z\to z_0}[f(z)+g(z)]=f_0+g_0$

(2) $\lim_{z\to z_0}[f(z)g(z)]=f_0 g_0$

(3) $\lim_{z\to z_0}[f(z)/g(z)]=f_0/g_0$, if $g_0\neq 0$.

♣ **These limits can be applied at infinity.**

5. **Definition of Continuity**

A function $w=f(z)$ is continuous at $z=z_0$ provided that the following conditions are both satisfied:

(a) $f(z_0)$ is defined;

(b) $\lim_{z\to z_0} f(z)$ exists, and

$$\lim_{z\to z_0} f(z)=f(z_0)$$

Example 8 Investigate the continuity at $z = i$ of the function

$$f(z) = \begin{cases} \dfrac{z^2+1}{z-i}, & z \neq i \\ 3i, & z = i \end{cases}$$

<Sol.>

Because $f(i)$ is defined, part (a) in our definition of continuity is satisfied.
Since

$$f(z) = \frac{z^2+1}{z-i} = \frac{(z+i)(z-i)}{(z-i)} = z+i, \quad \text{for} \quad z \neq i$$

$$\Rightarrow \qquad \lim_{z \to i} f(z) = 2i \neq f(i) = 3i$$

That is, condition (b) in our definition of continuity is not satisfied for $z = i$. Thus $f(z)$ is discontinuous at $z = i$.

§2-2 The Derivative of a Function, Differentiation Formula, and Cauchy's -Riemann Theorem

1. 吾人沿用實變中之定義，定義複變函數之微分如下所示：

$$w' \equiv \frac{d}{dz} w \equiv \frac{d}{dz} f(z) \equiv f'(z)$$

$$\equiv \lim_{\Delta z \to 0} \frac{\Delta f(z)}{\Delta z} \equiv \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$

If $f'(z_0)$ exist $\Rightarrow$ $f(z)$ is called differentiable at $z = z_0$.
上式所定義之導數若存在且與 $\Delta z \to 0$ 之方式無關者，則稱 $f(z)$ 在點 z 為可微分。

Example 1 利用複變函數之微分定義，求函數 $w \equiv f(z) \equiv z^3 - 2z$ 在 i) $z = z_0$ ii) $z = -1$ 處之導數。

<Sol.>

i) 由定義知

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{(z_0+\Delta z)^3 - 2(z_0+\Delta z) - (z_0{}^3 - 2z_0)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{z_0^3 + 3z_0^2\Delta z + 3z_0(\Delta z)^2 + (\Delta z)^3 - 2z_0 - 2\Delta z - z_0{}^3 + 2z_0}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left[3z_0{}^2 + 3z_0\Delta z + (\Delta z)^2 - 2 \right]$$

$$= 3z_0{}^2 - 2$$

此微分值與 $\Delta z \to 0$ 之方式無關。

♣ $f(z)$ 直接對 z 微分得通式： $f'(z) = 3z^2 - 2$ (對任何 z 值)，由此式亦可得出相同結果。

ii) 同理，由定義知

$f'(-1) = 1$

2. 若某一單值函數在 z 平面內區域 R 之各點，其導數 $f'(z)$ 均確切存在時，則稱該函數在該區域為可解析（analytic）。

* 此函數稱為解析函數（Analytic Function）、或稱正則函數（Regular Functions）、或全純函數（Holomorphic Functions）。

* 一函數 $f(z)$ 在點 z_0 為解析函數時，則必存在一鄰域 $|z - z_0| < \delta$ ，而使其中各點之 $f'(z)$ 均存在。

* 若單值函數在全部 z 平面之各點均可解析，則稱此函數為完全函數（Entire Function）。

Example 2 試求出函數 $f(z)=\dfrac{1+z}{1-z}$ 為解析函數之區域。

<Sol.>

由於

$$\frac{d}{dz}f(z)=\frac{d}{dz}\frac{1+z}{1-z}$$
$$=\lim_{\Delta z\to 0}\frac{\dfrac{1+(z+\Delta z)}{1-(z+\Delta z)}-\dfrac{1+z}{1-z}}{\Delta z}$$
$$=\lim_{\Delta z\to 0}\frac{2}{(1-z-\Delta z)(1-z)}$$
$$=\frac{2}{(1-z)^2}$$

1) 若 $z\neq 1$ ，該函數之導數與 $\Delta z\to 0$ 之方式無關，而為解析函數。

2) 若 $z=1$，導數不存在，故為非解析函數，此 $z=1$ 點通常稱為該函數之奇異點（Singular point），又稱為極點（pole）。

Example 3 若 $f(z)$ 在 z_0 為解析函數（即可微分），試證該函數必為連續函數，並舉例說明具連續性之函數並非一定可微分者，即並非一定為解析函數。

<pf.> 因

$$f(z_0+\Delta z)-f(z_0)=\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\Delta z$$ ，其中 $\Delta z\neq 0$ 。

故知

$$\lim_{\Delta z\to 0}\left[f(z_0+\Delta z)-f(z_0)\right]$$
$$=\lim_{\Delta z\to 0}\frac{f(z_0+\Delta z)-f(z_0)}{\Delta z}\cdot\lim_{\Delta z\to 0}\Delta z$$
$$=f'(z_0)\cdot 0=0$$
$$=\lim_{\Delta z\to 0}f(z_0+\Delta z)-f(z_0)$$

