Chapter One
Complex Numbers

§1-1 Basic Conception and Definition

1. If $x, y \in R$, $i=\sqrt{-1}$, then

   $z = x + iy$

   is called a complex numbers, and

   $\left.\begin{array}{l} x = \operatorname{Re} z \\ y = \operatorname{Im} z \end{array}\right\}$ 仍為實數

2. If $x_1, y_1, x_2, y_2, \in R$, and $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$. Suppose that

   $z_1 = z_2 \quad \Leftrightarrow \quad x_1 = x_2 \quad y_1 = y_2$

3. If $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$, then

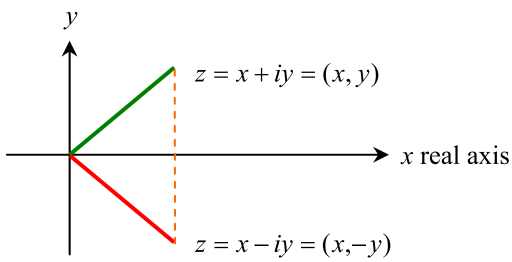
   a) $z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\, i$

   b) $z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)\, i$

   c) $z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)\, i$

   d) $\dfrac{z_1}{z_2} = \dfrac{x_1 x_2 + y_1 y_2}{x_2^{\,2} + y_2^{\,2}} + \dfrac{x_2 y_1 - x_1 y_2}{x_2^{\,2} + y_2^{\,2}}\, i$

4. If $z = x + iy$, then $\overline{z} = x - i y$ is called **conjugate** of z.



   1) $z + \overline{z} = 2x = 2 \operatorname{Re} z$

   2) $z \cdot \overline{z} = x^2 + y^2 = |z|^2$

   3) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

   <proof>：

   If $z_1 = x_1 + i y_1$, $z_2 = x_2 + i y_2$

   $\Rightarrow \quad z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)\, i$

   $\Rightarrow \quad \overline{z_1 + z_2} = (x_1 + x_2) - i\,(y_1 + y_2)$

   $= (x_1 - i y_1) + (x_2 - i y_2)$

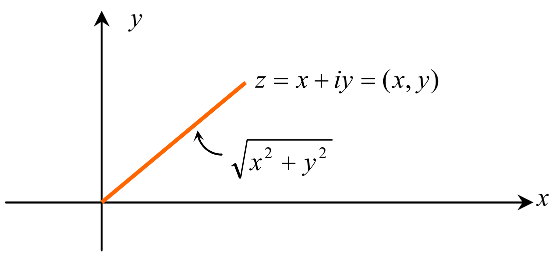
   $= \overline{z_1} + \overline{z_2}$

   4) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

   5) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \dfrac{\overline{z_1}}{\overline{z_2}}$

5. If $z = x + iy \quad \Rightarrow$ then $|z| = \sqrt{x^2 - y^2}$

   $|z| =$ the absolute value of z or called modulus



由上圖所示亦可知 $|z|=\sqrt{x^2+y^2}$ = The distance from point (x, y) to the origin (0, 0).

1) $z\cdot\bar{z}=x^2+y^2=|z|^2 \quad\Rightarrow\quad |z|=\sqrt{z\bar{z}}$

2) $|z|=\sqrt{x^2+(-y)^2}=\sqrt{x^2+y^2}=|\bar{z}|$

3) $|z_1\cdot z_2|=|z_1|\cdot|z_2|$

<proof>：

$$\begin{aligned}|z_1\cdot z_2|^2&=(z_1z_2)\cdot(\overline{z_1z_2})\\&=(z_1\cdot z_2)\cdot(\overline{z_1}\,\overline{z_2})\\&=(z_1\cdot\overline{z_1})\cdot(z_2\cdot\overline{z_2})\\&=|z_1|^2\cdot|z_2|^2\end{aligned}$$

$\Rightarrow\quad |z_1\cdot z_2|=|z_1|\cdot|z_2|$ **(Another proof: refer to p. 14 in the textbook of A. David Wunsch.)**

4) $\left|\dfrac{z_1}{z_2}\right|=\dfrac{|z_1|}{|z_2|}$

Example 1 Find $\left|\dfrac{(3+4i)^5}{(1+i\sqrt{3})}\right|=?$

<Sol.>

Using property (4), we have

$$\left|\frac{(3+4i)^5}{(1+i\sqrt{3})}\right|=\frac{\left|(3+4i)^5\right|}{\left|1+i\sqrt{3}\right|}=\frac{\left(\sqrt{3^2+4^2}\right)^5}{\sqrt{1^2+\left(\sqrt{3}\right)^2}}=\frac{5^5}{2}$$

6. If $z=x+iy \quad\Rightarrow\quad |z|=\sqrt{x^2+y^2}$ **(Pythagorean expression)**

$\Rightarrow\quad |z|\geq\sqrt{x^2}=|x|\geq x=\operatorname{Re} z$

So, we obtain the following inequality

$|z|\geq\operatorname{Re} z$

1) Triangle Inequality I: $|z_1+z_2|\leq|z_1|+|z_2|$

<proof>：

$$\begin{aligned}|z_1+z_2|^2&=(z_1+z_2)(\overline{z_1+z_2})\\&=(z_1+z_2)(\overline{z_1}+\overline{z_2})\\&=z_1\overline{z_1}+z_1\overline{z_2}+\overline{z_1}z_2+z_2\overline{z_2}\\&=|z_1|^2+|z_2|^2+z_1\overline{z_2}+z_2\overline{z_1}\\&=|z_1|^2+|z_2|^2+2\operatorname{Re}(z_1\overline{z_2})\\&\leq|z_1|^2+|z_2|^2+2\left|z_1\overline{z_2}\right|\\&=|z_1|^2+|z_2|^2+2|z_1|\left|\overline{z_2}\right|\\&=|z_1|^2+|z_2|^2+2|z_1||z_2|\\&=\left(|z_1|+|z_2|\right)^2\end{aligned}$$

