Prandtle/Blasius Solution

Prandtle used boundary layer concept and imposed approximation (valid for large Reynolds number flows) to simplify the governing Navier-Stokes equations. H. Blasius (1883-1970), one of Prandtl's students, solved these simplified equations.

從Navier-Stokes Equations 開始

The Navier-Stokes Equations

❖ Under **incompressible flow with constant viscosity conditions**, the Navier-Stokes equations are reduced to:

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\rho g_x+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\rho g_y+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\rho g_z+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)$$

這是已經過「不可壓縮」且「黏度是constant」假設後，所得到的簡化後的Navier-Stokes equations

Prandtle/Blasius Solution 1/10

Prandtle進一步簡化…請注意簡化過程的假設…

- The details of viscous incompressible flow past any object can be obtained by solving the governing Navier-Stokes equation.
- For **steady**, **two dimensional** laminar flow with **negligible gravitational effects**, these equations reduced to the following

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

- In addition, the conservation of mass

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

綜合假設…不可壓縮、2D、重力效應忽略、黏度=constant 、steady
注意每一假設後，方程式會如何改變…

No analytical solution

三個未知數
需要三個方程式

Prandtle/Blasius Solution 2/10

再進一步簡化：（1）垂直板的速度遠低於平行板的速度（2）邊界層很薄，y方向梯度當然遠大於x方向梯度

❖ Simplification…….

Since the boundary layer is thin, it is expected that the component of velocity normal to the plate is much smaller than the parallel to the plate and that the rate of change of any parameter across the boundary layer should be much greater than that along the flow direction. That is

$$v \ll u \quad \text{and} \quad \frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$$

得再加no pressure variations

$$\begin{cases} u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} \\ \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0 \end{cases}$$

仍無解析解

剩下兩個未知數兩個方程式

Prandtle/Blasius Solution 3/10

總結簡化過程後的結果

Second order partial differential equations

Governing equations

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=+\nu\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

Boundary conditions

$$y=0,\quad u=0,\quad v=0$$

看似簡單，但…

$$y=\infty,\quad u=U,\quad \frac{\partial u}{\partial y}=0$$

Solution ? …… ***are extremely difficult to obtain.***

Prandtle/Blasius Solution 4/10

徒弟出手……將偏微分轉成常微分

- Blasius reduced the partial differential equations to an ordinary differential equation…

The velocity profile, u/U, should be similar for all values of x. Thus the velocity profile is of the dimensionless form

1 $$\frac{u}{U} = g(\eta) \quad \text{where} \quad \eta \propto \frac{y}{\delta}$$

設定無因次速度分佈曲線u/U，與無因次板垂直向距離y/δ；u/U僅與y/δ有關，關係不明，也是我們想要探討的

見下頁

Is an unknown function to be determined.

Prandtle/Blasius Solution 5/10

設定一個無因次變數與stream function

Set a dimensionless similarity variable

2 $\eta = \left(\frac{U}{\nu x}\right)^{1/2} y$ and the stream function 3 $\psi = (\nu x U)^{1/2} f(\eta)$

Is an unknown function to be determined.

The velocity component

$$u = \frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial \eta}\frac{\partial \eta}{\partial y} = \sqrt{\nu x U} f'(\eta)\sqrt{\frac{U}{\nu x}} = Uf'(\eta)$$

$$v = -\frac{\partial \psi}{\partial x} = -[\sqrt{\nu x U}\frac{\partial f}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U}{x}} f] = [\sqrt{\nu x U}\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{1}{2}\sqrt{\frac{\nu U}{x}} f] = \frac{1}{2}\left(\frac{\nu U}{x}\right)^{1/2}(\eta f' - f)$$

⟶ 代入方程式 $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = +\nu\frac{\partial^2 u}{\partial y^2}$ $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