即

$$\lim_{\Delta z\to 0}f(z_0+\Delta z)=f(z_0)$$

故知

$$\lim_{z\to z_0}f(z)=f(z_0)$$

因此， $f(z)$ 在 $z=z_0$ 為連續。

H.W. 1 試證函數 $f(z)=\overline{z}$ 在 $z=z_0$ 雖為連續函數，但並非為解析函數。

Example 4 已知 $w=f(z)=z^3-2z^2$ ，求 Δw ， dw 及 $\Delta w-dw$ 。

<Sol.>

$$\Delta w=f(z+\Delta z)-f(z)$$
$$=\left[(z+\Delta z)^3-2(z+\Delta z)^2\right]-(z^3-2z^2)$$
$$=z^3+3z^2\Delta z+3z(\Delta z)^2+(\Delta z)^3-2z^2-4z(\Delta z)-2(\Delta z)^2-z^3+2z^2$$
$$=(3z^2-4z)\Delta z+(3z-2)(\Delta z)^2+(\Delta z)^3$$
$$=(3z^2-4z)\Delta z+\epsilon\Delta z$$

其中 $\epsilon=(3z-2)\Delta z+(\Delta z)^2$ 。

當 $\Delta z\to 0$ ， $\epsilon\to 0$ 。

$$\frac{dw}{dz}=\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=3z^2-4z$$
$$dw=(3z^2-4z)dz$$

Δz 與 dz 若為同一增量，即 $\Delta z = dz$ ，而有 $\Delta w - dw = \epsilon \Delta z$ ，其中

$$\epsilon = (3z-2)\Delta z + (\Delta z)^2$$

其極微程度較 Δz 為高。但需注意 $\Delta w \neq dw$ ，dw 稱為函數 w 之微分式(Differential)，dw 又稱為 Δw 之主部(principal part)。

3. If $f(z) = u(x,y) + i\,v(x,y)$ is differentiable at $z = z_0$ in the region R。

$\Rightarrow$ $u(x,y)$ and $v(x,y)$ must satisfy the condition of ***Cauchy-Riemann Equation***:

$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\ \dfrac{\partial v}{\partial x} = -\dfrac{\partial u}{\partial y} \end{cases}$$

<pf.> 必要條件：

因為 $w \equiv f(z) = u(x,y) + i\,v(x,y)$ ，故

$$f(z+\Delta z) - f(z) = f[x+\Delta x,\ y+\Delta y]$$
$$= u(x+\Delta x,\ y+\Delta y) + i\,v(x+\Delta x,\ y+\Delta y)$$

而由定義可知

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$$
$$= \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{u(x+\Delta x,\ y+\Delta y) + i\,v(x+\Delta x,\ y+\Delta y) - u(x,y) - i\,v(x,y)}{\Delta x + i\Delta y}$$

i) Let $\Delta y \to 0$ first, then $\Delta x \to 0$, we can

$$\Rightarrow \quad f'(z) = \lim_{\Delta x \to 0} \frac{[u(x+\Delta x,y) + i\,v(x+\Delta x,\ y)] - [u(x,y) + i\,v(x,y)]}{i\Delta y}$$
$$= \lim_{\Delta x \to 0} \frac{u(x+\Delta x,y) - u(x,y)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x+\Delta x,y) - v(x,y)}{\Delta x}$$
$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \quad ------(1)$$

ii) Let $\Delta x \to 0$ first, and then $\Delta y \to 0$, we can

$$\Rightarrow \quad f'(z) = \lim_{\Delta y \to 0} \frac{[u(x,\ y+\Delta y) + i\,v(x,\ y+\Delta y)] - [u(x,y) + i\,v(x,y)]}{i\Delta y}$$
$$= \lim_{\Delta y \to 0} \frac{u(x,\ y+\Delta y) - u(x,y)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x,\ y+\Delta y) - v(x,y)}{i\Delta y}$$
$$= \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$
$$= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y} \quad ------(2)$$

若為解析函數，則(1)及(2)式必相等。故知

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

充分條件：

若 $\dfrac{\partial u}{\partial x}$ 及 $\dfrac{\partial u}{\partial y}$ 具連續性，則

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x,y)$$
$$= [u(x+\Delta x, y+\Delta y) - u(x, y+\Delta y)] + [u(x, y+\Delta y) - u(x,y)]$$
$$= \left[\frac{\partial u}{\partial x} + \epsilon_1\right]\Delta x + \left[\frac{\partial u}{\partial y} + \eta_1\right]\Delta y$$
$$= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \eta_1 \Delta y$$

其中 $\Delta x \to 0$ ， $\Delta y \to 0$ ； $\epsilon_1 \to 0$ ， $\eta_1 \to 0$ 。

又因 $\dfrac{\partial v}{\partial x}$ 及 $\dfrac{\partial v}{\partial y}$ 具連續性，則

$$\begin{aligned}\Delta v &= v(x+\Delta x, y+\Delta y) - v(x,y) \\ &= \left[v(x+\Delta x, y+\Delta y) - v(x, y+\Delta y)\right] + \left[v(x, y+\Delta y) - v(x,y)\right] \\ &= \left[\frac{\partial v}{\partial x} + \epsilon_2\right]\Delta x + \left[\frac{\partial v}{\partial y} + \eta_2\right]\Delta y \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_2 \Delta x + \eta_2 \Delta y\end{aligned}$$

其中 $\Delta x \to 0$ ， $\Delta y \to 0$ ； $\epsilon_2 \to 0$ ， $\eta_2 \to 0$ 。

因此，

$$\Delta w = \Delta u + i\Delta v = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y + \epsilon\Delta x + \eta\Delta y$$

此處， $\Delta x \to 0$ ， $\Delta y \to 0$ ； $\epsilon \equiv \epsilon_1 + i\epsilon_2 \quad \to \quad 0$ ，而且 $\eta \equiv \eta_1 + i\eta_2 \to 0$ 。