$\Rightarrow\quad |z_1+z_2|\leq|z_1|+|z_2|$

2) Triangular Inequality II: $|z_1-z_2|\leq|z_1|+|z_2|$

3) Triangular Inequality III: $|z_1+z_2|\geq\big||z_1|-|z_2|\big|$

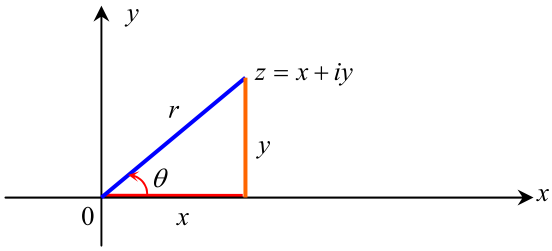
♣ **General case:**

$$|z_1+z_2+\cdots+z_n|\leq|z_1|+|z_2|+\cdots+|z_n|$$

7. $z = x + iy = r(\cos\theta + i\sin\theta)$

↑ Rectangular form　　　↑ Polar form

其所代表的意義如下所示：



其中

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

where　r：is the modulus of z

θ：is the argument of z　(Denoted by $\theta = \arg(z)$)

and　$r = |z| = \sqrt{x^2 + y^2}$

$$\cos\theta = \frac{x}{\sqrt{x^2 + y^2}}$$

$$\sin\theta = \frac{y}{\sqrt{x^2 + y^2}}$$

1) $z = r(\cos\theta + i\sin\theta) = r[\cos(\theta + 2n\pi) + i\sin(\theta + 2n\pi)]$, where $n \in I$.
2) If $-\pi < \theta \le \pi \Rightarrow \theta$ is the principal argument of z. （主幅角）
3) Euler's Formula：

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$\Rightarrow z = r(\cos\theta + i\sin\theta) = r e^{i\theta}$$

4) **DeMoivre's Theorem**

$$(e^{i\theta})^n = e^{in\theta} \text{ or } (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

<proof>：

$$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$$
$$= e^{in\theta}$$
$$= \cos n\theta + i\sin n\theta$$

5) If　$z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

then　$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

and　$\dfrac{z_1}{z_2} = \dfrac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

$$= \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

Example 1　Find $(1 + i\sqrt{3})^{11} = ?$

<Sol.>

1. $1 + i\sqrt{3} = 2\angle(\pi/3)$
2. $\left(1 + i\sqrt{3}\right)^{11} = \left(2\angle(\pi/3)\right)^{11} = 2^{11}\left[\cos\left(\frac{11\pi}{3}\right) + i\sin\left(\frac{11\pi}{3}\right)\right] = 2^{10} - i2^{10}\sqrt{3}$

H.W. 1 (a) By considering the expression $(p-q)^2$, where p and q are nonnegative real numbers, show that

$$p+q \le \sqrt{2}\sqrt{p^2+q^2}\ .$$

(b) Use the preceding result to show that for any complex number z we have

$$|\mathrm{Re}(z)|+|\mathrm{Im}(z)| \le \sqrt{2}\,|z|\,.$$

(c) Verify the preceding result for $z=1-i\sqrt{3}$.

(d) Find a value for z such that the equality sign holds in (b).

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.3, Prob. 41, Pearson Education, Inc., 2005.】

H.W. 2 (a) By considering the product of $1+ia$ and $1+ib$, and the argument of each factor, show that

$$\tan^{-1}(a)+\tan^{-1}(b)=\tan^{-1}\left(\frac{a+b}{1-ab}\right)$$

where a and b are real numbers.

(b) Use the preceding formula to prove that

$$\pi = 4\left[\tan^{-1}\left(\frac{1}{2}\right)+\tan^{-1}\left(\frac{1}{3}\right)\right]$$

Check this result with a pocket calculator.

(c) Extend the technique used in (a) to find a formula for $\tan^{-1}(a)+\tan^{-1}(b)+\tan^{-1}(c)$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.3, Prob. 43, Pearson Education, Inc., 2005.】

H.W. 3 (a) Consider the inequality $|z_1+z_2|^2 \le |z_1|^2+|z_2|^2+2|z_1||z_2|$. Prove this expression by algebraic means (no triangle).