Prandtle/Blasius Solution 6/10

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=+\nu\frac{\partial^2 u}{\partial y^2} \qquad \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$

$$\frac{\partial u}{\partial x}=-\frac{U}{2x}\eta\frac{d^2f}{d\eta^2} \qquad \frac{\partial u}{\partial y}=U\sqrt{U/\nu x}\frac{d^2f}{d\eta^2} \qquad \frac{\partial^2 u}{\partial y^2}=\frac{U^2}{\nu x}\frac{d^3f}{d\eta^3}$$

$$2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=2f'''+ff''=0$$

原來要解u與v，現在要解f、f’ 、f”

Prandtle/Blasius Solution 7/10

$$2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=2f'''+ff''=0$$

Nonlinear, third-order ordinary differential equation

With boundary conditions

f = f' = 0 at η = 0 邊界條件

f' → 1 at $\eta \to \infty$

Solution ? No analytical solution !

Easy to integrate to obtain numerical solution

數值解

Blasius solved it using a power series expansion about

η = 0 …Blasius solution

利用power series expansion，解出f、f'、f"

Prandtle/Blasius Solution 8/10

Numerical solution of

$$2\frac{d^3f}{d\eta^3}+f\frac{d^2f}{d\eta^2}=2f'''+ff''=0$$

TABLE 9.1
Laminar Flow along a Flat Plate (the Blasius Solution)

$\eta = y(U/\nu x)^{1/2}$	$f'(\eta) = u/U$	η	$f'(\eta)$
0	0	3.6	0.9233
0.4	0.1328	4.0	0.9555
0.8	0.2647	4.4	0.9759
1.2	0.3938	4.8	0.9878
1.6	0.5168	5.0	0.9916
2.0	0.6298	5.2	0.9943
2.4	0.7290	5.6	0.9975
2.8	0.8115	6.0	0.9990
3.2	0.8761	∞	1.0000

Table 9.1 The Function $f(\eta)$ for the Laminar Boundary Layer along a Flat Plate at Zero Incidence

$\eta = y\sqrt{\frac{U}{\nu x}}$	f	$f' = \frac{u}{U}$	f''
0	0	0	0.3321
0.5	0.0415	0.1659	0.3309
1.0	0.1656	0.3298	0.3230
1.5	0.3701	0.4868	0.3026
2.0	0.6500	0.6298	0.2668
2.5	0.9963	0.7513	0.2174
3.0	1.3968	0.8460	0.1614
3.5	1.8377	0.9130	0.1078
4.0	2.3057	0.9555	0.0642
4.5	2.7901	0.9795	0.0340
5.0	3.2833	0.9915	0.0159
5.5	3.7806	0.9969	0.0066
6.0	4.2796	0.9990	0.0024
6.5	4.7793	0.9997	0.0008
7.0	5.2792	0.9999	0.0002
7.5	5.7792	1.0000	0.0001
8.0	6.2792	1.0000	0.0000

Prandtle/Blasius Solution 9/10

❖ 依據定義，從 Blasius solution（ Table 9.1）解出

Boundary layer thickness

$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Ux/\nu}} = \frac{5.0}{\sqrt{Re_x}}$$