利用 Cauchy-Riemann Equation 可得

$$\begin{aligned}\Delta w &= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\Delta x + \left[-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right]\Delta y + \varepsilon\Delta x + \eta\Delta y \\ &= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right](\Delta x + i\Delta y) + \epsilon\Delta x + \eta\Delta y\end{aligned}$$

故知

$$\frac{dw}{dz} = \lim_{\Delta z \to 0}\frac{\Delta w}{\Delta z} = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}}\left[\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \frac{\epsilon\Delta x + \eta\Delta y}{\Delta x + i\Delta y}\right] = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

因此，該函數之導數存在且為唯一，即 $f(z)$ 為解析函數。

♣ Given $w \equiv f(z) \equiv u(x,y) + iv(x,y)$, then the derivative of $f(z)$ can be written as

$$\begin{aligned}\frac{dw}{dz} &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\end{aligned}$$

♣ When $f(z)$ and $g(z)$ are differentiable for some *z*, there are several important identities as shown below:

1) $\dfrac{d}{dz}\left[f(z) \pm g(z)\right] = f'(z) \pm g'(z)$

2) $\dfrac{d}{dz}\left[f(z)g(z)\right] = f'(z)g(z) \pm f(z)g'(z)$

3) $\dfrac{d}{dz}\left[\dfrac{f(z)}{g(z)}\right] = \dfrac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$, provided $g(z) \neq 0$

4) $\dfrac{d}{dz}f\left(g(z)\right) = \dfrac{df}{dg}g'(z)$

♣ **L'Hôspital's Rule**

If $g(z_0) = 0$ and $h(z_0) = 0$, and if $g(z)$ and $h(z)$ are differentiable at z_0 with $h'(z_0) \neq 0$, then

$$\lim_{z \to z_0}\frac{g(z)}{h(z)} = \frac{g'(z_0)}{h'(z_0)}$$

<pf.> ***Refer to*** **A. David Wunsch, *Complex Variable with Applications*, 3rd ed., pp. 72-73, Pearson Education, Inc., 2005**.

♦ If $g(z)$ and $h(z)$, and their first n derivatives vanish at z0 and $h^{n+1}(z_0) \neq 0$, then with $h'(z_0) \neq 0$, then

$$\lim_{z\to z_0}\frac{g(z)}{h(z)}=\frac{g^{(n+1)}(z_0)}{h^{(n+1)}(z_0)}$$

Example 5 Investigate the differentiability of $f(z)=z^2=(x+iy)^2=x^2-y^2+i2xy$.

<Sol.>

Since $u(x,y)=x^2-y^2$, $v(x,y)=2xy$, we have

$$\frac{\partial u}{\partial x}=2x=\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial x}=2y=-\frac{\partial u}{\partial y}$$

The $u(x,y)$ and $v(x,y)$ satisfy the Cauchy-Riemann equation. Hence, $f'(z)$ exists for all z. The derivative of $f(z)$ is

$$f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=2x+i2y=2z$$

Example 6 Investigate the differentiability of $f(z)=z\bar{z}=|z|^2$.

<Sol.>

Since $f(z)=z\bar{z}=|z|^2=x^2+y^2$, we have $u(x,y)=x^2+y^2$, $v(x,y)=0$.

$$\Rightarrow \begin{aligned}&\frac{\partial u}{\partial x}=2x\neq\frac{\partial v}{\partial y}=0\\ &\frac{\partial u}{\partial y}=2y\neq-\frac{\partial v}{\partial x}=0\end{aligned}$$

This function of z possesses a derivative only for $z=0$.

Example 7 Find $\lim_{z\to 2i}\frac{z-2i}{z^4-16}$.

<Sol.> $1/(-32i)$

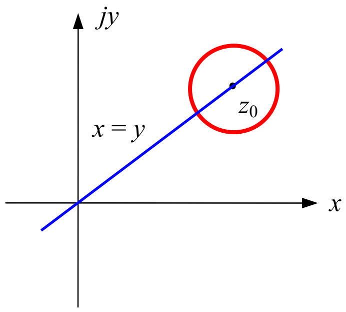
Example 8 For what values of z is the function $f(z)=x^2+iy^2$ analytic ?

<Sol.>

Since $u(x,y)=x^2$ and $v(x,y)=y^2$, we have

$$\frac{\partial u}{\partial x}=2x=2y=\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial v}{\partial x}=0=-\frac{\partial u}{\partial y}$$

Thus, $f(z)$ is differentiable only for values of z that lie along the straight line $x=y$. If z_0 lies on this line, any circle centered at z_0 will contain points for which $f'(z)$ does not exist. Thus, $f(z)$ is **nowhere analytic**.



Example 9 For what values of z does $f(z)=\frac{z^3+2}{z^2+1}$ fail to be analytic ?

<Sol.> For z satisfying $z^2+1=0$ or $z=\pm i$. Thus, $f(z)$ has singularities at $+i$ and $-i$.

H.W. 2 Find the derivative of the function $f(z)=z^2+\bar{z}^2+2\bar{z}$ whenever the derivative exists.

<Ans.> The derivative of $f(z)$ only exists at one point $(-1, 0)$, i.e., $f'(z)|_{(-1,0)}=-2$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.1, Problem 12, Pearson Education, Inc., 2005.】

H.W. 3 函數 $f(z) \equiv \mathrm{Re}(z) = x$ 是否為解析函數?

<Ans.> 否

H.W. 4 Given an analytic function $f(z) = u(x, y) + i\,v(x, y)$, if $u(x, y) = x^2 - y^2$, find $f(z) = ?$

<Ans.> $f(z) = z^2 + K$, where K is a constant.