<Hint> Note that $|z_1+z_2|^2=(z_1+z_2)\left(\overline{z_1+z_2}\right)=(z_1+z_2)\left(\overline{z_1}+\overline{z_2}\right)$. Multiply out $(z_1+z_2)\left(\overline{z_1}+\overline{z_2}\right)$, and use the facts that for a complex number, say, w

$$w+\overline{w}=2\,\mathrm{Re}(w) \quad \text{and} \quad |\mathrm{Re}(w)| \le |w|\,.$$

(b) Observe that $|z_1|^2+|z_2|^2+2|z_1||z_2|=(|z_1|+|z_2|)^2$. Show that the inequality proved in part (a) leads to the triangle inequality

$$|z_1+z_2| \le |z_1|+|z_2|\,.$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.3, Prob. 44, Pearson Education, Inc., 2005.】

8. **Fractional Power**

If $w=z^{1/n} \Rightarrow z=w^n$

$\Rightarrow$ w is the n-th root of z

1) If $\begin{cases} z=r\,(\cos\theta+i\sin\theta) \\ w=R\,(\cos\varphi+i\sin\varphi) \end{cases}$

$$\Rightarrow \quad r\,[\cos(\theta+2k\pi)+i\sin(\theta+2k\pi)]$$
$$=r\,(\cos\theta+i\sin\theta)$$
$$=R^n(\cos n\varphi++i\sin n\varphi)$$

$$\Rightarrow \begin{cases} R=\sqrt[n]{r} \\ \varphi=\dfrac{\theta+2k\pi}{n} \end{cases}, \quad k=0,\ 1,\ 2,\cdots,\ (n-1),$$

♣ General case:

$$z^{n/m}=\left(\sqrt[m]{r}\right)\left[\cos\left(\frac{n}{m}\theta+\frac{2kn\pi}{m}\right)+i\sin\left(\frac{n}{m}\theta+\frac{2kn\pi}{m}\right)\right], \quad k=0,\ 1,\ \cdots,\ m-1$$

Example 1 $(-1)^{1/2}=\sqrt{1}\left[\cos\left(\frac{\pi}{2}+k\pi\right)+i\sin\left(\frac{\pi}{2}+k\pi\right)\right],\quad k=0,\ 1\quad\Rightarrow\quad(-1)^{1/2}=i,\ -i$

Example 2 $(1+i\sqrt{3})^{1/5}=\sqrt[5]{2}\left[\cos\left(\frac{\pi}{15}+k\frac{2\pi}{5}\right)+i\sin\left(\frac{\pi}{15}+k\frac{2\pi}{5}\right)\right],\quad k=0,\ 1,\ 2,\ 3,\ 4$

$$\Rightarrow\quad\begin{aligned}&(1+i\sqrt{3})^{1/5}=1.123+i0.241,\quad k=0\\&(1+i\sqrt{3})^{1/5}=0.120+i1.142,\quad k=1\\&(1+i\sqrt{3})^{1/5}=-1.049+i0.467,\quad k=2\\&(1+i\sqrt{3})^{1/5}=-0.769-i0.854,\quad k=3\\&(1+i\sqrt{3})^{1/5}=1.574-i0.995,\quad k=4\end{aligned}$$

Example 3 Solve the following equation for w:

$$w^{4/3}+2i=0$$

<Sol.>

We have

$$w^{4/3}=-2i\quad\Rightarrow\quad w=\left(-2i\right)^{3/4}\ \text{with}\ \ n=3,\quad m=4,\quad r=|z|=|-2i|=2\ \text{ and }\ \theta=\arg(-2i)=-\pi/2\,.$$

Thus,

$$w=(-2i)^{3/4}=\left(\sqrt[4]{2}\right)^{3}\left[\cos\left(\frac{3}{4}\left(\frac{-\pi}{2}\right)+2k\frac{3}{4}\pi\right)+i\sin\left(\frac{\pi}{15}+k\frac{2\pi}{5}\right)\right],\quad k=0,\ 1,\ 2,\ 3$$

$$\Rightarrow\quad\begin{aligned}&w=\left(\sqrt[4]{2}\right)^{3}\angle(-3\pi/8),\quad k=0\\&w=\left(\sqrt[4]{2}\right)^{3}\angle(9\pi/8),\quad k=1\\&w=\left(\sqrt[4]{2}\right)^{3}\angle(21\pi/8),\quad k=2\\&w=\left(\sqrt[4]{2}\right)^{3}\angle(33\pi/8),\quad k=3\end{aligned}$$

H.W. 4 (a) With the aid of DeMoivre's theorem, express $\sin 3\theta$ as a real sum of terms containing only functions like $\cos^m\theta\sin^n\theta$, where m and n are nonnegative integers.

(b) Repeat the above, but use $\cos 3\theta$ instead of $\sin 3\theta$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 6, Pearson Education, Inc., 2005.】

H.W.5 (a) Using the DeMoivre's theorem, the binomial formula, and an obvious trigonometric identity, show that for integer n,

$$\cos n\theta=\mathrm{Re}\left[\sum_{k=0}^{n}\left(\cos^{n-k}\theta\right)\left(\sqrt{1-\cos^2\theta}\right)^{k}i^{k}\frac{n!}{(n-k)!k!}\right]$$

(b) Show that the preceding expression can be rewritten as

$$\cos n\theta=\sum_{m=0}^{n/2}\left(\cos\theta\right)^{n-2m}\left(1-\cos^2\theta\right)^{m}(-1)^{m}\frac{n!}{(n-2m)!(2m)!},\ \text{if } n \text{ is even}$$

and

$$\cos n\theta=\sum_{m=0}^{(n-1)/2}\left(\cos^{n-2m}\theta\right)\left(1-\cos^2\theta\right)^{m}(-1)^{m}\frac{n!}{(n-2m)!(2m)!},\ \text{if } n \text{ is odd.}$$

(c) The preceding formula are useful because they allow us to express $\cos n\theta$, where $n\geq 0$ is any integer, in a finite series involving only powers of $\cos\theta$ with the highest power equaling n. For example, show that $\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1$ by using one of the above formulas.