要知道是怎麼來的，千萬不要背

$$Re_x = Ux/\nu$$

Displacement thickness

$$\frac{\delta^*}{x} = \frac{1.721}{\sqrt{Re_x}}$$

Prandtle/Blasius Solution 10/10

Momentum thickness

$$\frac{\Theta}{x} = \frac{0.664}{\sqrt{Re_x}}$$

Shear stress

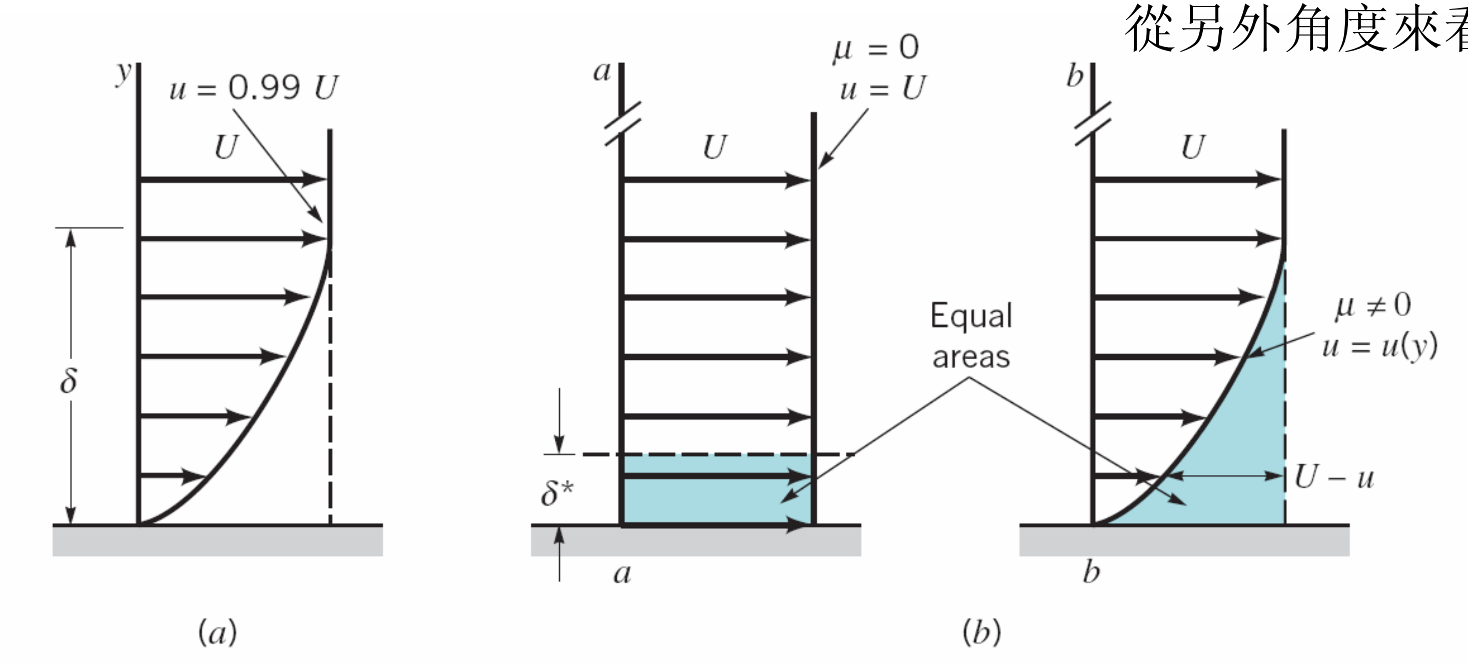
$$\tau_w = \mu \frac{du}{dy}\Big|_{y=0} = 0.332 U^{3/2} \sqrt{\frac{\rho\mu}{x}} = \frac{0.332 \rho U^2}{\sqrt{Re_x}}$$

Boundary Layer Thickness

邊界層厚度？定義？範圍？

- Standard Boundary layer thickness
- Boundary layer displacement thickness
- Boundary layer momentum thickness