H.W. 5 Given a analytic function $f(z) = u(x, y) + i\,v(x, y)$, if $u(x, y) = e^{-x}(x\sin y - y\cos y)$, find $f(z)$, $f'(z)$, and $v(x, y) = ?$

<Ans.> $f(z) = ie^{-z}z + K$, where K is a constant; $f'(z) = ie^{-z}(1-z)$; $v(x, y) = e^{-x}(y\sin y + x\cos y) + c$, where c is a constant.

H.W. 6 Given a analytic function $f(z) = u(x, y) + i\,v(x, y)$, show that the derivative $f'(z)$ can be expressed as

$$f'(z) = \left.\frac{\partial u}{\partial x}\right|_{(x,y)\to(z,0)} - i\left.\frac{\partial u}{\partial y}\right|_{(x,y)\to(z,0)}$$

and

$$f'(z) = \left.\frac{\partial v}{\partial y}\right|_{(x,y)\to(z,0)} + i\left.\frac{\partial v}{\partial x}\right|_{(x,y)\to(z,0)}$$

【本題摘自：朱越生，*工程數學*，下冊，正中書局，頁次：1387-1388，1972。】

H.W. 7 Find the derivative at $z = \pi + 2i$ of the function $[\sin x\cosh y + i\cos x\sinh y]^5$.

<Ans.> $f'(z)\big|_{z=\pi+2i} = 5(\sinh 2)^4(-\cosh 2)$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.4, Problem 7, Pearson Education, Inc., 2005.】

4. **底下我們介紹一些解析函數之性質：**

1) If $f(x)$ is differentiable at every point in $|z - z_0| < \delta$

   $\Rightarrow$ $f(z)$ is analytic at $z = z_0$

2) If $f(z)$ is analytic at every point in a domain D

   $\Rightarrow$ $f(z)$ is analytic in D

3) If $f(z) = u(x, y) + i\,v(x, y)$ is analytic in D, then

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \\ &= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \\ &= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \end{aligned}$$

4) If $f(z)$ is analytic $\Rightarrow$ $f'(z), f''(z), f'''(z)\cdots, f^{(n)}(z), \cdots,$ exist.

5) Laplace Equation:

   If $f(z) = u(x, y) + i\,v(x, y)$ is analytic, then

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

   <pf.> Since $f(z)$ is analytic, hence

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y} \ , \ \frac{\partial^2 u}{\partial y^2}=-\frac{\partial^2 v}{\partial y\partial x}=-\frac{\partial^2 v}{\partial x\partial y}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

同理，

$$\frac{\partial^2 v}{\partial x^2}=-\frac{\partial^2 u}{\partial x\partial y} \ , \ \frac{\partial^2 v}{\partial y^2}=\frac{\partial^2 u}{\partial y\partial x}=\frac{\partial^2 v}{\partial x\partial y}$$

$$\Rightarrow \quad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0$$

$$\Rightarrow \quad \nabla^2 u=0 \text{ and } \nabla^2 v=0$$

6) If an analytic function $f(z)$ whose both real part and imaginary part have continuous second order partial derivatives at every point in the region R and satisfy Laplace's Equation. The function is said to be harmonic in that region.（The functions satisfying Laplace equation are called harmonic functions(諧和函數).）
因此，解析函數之實部及虛部[$u(x,y)$ and $v(x,y)$]稱為共軛諧和函數（Conjugate Harmonic Function）。

7) If $f(z)=u(x,y)+i\,v(x,y)$ is analytic,
$\Rightarrow$ $u(x,y)$ and $v(x,y)$ are harmonic functions

8) If $f(z)=u(x,y)+i\,v(x,y)$ is analytic
$\Rightarrow$ $u(x,y)=c_1$, $v(x,y)=c_2$ are orthogonal each other in w-plane
在 z-plane 我們稱 $u(x,y)=c_1$, $v(x,y)=c_2$ 之曲線為 z 平面之階層曲線（Level curves）。

<pf.> Since $f(z)=u(x,y)+i\,v(x,y)$ is analytic,

hence $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

曲線族 $u(x,y)=c_1$ 之斜率為

$$m_1=\left.\frac{dy}{dx}\right|_{u=c_1}=-\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial y}}$$

曲線族 $v(x,y)=c_2$ 之斜率為

$$m_2=\left.\frac{dy}{dx}\right|_{v=c_2}=-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}$$

因此

$$m_1\cdot m_2=\left[-\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial y}}\right]\left[-\frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}}\right]=-1$$

Thus, $u(x,y)=c_1$, $v(x,y)=c_2$ are orthogonal each other.

Example 10 Consider the function

$$f(z)=\frac{1}{2}\ln\left(x^2+y^2\right)+i\arg(z)$$

where we use the principal value of arg(z). Thus, $-\pi<\arg(z)\le\pi$. Show that this function satisfies the Cauchy-Riemann equations in any domain not containing the origin and/or points on the negative axis (where arg(z) is discontinuous), and that the above property holds for this function.