(d) If we replace $\cos\theta$ with x in either of the formulas derived in part (b), we obtain polynomials in the variable x of degree n. These are called ***Tchebyshev polynomials***, $T_n(x)$, after their inventor, Pafnuty

Tchebyshev (1821-1894), a Russian who is famous for his research on prime numbers. There are various spelling of his name, some beginning with C. Show that

$$T_5(x) = 16x^5 - 20x^3 + 5x$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 7, Pearson Education, Inc., 2005.】

H.W.6 Find all solutions of the equation: $w^4 + w^2 + 1 = 0$. **【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 25, Pearson Education, Inc., 2005.】**

H.W. 7 Prove that $\left(\dfrac{1+i\tan\theta}{1-i\tan\theta}\right)^n = \dfrac{1+i\tan n\theta}{1-i\tan n\theta}$, where n is any integer. **【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 8, Pearson Education, Inc., 2005.】**

H.W. 8 (a) Show that $z^{n+1} - 1 = (z-1)(z^n + z^{n-1} + \cdots + z + 1)$,where $n \ge 0$ is an integer and z is any complex number.

The preceding implies that

$$\frac{z^{n+1}-1}{z-1} = z^n + z^{n-1} + \cdots + z + 1, \quad \text{for} \quad z \ne 1 \qquad \textbf{(A)}$$

which the reader should recognize as the sum of a geometric series.

(b) Use the preceding result to find and plot all solutions of $z^4 + z^3 + z^2 + z + 1 = 0$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 27, Pearson Education, Inc., 2005.】

H.W. 9 Use the formula for the sum of a geometric series in **H.W. 8** and DeMoivre's theorem to derive the following formulas for $0 < \theta < 2\pi$:

$$1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta = \frac{\cos(n\theta/2)\sin[(n+1)\theta/2]}{\sin(\theta/2)},$$

$$\sin\theta + \sin 2\theta + \sin 3\theta + \cdots + \sin n\theta = \frac{\sin(n\theta/2)\sin[(n+1)\theta/2]}{\sin(\theta/2)}$$

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 34, Pearson Education, Inc., 2005.】

H.W. 10 (a) If $z = r(\cos\theta + i\sin\theta)$, show that the sum of the values of $z^{1/n}$ is given by

$$\sum_{k=0}^{n-1} \sqrt[n]{r}\left[\cos(\theta/n) + i\sin(\theta/n)\right] \times \left[\cos(2\pi/n) + i\sin(2\pi/n)\right]^k, \text{ for } 0 \le \theta \le 2\pi$$

(b) Show that the sum of the values of $z^{1/n}$ is zero. Do this by rewriting Eq. (A) in part (a) of **H.W. 8**, but with $n-1$ used in place of n, and employing the formula of part (a).

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 28, Pearson Education, Inc., 2005.】

H.W. 11 If n is an integer greater than or equal to 2, prove that:

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cdots + \cos\left[\frac{2(n-1)\pi}{n}\right] = -1,$$

and that

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \cdots + \sin\left[\frac{2(n-1)\pi}{n}\right] = 0$$

<Hint> Use the result of **H.W. 10 (b)** and take $z = 1$.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 35, Pearson Education, Inc., 2005.】

H.W. 12 Most computational packages such as MATLAB have a numerical means of finding all the roots of any polynomial equation like $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ where n is a positive integer and the coefficients a_n, a_{n-1}, etc. are arbitrary known complex numbers. In **H.W. 8**, we learned an analytic method for solving this equation if all the coefficients are unity; we look at the roots of $z^{n+1} - 1 = 0$ for $z \neq 1$. Using MATLAB, or something comparable, solve the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ and compare your solution with that obtained from the method of **H.W. 8**.

【本題摘自：A. David Wunsch, *Complex Variable with Applications*, 3rd ed., Exercise 1.4, Prob. 36, Pearson Education, Inc., 2005.】

<Ans.>

MATLAB code:

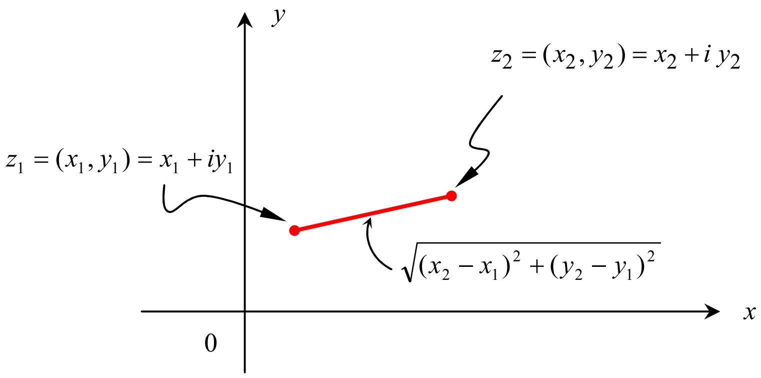
```
w=[1 1 1 1 1 1];
roots(w)

ans =

   0.5000 + 0.8660i
   0.5000 - 0.8660i
  -1.0000
  -0.5000 + 0.8660i
  -0.5000 - 0.8660i
```