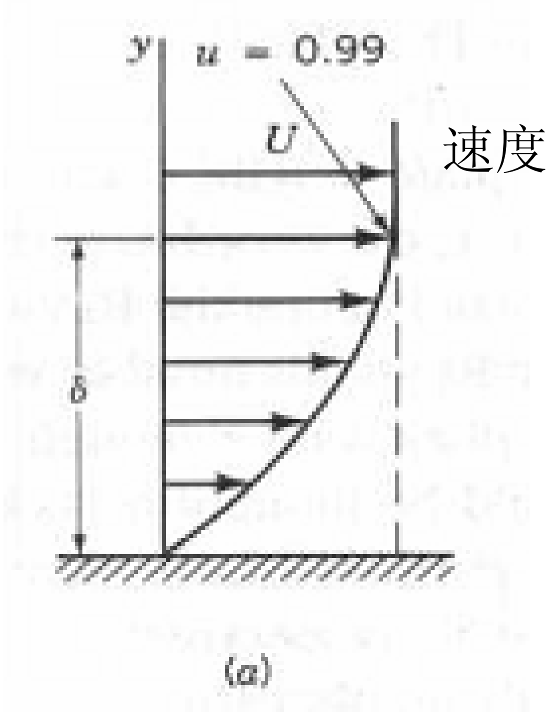
從另外角度來看邊界層



Standard Boundary Layer Thickness

❖ The standard boundary layer thickness is the distance from the plate at which the fluid velocity is within some arbitrary value of the upstream velocity.

用速度來界定



速度達到free stream的99%

$y = \delta$ where u=0.99 U

Boundary Layer Displacement Thickness

邊界層產生阻止流體的效應，mass flux與momentum flux都少下來，少的部分**相當於多厚？**

- The boundary layer retards the fluid, so that the mass flux and momentum flux are both less than they would be in the absence of the boundary layer.
- The displacement distance is the distance the plate would be moved so that the loss of mass flux (due to reduction in uniform flow area) is equivalent to the loss the boundary layer causes.

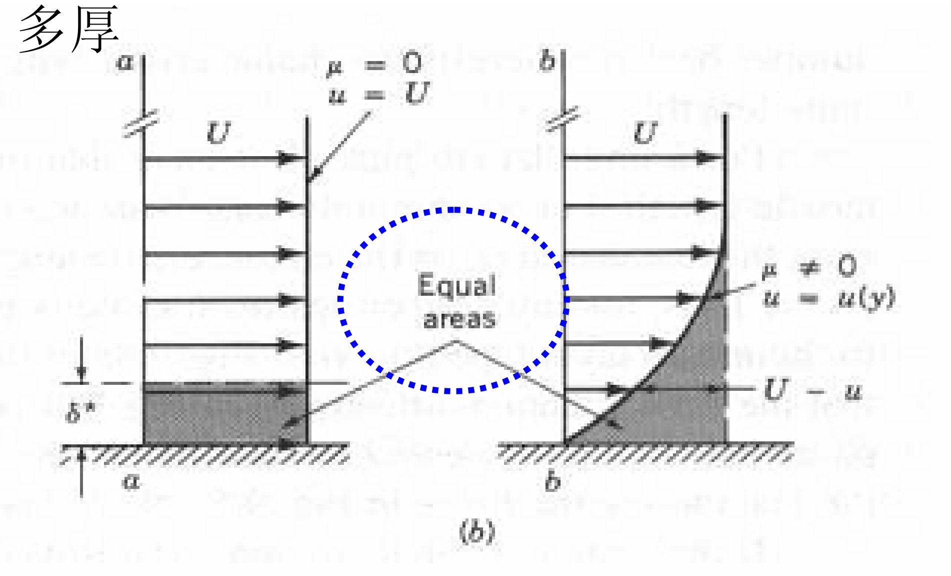
Mass flux短少　　短少的量相當於..多厚

The loss due to the boundary layer

$$\rho\delta^* U w = \int_0^\infty \rho(U - u) w\, dy$$

$$\Rightarrow \delta^* = \int_0^\infty \left(1 - \frac{u}{U}\right) dy$$

此厚度稱為Displacement thickness



(b)

Boundary Layer Momentum Thickness

- The momentum thickness is the distance the plate would be moved so that the loss of momentum flux is equivalent to the loss the boundary layer actually causes.

The loss of momentum due to the boundary layer

Momentum flux減少

$$\rho w U^2 \Theta = \int_0^\infty \rho w u(U-u)dy$$

短少的量（Momentum flux）相當於..多厚

$$\Rightarrow \Theta = \int_0^\infty \frac{u}{U}\left(1-\frac{u}{U}\right)dy$$

此厚度稱為Momentum thickness