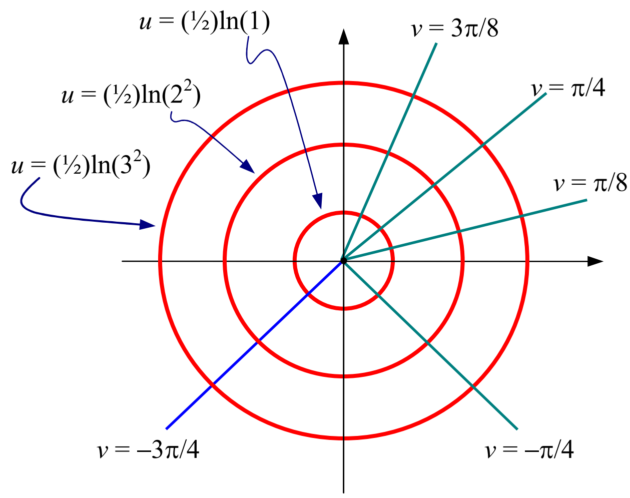
<pf.> Let $u(x,y)=\frac{1}{2}\ln\left(x^2+y^2\right)$ and $v(x,y)=\arg(z)=\tan^{-1}(y/x)=\cot^{-1}(x/y)$ .The multivalued functions $\tan^{-1}(y/x)$ and $\cot^{-1}(x/y)$ are evaluated so that v will be the principal value of arg(z).

When x = 0 we employ $\cot^{-1}(x/y)$; while when x = 0 we employ $\tan^{-1}(y/x)$.

Differentiating both u and v, we observe that the Cauchy-Riemann equations are satisfied. That is,

$$\frac{\partial u}{\partial x}=\frac{x}{x^2+y^2}=\frac{\partial v}{\partial y}\quad\text{and}\quad\frac{\partial u}{\partial y}=\frac{y}{x^2+y^2}=-\frac{\partial v}{\partial x}$$

Loci along which u is constant are merely circles, i.e., $u(x,y)=\text{constant}\ \Rightarrow\ x^2+y^2=C$. The curves along which v is constant are merely rays extending outward from origin of the z-plane, i.e., $c(x,y)=\text{constant}\ \Rightarrow\ x/y=K$. These families of curves of the above-mentioned forms are shown in the following figure. The orthogonality of the intersections should be apparent.



9) If $f(z)=u(x,y)+i\,v(x,y)$ is analytic

$\Rightarrow$ The derivative of $f(z)$ is independent of $\bar{z}$.

<pf.> Let $z=x+i\,y$

then $\bar{z}=x-i\,y$

Thus, $x=\dfrac{z+\bar{z}}{2}$, $y=\dfrac{z-\bar{z}}{2i}$

so,
$$\frac{\partial f(z)}{\partial\bar{z}}=\frac{\partial u(x,y)}{\partial\bar{z}}+i\frac{\partial v(x,y)}{\partial\bar{z}}$$
$$=\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial\bar{z}}\right]+i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial\bar{z}}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\bar{z}}\right]$$
$$=\left[\frac{1}{2}\frac{\partial u}{\partial x}-\frac{i}{2}\frac{\partial u}{\partial y}\right]+i\left[\frac{1}{2}\frac{\partial v}{\partial x}-\frac{i}{2}\frac{\partial v}{\partial\bar{z}}\right]$$
$$=\frac{1}{2}\frac{\partial u}{\partial x}-\frac{i}{2}\frac{\partial v}{\partial x}+\frac{i}{2}\frac{\partial v}{\partial x}-\frac{1}{2}\frac{\partial u}{\partial x}=0$$

(Since $f(z)=u(x,y)+i\,v(x,y)$ is analytic, $\Rightarrow \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$.)

$\Rightarrow$ The derivative of $f(z)$ is independent of $\bar{z}$.

10) If $u(x,y)$ is a harmonic function and $f(z)=u(x,y)+i\,v(x,y)$ is analytic, then the $v(x,y)$ is called the conjugate harmonic-function of $u(x,y)$, vice versa.

For example If $u(x,y)=x^2-y^2+2x$, find the conjugated harmonic function of $v(x,y)$.

<Sol.> Let $v(x,y)$ be the conjugate harmonic function of $u(x,y)$

$\Rightarrow$ $f(z)=u(x,y)+i\,v(x,y)$ is analytic

$\Rightarrow$ $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$

Hence,

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}=2x+2$$

$\Rightarrow$ $v(x,y)=2xy+2y+\varphi(x)$

and

$$\frac{\partial v}{\partial x}=\partial y+\varphi'(x)=-\frac{\partial u}{\partial y}=2y$$

Thus, we see that $\varphi'(x)=0 \;\Rightarrow\; \varphi(x)=c$.

Then, we obtained that

$$v(x,y)=2xy+2y+c$$

where c is a constant.

$*$ $\Rightarrow$

$$\begin{aligned} f(z)&=u(x,y)+i\,v(x,y)\\ &=[x^2-y^2+2x]+i[2xy+2y+c]\\ &=[x^2-y^2+i2xy]+2[x+iy]+ic\\ &=z^2+2z+ic \end{aligned}$$

故知解析函數 $f(z)$ 只含 z，而沒 $\bar{z}$ 之成份。

11) If $f(z)=u(x,y)+i\,v(x,y)$ is analytic,

$\Rightarrow$ $\dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y}$ and $\dfrac{\partial v}{\partial x}=-\dfrac{\partial u}{\partial y}$

Let the functions u, v be in terms of polar form of $(r,\ \theta)$ such that

$$x=r\cos\theta,\quad y=r\sin\theta$$

$\Rightarrow$ $\dfrac{\partial u}{\partial r}=\dfrac{1}{r}\dfrac{\partial v}{\partial\theta}$ and $\dfrac{\partial v}{\partial r}=-\dfrac{1}{r}\dfrac{\partial u}{\partial\theta}$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.4, Problem 23, Pearson Education, Inc., 2005.】