♣ We can also use $z^6 - 1 = 0$ to solve this problem, i.e.,

$$z = 1^{1/6} = \cos\left(\frac{2k\pi}{6}\right) + i\sin\left(\frac{2k\pi}{6}\right), \quad k = 1, 2, 3, 4, 5, \text{ but not } k = 0.$$

$$\Rightarrow \begin{aligned} z = 1^{1/6} &= \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right) = 0.5 + 0.866i, && k = 1 \\ &= \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = -0.5 + 0.866i, && k = 2 \\ &= \cos(\pi) + i\sin(\pi) = -1 + 0i, && k = 3 \\ &= \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = -0.5 - 0.866i, && k = 4 \\ &= \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right) = 0.5 - 0.866i, && k = 5 \end{aligned}$$

9. Conception of Distance



$|z_2 - z_1| =$ The distance from z_1 to z_2

Since $z_2 - z_1 = (x_2 - x_1) + i\,(y_2 - y_1)$

so $|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

10. Points, Sets, Loci, and Regions in the Complex Plane

Ex. $\mathrm{Re}(z) = \mathrm{Re}(x + iy) = 1 \Rightarrow x = 1 \Rightarrow$ a vertical **line** in the z-plane.

Ex. Inequality $\mathrm{Re}(z) < 1 \Rightarrow$ a **region** to the left of the vertical line $x = 1$ in the z-plane

Ex. $-2 \le \mathrm{Re}(z) \le 1 \Rightarrow$ **infinite strip** in the z-plane

Ex. $\mathrm{Re}(z) \le \mathrm{Im}(z) \Rightarrow$ the **shaded region** shown in the figure and includes the boundary line $x = y$.

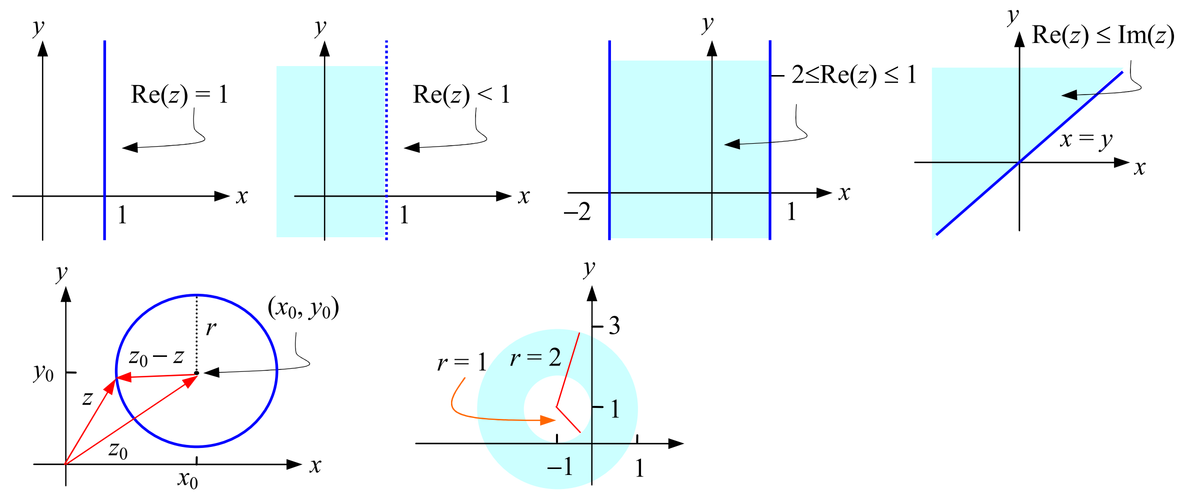
Ex. Let $z_0 = x_0 + iy_0$ be a complex number. $|z - z_0| = r$, where $r > 0 \Rightarrow$ a **circle** in the z-plane.

Ex. Let r_1 and r_2 be a pair of nonnegative real numbers and $r_1 < r_2$.
$r_1 < |z - z_0| < r_2 \Rightarrow$ an **annulus** of inner radius r_1, outer radius r_2, and center z_0 in the z-plane

Example 1 What region is described by the inequality $1 < |z + 1 - i| < 2$?

<Sol.>

We can write this as $r_1 < |z - z_0| < r_2$, where $r_1 = 1,\ r_2 = 2,\ z_0 = -1 + i$. The region described is the shaded annulus area, but not including the circles shown in the following figure.



1) sets ≡ collections of points
2) The points belonging to a set are called its *members* or *elements*.
3) A *neighborhood* of radius r of a point z_0 is the collection of all the points inside a circle, of radius r, centered at z_0.
4) A *deleted neighborhood* of z_0 consists of the points inside a circle centered at z_0 but excludes the point z_0 itself.
5) An *open set* is one in which every member of the set has same neighborhood, all of whose points lie entirely within the set.
6) A *connected set* is one in which any two points of the set can be joined by some path of straight line segments, all of whose points belong to the set.
7) A *domain* is an open connected set.
8) A *simply connected domain* contains no holes, but a *multiply connected domain* has one or more holes.
9) A *boundary point* of a set is a point whose every neighborhood contains at least one point belonging to the set and one point not belonging to the set.
10) An *interior point* of a set is a point having some neighborhood, all of whose elements belong to the set.
11) An *exterior point* of a set is a point having a neighborhood, all of whose elements do not belong to the set.
12) Let P be a point whose every deleted neighborhood contains at least one element of a set S. We say that P is an *accumulation (limit) point* of S.

13) The *null set* contains no points. (*empty set*)
14) A *region* is a domain plus possibly some, none, or all the boundary points of the domain.
15) A *closed region* consists of a domain plus all the boundary points of the domain.
16) A *bounded set* is one whose points can be enveloped by a circle of some finite radius.
17) A *bounded closed region* is called a *compact* region.

Ex. The set $|z|<1$ is open.

Ex. See: **A. David Wunsch, *Complex Variable with Applications*, 3rd ed., pp. 1-44, Pearson Education, Inc., 2005**.

♣ The complex number infinity and the point at infinity

***Stereographic projection*:**

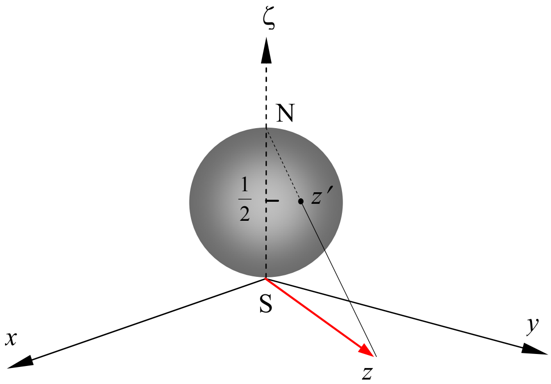
Consider the z-plane, with a third orthogonal axis, the ζ-axis, added on. A sphere of radius 1/2 is placed with center at $x = 0$, $y = 0$, and $\zeta = 1/2$. The north pole, N, lies at $x = 0$, $y = 0$, and $\zeta = 1$ while the south pole, S, is at $x = 0$, $y = 0$, and $\zeta = 0$. This is called the *Riemann number sphere*.

Let us draw a straight line from N to the point in the xy-plane that besides N, represents a complex number z. This line intersects the sphere at exactly one point, which we label z'. We say that z' is the project on the sphere of z.

1) N on the sphere corresponds to the point at infinity.
2) Except when explicitly stated, we will not regard infinity as a number in any sense in this text.
3) We will treat infinity as a number satisfying these rules:

$$\frac{z}{\infty}=0;\quad z\pm\infty=\infty,\quad (z\neq\infty);\quad \frac{z}{0}=\infty,\quad (z\neq 0);$$

$$z\cdot\infty=\infty,\quad (z\neq 0);\quad \frac{\infty}{z}=\infty,\quad (z\neq\infty)$$



1-2 Some Examples

1. If z_1+z_2 and z_1z_2 are both real.

$\Rightarrow$ z_1 and z_2 are both real or $z_1=-z_2$

<proof> :

Let $\begin{cases} z_1+z_2=r_1 \\ z_1z_2=r_2 \end{cases}$, $r_1, r_2\in R$

It is sufficient to prove that the results for the case when both z_1 and z_2 are not zero. For example, $z_2\neq 0$. Then,

$$\begin{aligned} z_1 &= \frac{r_2}{z_2} \\ &= \frac{r_2\bar{z}_2}{z_2\bar{z}_2} \\ &= \frac{r_2}{|z_2|^2}\bar{z}_2 \\ &= r_3\bar{z}_2 \end{aligned}$$

where $r_3=\dfrac{r_2}{|z_2|^2}\in R$

Thus, $z_1 + z_2 = r_3 z_2 + z_2$

$= (r_3 + 1) z_2$

$= r_1 \in R$

But, we know that

$\operatorname{Im} r_1 = 0$

So,

$$\operatorname{Im}[(r_3+1)z_2] = (r_3+1)\operatorname{Im}(z_2) = \operatorname{Im} r_1 = 0$$

$\Rightarrow \quad r_3 = -1 \quad$ or $\quad \operatorname{Im} z_2 = 0$

1) If we take $\operatorname{Im} z_2 = 0 \Rightarrow z_2$ is real.
   Since $z_1 + z_2 = r_1 \Rightarrow z_1$ is also real.
2) If we take $r_3 = -1$
   $\Rightarrow z_1 = r_3 z_2 = -z_2$

2. If $z = 1 + \sqrt{3}i$，express the z in polar form completely.

<Sol.>： $r = \sqrt{x^2 + y^2} = \sqrt{1^2 + \sqrt{3}^2} = 2$

and $\begin{cases} \cos\theta = \dfrac{1}{\sqrt{2}} \\ \sin\theta = \dfrac{\sqrt{3}}{2} \end{cases} \Rightarrow \theta = \dfrac{\pi}{3} + 2n\pi$

$\therefore \quad z = 2[\cos(\frac{\pi}{3} + 2n\pi) + i\sin(\frac{\pi}{3} + 2n\pi)]$

If $-\pi < \theta \le \pi$，then

$z = 2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$

3. Find $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^{100}$

<Sol.>： $\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$

$$\Rightarrow \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)^{100} = \left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{100} = \cos\frac{100\pi}{3} + i\sin\frac{100\pi}{3} = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$$

4. Find $\left|\dfrac{(3+4i)(1+\sqrt{3}i)}{5+12i}\right|$

<Sol.>： $\left|\dfrac{(3+4i)(1+\sqrt{3}i)}{5+12i}\right|$

$$=\frac{|3+4i|\cdot|1+\sqrt{3}i|}{|5+12i|}=\frac{5\times 12}{13}=\frac{10}{13}$$

Note that don't express in terms of polar form.

5. Find the square root of $3+4i$

<Sol.>： We will solve this problem in two ways.