<pf.> 由偏微分方程式之連鎖律，知

$$\begin{aligned} \frac{\partial u}{\partial r}&=\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\\ &=\frac{\partial \varphi}{\partial x}\cos\theta+\frac{\partial u}{\partial y}\sin\theta \end{aligned}$$

$$\begin{aligned} \frac{\partial v}{\partial\theta}&=\frac{\partial v}{\partial x}\frac{\partial x}{\partial\theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial\theta}\\ &=\frac{\partial v}{\partial x}[-r\sin\theta]+\frac{\partial u}{\partial y}[r\,coos\theta]\\ &=\frac{\partial u}{\partial y}[r\sin\theta]+\frac{\partial u}{\partial y}[r\cos\theta] \end{aligned}$$

$$\Rightarrow \quad \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

同理，

$$\frac{\partial v}{\partial r}=\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$=\frac{\partial v}{\partial x}\cos\theta+\frac{\partial v}{\partial y}\sin\theta$$

$$\frac{\partial u}{\partial \theta}=\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$=\frac{\partial u}{\partial x}[-r\sin\theta]+\frac{\partial u}{\partial y}[r\cos\theta]$$

$$=\frac{\partial v}{\partial y}[r\sin\theta]+\frac{\partial v}{\partial y}[-r\cos\theta]$$

$$\Rightarrow \quad \frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

12) If $w=f(z)=u(x,y)+i\,v(x,y)$ is analytic, and let $x=r\cos\theta$, $y=r\sin\theta$, then

$$\frac{dw}{dz}=[\cos\theta-i\sin\theta]\frac{\partial w}{\partial \theta}$$

$$\frac{dw}{dz}=-\frac{i}{r}[\cos\theta-i\sin\theta]\frac{\partial w}{\partial \theta}$$

<pf.> Since $w=f(z)=u(x,y)+i\,v(x,y)$ is analytic,

$$\Rightarrow \quad \frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \text{ and } \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$$

而且知

$$\frac{dw}{dz}=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial y}$$

故知

$$\frac{\partial w}{\partial r}=\frac{\partial u}{\partial r}+i\frac{\partial v}{\partial r}$$

$$=\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right]+i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial r}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right]$$

$$=\left[\frac{\partial u}{\partial x}\cos\theta-\frac{\partial v}{\partial x}\sin\theta\right]+i\left[\frac{\partial v}{\partial x}\cos\theta+\frac{\partial u}{\partial x}\sin\theta\right]$$

$$=\frac{\partial u}{\partial x}[\cos\theta+i\sin\theta]+i\frac{\partial v}{\partial x}[\cos\theta+i\sin\theta]$$

$$=\frac{dw}{dz}[\cos\theta+i\sin\theta]$$

$$\Rightarrow \quad \frac{dw}{dz}=\frac{1}{\cos\theta+i\sin\theta}\frac{\partial w}{\partial r}$$

$$=[\cos\theta+i\sin\theta]\frac{\partial w}{\partial r}$$

同理，

$$\frac{\partial w}{\partial \theta}=\frac{\partial u}{\partial \theta}+i\frac{\partial v}{\partial \theta}$$

$$=\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}\right]+i\left[\frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}\right]$$

$$= \left[\frac{\partial u}{\partial x}(-r\sin\theta) + \frac{\partial u}{\partial y}(r\cos\theta\right] + i\left[\frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial x}(r\cos\theta)\right]$$

$$= (-r\sin\theta)\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + (r\cos\theta)\left[\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right]$$

$$= (-r\sin\theta)\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right] + (i\,r\cos\theta)\left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\left[-r\sin\theta + i\,r\cos\theta\right]$$

$$= \left[\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right]\left[r\,i(\cos\theta - i\sin\theta)\right]$$

$$\Rightarrow \quad \frac{dw}{dz} = \frac{1}{r\,i}\frac{1}{\cos\theta + i\sin\theta}\frac{\partial w}{\partial\theta}$$

$$= -\frac{i}{r}(\cos\theta - i\sin\theta)\frac{\partial w}{\partial\theta}$$

13) If $f(z) = u(x,y) + i\,v(x,y)$ is analytic,

$$\Rightarrow \quad |f'(z)|^2 = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

<pf.> Since $f(z) = u(x,y) + i\,v(x,y)$ is analytic, then

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Hence,

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial u}{\partial y} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial(u,v)}{\partial(x,y)}$$

5. Let $u = u(x,y)$ be a harmonic function in a domain. D which does not include the point $z = 0$. Show that the Laplace equation in polar form coordinates is given in D by

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial r}(r\frac{\partial u}{\partial r}) + \frac{1}{r}\frac{\partial^2 u}{\partial\theta^2} = 0$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.5, Problem 20, Pearson Education, Inc., 2005.】

<pf.> Let $f(z) = u(x,y) + i\,v(x,y)$ be an analytic function,

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

And then we prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

Let $x=r\cos\theta$, $y=r\sin\theta$

$$\Rightarrow \quad \frac{\partial u}{\partial r}=\frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r}=-\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Hence,

$$\frac{\partial u}{\partial \theta}=-r\frac{\partial v}{\partial r}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial \theta^2}=-r\frac{\partial^2 v}{\partial \theta \partial r}=-r\frac{\partial^2 v}{\partial r \partial \theta}$$

所以，

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 u}{\partial \theta^2}$$

$$=\frac{\partial}{\partial r}\left(\frac{\partial v}{\partial \theta}\right)+\frac{1}{r}\left(-r\frac{\partial^2 v}{\partial r \partial \theta}\right)$$

$$=\frac{\partial^2 v}{\partial r \partial \theta}-\frac{\partial^2 v}{\partial r \partial \theta}=0$$