1) Let $z=(3+4i)^{\frac{1}{2}}$

$\Rightarrow$ $z^2=3+4i$

$\Rightarrow$ $x^2-y^2+2xyi=3+4i$

$$\Rightarrow \begin{cases} x^2-y^2=3 \\ xy=2 \end{cases}$$

But $(x^2+y^2)^2=(x^2-y^2)^2+2x^2y^2$

$=9+16$

$=25$

$$\Rightarrow \begin{cases} x^2+y^2=5 \\ x^2-y^2=3 \end{cases}$$

$$\Rightarrow \begin{cases} x^2=4 \\ y^2=1 \end{cases} \Rightarrow \begin{cases} x=\pm 2 \\ y=\pm 1 \end{cases}$$

Since $xy=2>0$，we know x and y are both positive or negative

$$\Rightarrow \begin{cases} x=2 \\ y=1 \end{cases} \text{ or } \begin{cases} x=-2 \\ y=-1 \end{cases}$$

$\Rightarrow$ $z_1=2+i$，$z_2=-2-i$

2) We can use the following formula:

$$w=\sqrt[n]{r}\left[\cos\frac{\theta+2k\pi}{n}+i\sin\frac{\theta+2k\pi}{n}\right]$$

where $k=0, 1, 2, \cdots\cdots, (n-1)$ ------(1)

Since $3+4i=5\left(\frac{3}{5}+\frac{4}{5}i\right)$

$\Rightarrow$ $r=5$, $\cos\theta=\frac{3}{5}$, $\sin\theta=\frac{4}{5}$

$0<\theta<\frac{\pi}{2}$, $n=2$

i) $k=0$

$$\Rightarrow z_1=\sqrt{5}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$$

$$=\sqrt{5}\left[\sqrt{\frac{1+\cos\theta}{2}}+i\sqrt{\frac{1-\cos\theta}{2}}\right]$$

$$=\sqrt{5}\left[\frac{2}{\sqrt{5}}+\frac{1}{\sqrt{5}i}\right]$$

$$=2+i$$

ii) $k=1$

$$\Rightarrow z_2=\sqrt{5}\left[(\cos(\frac{\theta}{2}+\pi)+i\sin(\frac{\theta}{2}+\pi)\right]$$

$$= \sqrt{5}\left[-\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right]$$

$$= \sqrt{5}\left[-\frac{2}{\sqrt{5}} - \frac{1}{\sqrt{5}\,i}\right]$$

$$= -2 - i$$

6. Find the fourth roots of $-2-2\sqrt{3}\,i$

<Sol.>：

Let $z = -2-2\sqrt{3}\,i$

Then $\begin{cases} r = |z| = 4 \\ \theta = \frac{4}{3}\pi \quad \text{or} \quad \theta = -\frac{2}{3}\pi \end{cases}$

Substituting the above data into equation (1), gives

$$z^{\frac{1}{4}} = \sqrt{2}\, e^{i\left[\frac{\pi}{3}+k\cdot(\frac{\pi}{2})\right]} \text{, where } \quad k = 0, 1, 2, 3$$

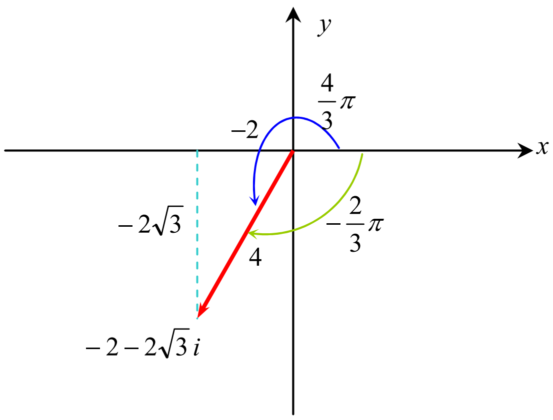
The four roots of z are

$z_1 = \sqrt{2}/2(1+i\sqrt{3})$

$z_2 = \sqrt{2}/2(-\sqrt{3}+i)$

$z_3 = \sqrt{2}/2(-1-i\sqrt{3})$

$z_4 = \sqrt{2}/2(\sqrt{3}-i)$



7. Denote the roots of equation $(z+1)^5 + z^5 = 0$ by z_k, $k = 0, 1, 2, 3, 4$.

Show that $\operatorname{Re} z_k = -\frac{1}{2}$, $k = 0, 1, 2, 3, 4$.

That is, all the roots lie on the line $x = -\frac{1}{2}$, which is parallel to the imaginary axis.

<pf.>： $(z+1)^5 + z^5 = 0$

$$\Rightarrow \left[\frac{z+1}{z}\right]^5 = -1$$

$$\Rightarrow -\frac{z+1}{z} = e^{i(\frac{2k\pi}{5})} \quad \text{---------} \quad (1)$$

where $k = 0, 1, 2, 3, 4$.

Let $\frac{2}{5}\pi = \lambda$, and solving for z in (1), we find the roots to be

$$z_k = -\frac{1}{1+e^{i\lambda k}}$$

$$= -\frac{1}{1+\cos\lambda k + i\sin\lambda k}$$

$$= -\frac{(1+\cos\lambda k) - i\sin\lambda k}{2(1+\cos\lambda k)}$$

$$= -\frac{1}{2} + i\frac{\sin\lambda k}{2(1+\cos\lambda k)} \quad , \quad k = 0, 1, 2, 3, 4$$

Then, we have

$$\operatorname{Re} z_k = -\frac{1}{2}$$

8. What is the graph of $|z+i| = 1.$?

<Sol.> : $|z+i| = |x+iy+i|$

$$= \sqrt{x^2+(y+1)^2}$$

$$= 1$$

$\Rightarrow \quad x^2+(y+1)^2 = 1$

Its graph is a circle, centered at $(0,-1)$ and the radius is $r = 1$.