H.W. 8 Refer to the above discussion. (a) Show that $u(r,\theta)=r^2\cos 2\theta$ is a harmonic function. (b) Find $v(r,\theta)$, the harmonic conjugate of $u(r,\theta)$, and show that it too satisfies Laplace's equation everywhere.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.5, Problem 20, Pearson Education, Inc., 2005.】

§2-3 Some Physical Applications of harmonic Functions

1. **Electrostatics**

1) Basic parameters:

F ≡ Coulomb force vector; **D** ≡ electric flux density vector; **E** ≡ electric field vector;
ϵ ≡ permittivity; q_0 ≡ charge; ϕ ≡ electrostatic potential

2) Basic relationships: $\mathbf{F}=q_0\dfrac{\mathbf{D}}{\epsilon}$ (Coulomb force law); $\mathbf{D}=\epsilon\mathbf{E}$ (Constitution relation);

$\mathbf{F}=q_0\mathbf{E}$ or $\mathbf{F}/q_0=\mathbf{E}$ (Electrostatic force); $\mathbf{E}=-\nabla\phi$

$\Rightarrow \quad \mathbf{D}=-\epsilon\nabla\phi$

3) System description:

The electric charges involved in the creation of the electric flux are assumed to exist along lines, or cylinders, of infinite extent that lie perpendicular to the *xy*-plane (assume to be ζ-axis).

⇒ The electric flux density vector **D** is parallel to the *xy*-plane.

⇒ Scalar component of **D**: $D_x=D_x(x,y)$ and $D_y=D_y(x,y)$

a) The electric flux density related to electric potential $\phi(x,y)$:

$$D_x=-\epsilon\frac{d\phi}{dx}, \qquad D_y=-\epsilon\frac{d\phi}{dy} \qquad \text{(A)}$$

b) Since $D_x=\epsilon E_x$ and $D_y=\epsilon E_y$, we have

$$E_x = -\frac{d\phi}{dx}$$
$$E_y = -\frac{d\phi}{dy} \qquad \text{(B)}$$

♣ *At any point in space where there is no electric charge, the electric flux density vector satisfies the same conservation equation (B) as does the heat flux density vector.*

4) The electrostatic potential is a harmonic function in any charge-free region:

$$\frac{d^2\phi}{dx^2} + \frac{d^2\phi}{dy^2} = 0$$

5) Define a complex electrostatic potential: $\Phi = \phi + i\psi$, whose real part is the actual electrostatic potential. The imaginary part is called the ***stream function***.

⇒ The electric flux density vector is tangent to the streamlines generated from ψ .

6) The electric flux density **D** and electric field vector **E** are the vectors corresponding to the following complex functions:

$$d(z) = D_x(x, y) + iD_y(x, y)$$
$$e(z) = E_x(x, y) + iE_y(x, y)$$

These are called the *complex electric flux density* and the *complex electric field*, respectively, and they satisfy

$$d(z) = -\epsilon\overline{\left(\frac{d\Phi}{dz}\right)} \quad \text{and} \quad e(z) = -\overline{\left(\frac{d\Phi}{dz}\right)}$$

Example 1 A complex potential is of the form

$\phi(x, y) = \text{Re}(Az + B)$, where A and B are real numbers.

Discuss its associated equipotentials, streamlines, and flux density in terms of electrostatics, heat conduction, and fluid flow.

<Sol.>

1) Potential function:

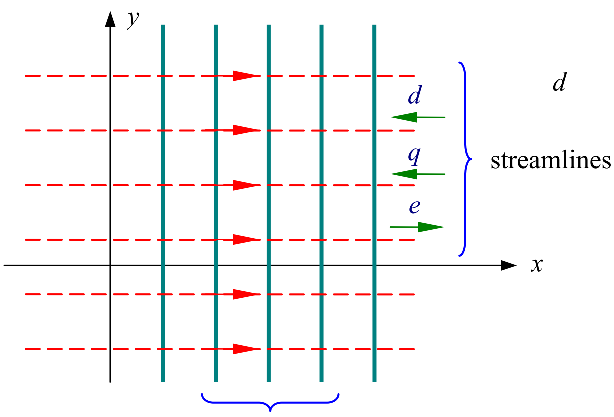
$$\phi(x, y) = \text{Re}(Az + B) = Ax + B \qquad (1)$$

2) Stream function:

$$\psi(x, y) = \text{Im}(Az + B) = Ay$$

The equipotentials (or isotherm) are the surfaces on which $\phi(x, y)$ assumes fixed values.

⇒ Figure shown below:



3) Complex electric flux density:

$$d = -\epsilon\overline{\frac{d}{dz}(Az + B)} = -\epsilon A = D_x + iD_y$$

⇒ $D_x = -\epsilon A$, $D_y = 0$

The electric flux density vector is parallel to the x-axis.

♣ If $\Phi(z)$ is the complex temperature, then the isotherms are the equipotentials in the above figure. The complex heat flux density is

$$q = -k\overline{\frac{d}{dz}(Az+B)} = -k\overline{A} = Q_x + iQ_y$$

which implies that $Q_x = -k\,A$ and $Q_y = 0$.

♣ If $\Phi(z)$ describes fluid flow, the fluid velocity is

$$V_x + iV_y = \overline{\frac{d}{dz}(Az+B)} = A$$

which implies that $V_x = A$ and $V_y = 0$.

H.W. 1 Suppose that $\Phi(z) = e^x \cos y + ie^x \sin y$ represents the complex potential, in volts, for some electrostatic configuration.

(a) Use the complex potential to find the complex electric field at $x=1$, $y=1/2$ (meter).

(b) Obtain the complex electric field at the same point by first finding and using the electrostatic potential $\phi(x,y)$.

(c) Assuming the configuration lies within a vacuum, find the components Dx and Dy of the electric flux density vector at $x=1$, $y=1/2$. In m.k.s uints, $\epsilon = 8.85\times 10^{-12}$ for vacuum.

(d) What is the value of ϕ at $x=1$, $y=1/2$? Using MATLAB, plot the equipotential surface passing through this point.

(e) What is the value of ψ at $x=1$, $y=1/2$? Using MATLAB, plot the streamline surface passing through this point.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.6, Problem 3, Pearson Education, Inc., 2005.】

<Ans.>

(a) Since $\Phi(z) = \underbrace{e^x \cos y}_{\phi} + i\underbrace{e^x \sin y}_{\psi}$

$$\Rightarrow \quad \frac{d\Phi(z)}{dz} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial y} = e^x\cos y + ie^x \sin y$$

$$\Rightarrow \quad e = -\overline{\frac{d\Phi(z)}{dz}} = -\left[e^x\cos y - ie^x\sin y\right]$$

Thus, we have

$$e\big|_{(1,1/2)} = -\left[e^x\cos y - ie^x\sin y\right]\Big|_{(1,1/2)} = -\left[e^1\cos(1/2) - ie^1\sin(1/2)\right]$$

$$= E_x + iE_y = -2.39 + i1.30$$

(b) Since $\phi(x,y) = e^x\cos y$, we have

$$E_x\big|_{(1,1/2)} = -\frac{\partial\phi}{\partial x}\bigg|_{(1,1/2)} = \left(-e^x\cos y\right)\Big|_{(1,1/2)} = -e^1\cos(1/2)$$

$$E_y\big|_{(1,1/2)} = -\frac{\partial\phi}{\partial y}\bigg|_{(1,1/2)} = \left(e^x\sin y\right)\Big|_{(1,1/2)} = e^1\sin(1/2)$$

Thus, the complex electric field is

$$\Phi = E_x + iE_y = -\left(e^1\cos(1/2) - ie^1\sin(1/2)\right) \quad \text{as in part (a).}$$

(c) Since $D_x = \epsilon E_x$ and $D_y = \epsilon E_y$, we have

$$D_x = -8.85\times 10^{-12} e^1\cos(1/2) \quad \text{and} \quad D_y = -8.85\times 10^{-12} e^1 \sin(1/2)$$

$$\Rightarrow \quad D_x = -21\times 10^{-12} \quad \text{and} \quad D_y = 11.5\times 10^{-12}$$

(d)

♣ MATLAB command:

```
k=[1/2 1];
% for H.W. 2
```

% part (d)
x=linspace(0,pi/2,1000);
y=acos(exp(1)*cos(1/2)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]); hold on

(e) Since $\psi = e^x \sin y = e^1 \sin(1/2)$, we have $y = \sin^{-1}\left[e^1 \sin(1/2) e^{-x}\right]$.

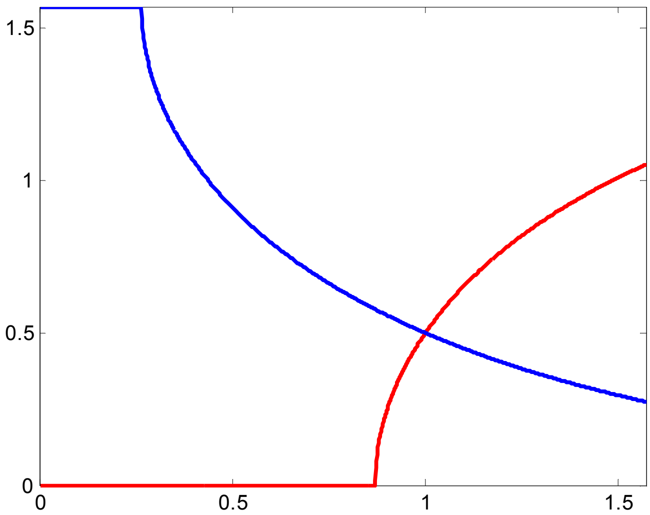
See the following attached plot. Suppose that $e^1 \sin(1/2) e^{-x} = 1$. We can solve for *x* as

$$x = \ln\left[e^1 \sin(1/2)\right] = 0.2684$$

For plotting, take $x \geq 0.2684$, to avoid taking $\sin^{-1}$ of a number > 1.

♣ MATLAB command:

% part (e)
y=asin(exp(1)*sin(1/2)*exp(-x));
plot(x,y);axis([0 pi/2 0 pi/2]); hold on
end
grid



H.W. 2 (a) Explain why $d(x,y) = y + ix$ can be the complex flux density in a charge-free region, but $d(x,y) = x + iy$ cannot.

(b) Assume that the complex electric flux density $d(x,y) = y + ix$ exists in a medium for which $\epsilon = 9 \times 10^{-12}$. Find the electrostatic potential $\phi(x,y)$. Assume $\phi(0,0) = 0$. Shetch the equipotentials $\phi(x,y) = 0$, $\phi(x,y) = 1/\epsilon$.

(c) Find the stream function $\psi(x,y)$. Assume $\psi(0,0) = 0$.

(d) Find the complex potential Φ and express it explicitly in terms of *z*.

(e) Find the components of the electric field at $x = 1, \quad y = 1$ by three different methods: from *d*, from $\Phi(z)$, and from $\phi(x,y)$. Show with a sketch the vector for this field and the equipotential passing through $x = 1, \quad y = 1$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 2.6, Problem 3, Pearson Education, Inc., 2005.】