9. What is the graph of $\left|\frac{z+1}{z-1}\right| = 1$.

<Sol.> : $\left|\frac{z+1}{z-1}\right| = \left|\frac{x+iy+1}{x+iy-1}\right|$

$$= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}$$

$$= 1$$

$\Rightarrow \quad (x+1)^2 = (x-1)^2$

$\Rightarrow \quad x = 0$

So, its graph is y axis.

10. What is the graph of $\left|\frac{z+1}{z-1}\right| = 2$.

<Sol.> : $\left|\frac{z+1}{z-1}\right| = \left|\frac{x+iy+1}{x+iy-1}\right|$

$$= \frac{\sqrt{(x+1)^2+y^2}}{\sqrt{(x-1)^2+y^2}}$$

$$= 2$$

$\Rightarrow \quad (x+1)^2+y^2 = 4[(x-1)^2+y^2]$

$\Rightarrow \quad 3x^2-10x+3y^2+3 = 0$

$\Rightarrow \quad x^2-\frac{10}{3}x+\frac{25}{3^2}+y^2 = \frac{16}{9}$

So, $\left(x-\frac{5}{3}\right)^2 + y^2 = \left(\frac{4}{3}\right)^2$

Its graph is a circle.

11. If $|\alpha|=1$, and $\alpha \neq \beta$, show that $\left|\dfrac{\alpha-\beta}{1-\beta\alpha}\right|=1$.

<Proof> : $\left|\dfrac{\alpha-\beta}{1-\beta\alpha}\right|^2=\left(\dfrac{\alpha-\beta}{1-\beta\alpha}\right)\overline{\left(\dfrac{\alpha-\beta}{1-\beta\alpha}\right)}$

$$=\frac{\alpha-\beta}{1-\beta\alpha}\cdot\frac{\overline{\alpha}-\overline{\beta}}{1-\overline{\beta}\overline{\alpha}}$$

$$=\frac{\alpha\overline{\alpha}+\beta\overline{\beta}-\overline{\alpha}\beta-\alpha\overline{\beta}}{1+\alpha\overline{\alpha}\beta\overline{\beta}-\overline{\alpha}\beta-\alpha\overline{\beta}}$$

$\because \ \alpha\overline{\alpha}=|\alpha|^2=1$

$$=\frac{1+\beta\overline{\beta}-\overline{\alpha}\beta-\alpha\overline{\beta}}{1+\beta\overline{\beta}-\overline{\alpha}\beta-\alpha\overline{\beta}}$$

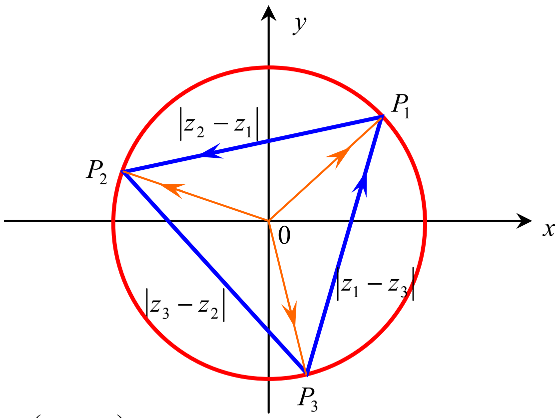
$$=1$$

$$\Rightarrow \quad \left|\frac{\alpha-\beta}{1-\beta\alpha}\right|=1$$

12. Suppose that $z_1, z_2\ z_3$ are three complex numbers, such that

$$\begin{cases}|z_1|=|z_2|=|z_3|=1\\ z_1+z_2+z_3=0\end{cases}$$

Show that z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the unit circle.



<Proof>:

Let $z_k=\overrightarrow{OP_k}$, $k=0, 1, 2, 3$

Then $\overrightarrow{P_1P_2}=z_2-z_1$

$\overrightarrow{P_2P_3}=z_3-z_2$

$\overrightarrow{P_3P_1}=z_1-z_3$

Since $z_1+z_2+z_3=0 \ \Rightarrow \ z_1=-(z_2+z_3)$

$$\Rightarrow \begin{cases}|z_2-z_1|=|2z_2+z_3|\\ |z_1-z_3|=|2z_3+z_2|\end{cases}$$

$$\Rightarrow \ |2z_2+z_3|^2-|2z_3+z_2|^2$$

$$=(2z_2+z_3)(\overline{2z_2+z_3})-(2z_3+z_2)(\overline{2z_3+z_2})$$

$$=3(|z_2|^2-|z_3|^2)$$

$$=0$$

Because $|z_1|=|z_2|=|z_3|=1 \ \Rightarrow \ |z_2|^2=|z_3|^2=1$

$\Rightarrow \ |z_2-z_1|=|z_1-z_3|$ ------ (1)

In a similar manner, we can verify that

$|z_2-z_1|=|z_3-z_2|$ ------ (2)

From equation (1) and (2), we obtain

$|z_2-z_1|=|z_3-z_2|=|z_1-z_3|$

Hence the triangle $\Delta P_1P_2P_3$ is equilateral.

Since $|z_1|=|z_2|=|z_3|=1$, the triangle $\Delta P_1P_2P_3$ can be inscribed in the unit circle.